NuHAG Numerical Harmonic Analysis Group

Banach Gelfand Triple: A Soft Introduction to Mild Distributions

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Theory of Functions and Functional Spaces, Nov. 2018

Goal of the Talk

This talk is to be understood as a contribution to the theory of function and functional (meaning generalized function) spaces. This theory has a long history and not only experts find a rich family of such spaces, including Banach spaces, but also topological vector spaces (in particular *nuclear Frechet spaces*) among them.
Among the Banach spaces certainly the Lebesgue Spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ are the most fundamental one. Their study has definitely shaped the development of early (linear) functional analysis, with case $p = 2$ being the prototype of a Hilbert spaces.
The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and it dual, the space $\mathcal{S}'(\mathbb{R}^d)$ of *tempered distributions* being the most prominent among them, but there are also the Gelfand-Shilov classes $\mathcal{S}_s^r$ providing the basis for the theory of *ultra-distributions*.

Focus on Abstract and Applied Harmonic Analysis

Although the tools described below are of some interest in the theory of pseudo-differential operators (for example) we will focus on questions which are of interest in the context of

1. Abstract Harmonic Analysis (i.e. for the treatment of function spaces over locally compact Abelian (LCA) groups);
2. Applied Harmonic Analysis, meaning engineering applications, where we have translation invariant linear systems (TILS) described as convolution operators using their *impulse response* resp. by the corresponding *transfer function*;
3. Numerical Harmonic Analysis, dealing with the DFT/FFT, sampling, approximation, and so on.

Banach Gelfand Triples

The concept of Banach Gelfand Triples is not new. It is a variant of the idea of a Gelfand Triple, with the chain

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d), \tag{1}$$

resp. variants of it, involving the Gelfand-Shilov spaces $\mathcal{S}_s^r$ which play an important role in the the theory of *ultra-distributions* and more recently time-frequency analysis. We will work with

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d). \tag{2}$$

We will often compare this situation with the embeddings

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \tag{3}$$

Comparing Things

In a nutshall we have these embeddings:

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d), \quad (4)$$

with the following differences:

1. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ and $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ are Banach spaces, not just topological vector spaces;
2. both spaces are Fourier invariant, but also under dilations;
3. $\boldsymbol{S}_0(\mathbb{R}^d)$ (or $\boldsymbol{S}_0'(\mathbb{R}^d)$) are *not* closed under differentiation;
4. nevertheless there is a kernel theorem;
5. useful for Fourier and time-frequency (or Gabor) analysis;
6. they are special cases of *modulation spaces*.

Modulation Spaces and Gabor Analysis

For those already familiar with some of the function spaces relevant for *time-frequency* (TFA) and *Gabor analysis*, the so-called modulation spaces (first presented at a conference in Kiev in 1983 (!)) we will provide some information about the key players within that theory, and how they can be used to simplify the approach to Fourier Analysis[1].
For those who want to learn about TFA and Gabor Analysis this talk should provide basic information about the fundamental players in this approach, which also appears to be a good way to deal with the problem of "generalized functions" (including Dirac measures, or Dirac combs and their Fourier transforms).

[1] See also my second talk here, entitled: `Mild distributions: A new setting making distributions more accessible to engineers`

The World of Modulation Spaces

Nowadays there we dispose over a rich variety of *Banach spaces of functions*, which for me in the spirit of Hans Triebel but also in the Russian tradition means: Banach space of smooth functions of (tempered) distributions.
The literature knows (rearrangement invariant) Banach function spaces: Here the membership of a function depends only on the level sets, i.e. on the measure of the sets $\{x \mid |f(x)| \geq \alpha\}$, as a function of $\alpha > 0$.
The membership of f in some Sobolev, or Besov-, or Triebel-Lizorkin spaces describes the smoothness, typically using a modulus of continuity, or *dyadic partitions of unity* (based on Paley-Littlewood Theory).

A glimpse into the overall zoo!

Comparing $\mathcal{S}(\mathbb{R}^d)$ with $S_0(\mathbb{R}^d)$

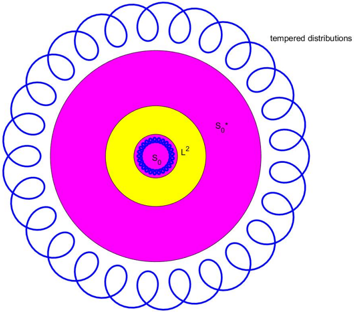


Figure: The Banach Gelfand Triple surrounded by the Schwartz space (inside) and the space of tempered distributions, containing everything.

All the spaces, including Wiener amalgams

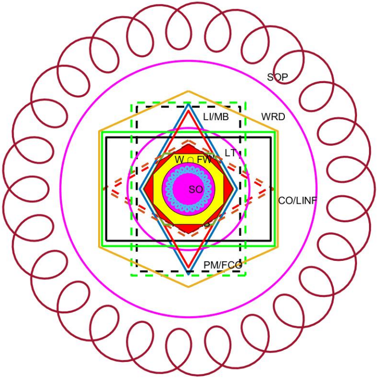


Figure: schwwienall1.jpg

What do we need for "generalized functions"

First of all we need a reservoir of "nice/ordinary" functions, usually called a space of test functions. Then we can proceed to look at the dual space which may include already some interesting objects which cannot be treated as ordinary functions.
If we start with the Banach algebra $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ we can identify (or call by definition) the dual spaces the space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (of bounded measures (regular Borel measures)).
But then $\boldsymbol{C}_0(\mathbb{R}^d)$ is not embedded into $\boldsymbol{M}_b(\mathbb{R}^d)$ in a natural way.
If we make the space space smaller, e.g. $\boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ (with the natural norm) we have already such an embedding, but *Wiener's Algebra* is much better. It is obtained by decomposing $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ into localized blocks and requiring absolute convergence.
Taking the Fourier transform of these local pieces and requiring membership of this TF-decomposition can already be used as a possible definition of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
An equivalent norm is in fact:

Various simple choices 1

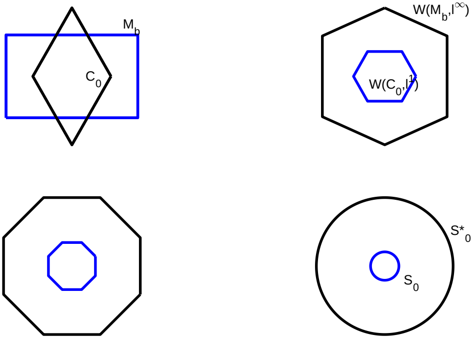


Figure: dualpairs2.eps

More elaborate choices

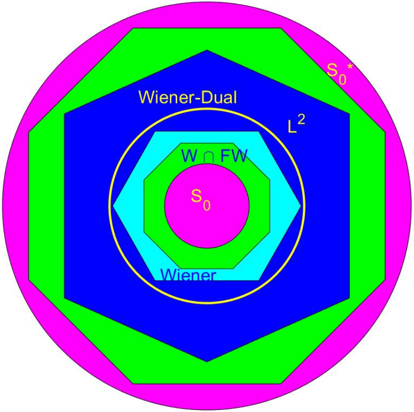


Figure: WienSOSOP2.jpg

New and Classical Spaces

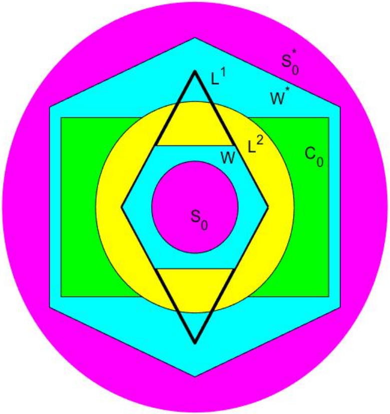
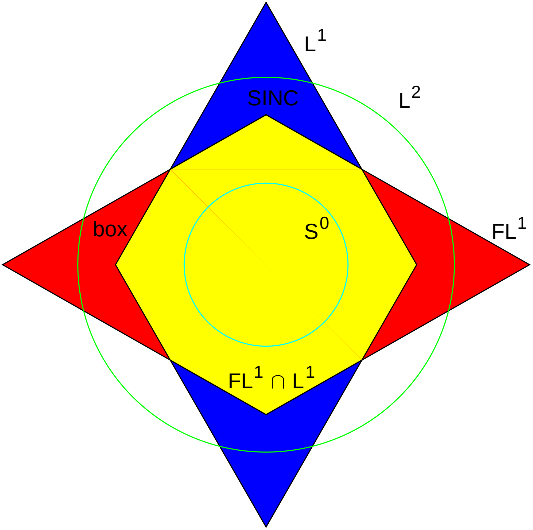
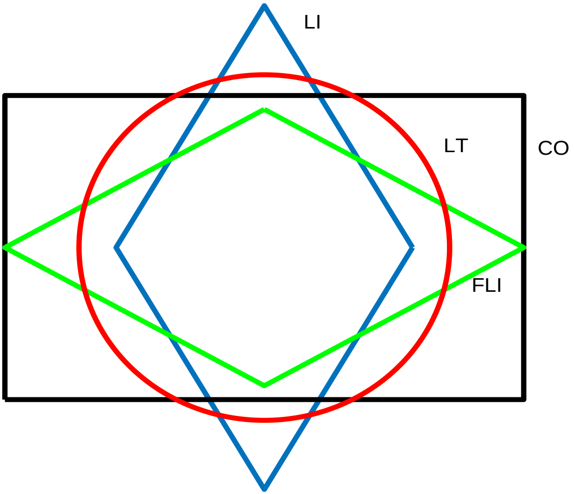


Figure: LILTSOBWien.jpg

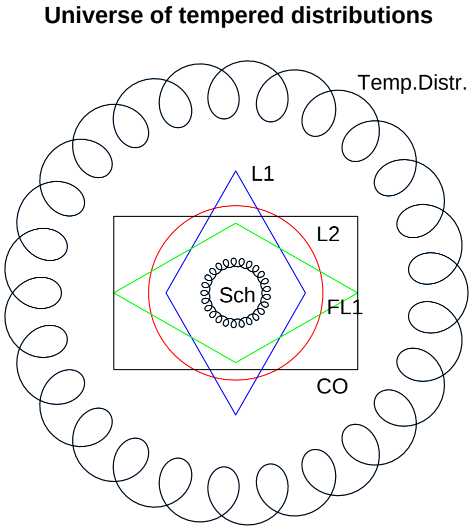
A closeup on the known spaces



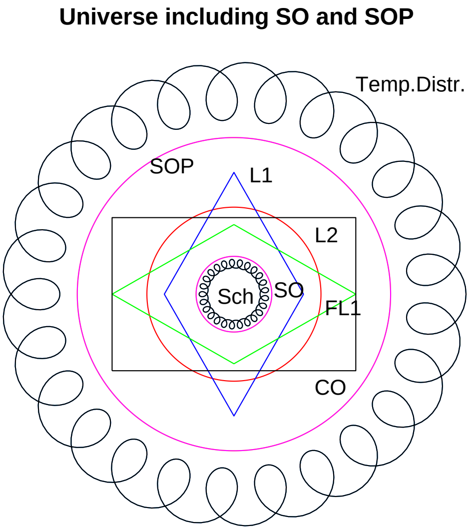
A schematic description of the situation: L^1, L^2

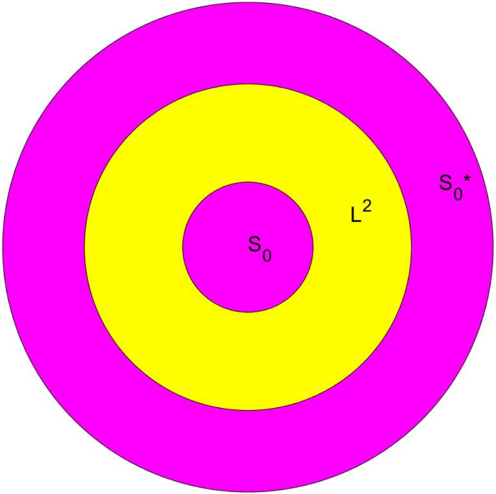


A schematic description of the situation: L^1, L^2, C_0



A schematic description that we are going for




S0*
L2
S0

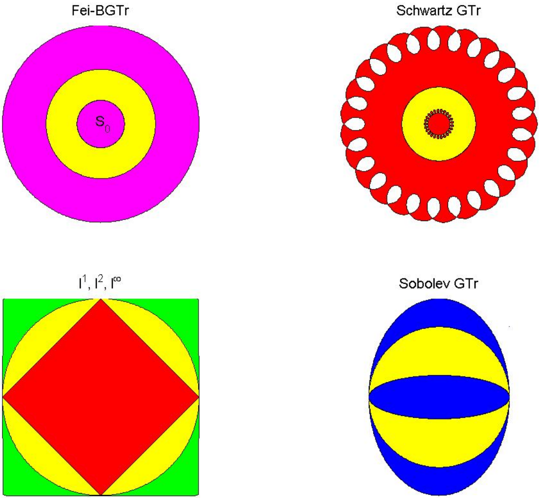


Figure: BanGelfTrip1.jpg: different Gelfand Triples: The S_0-triple,

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$ which is then IN ADDITION w^*-w^*--continuous!

A pictorial presentation

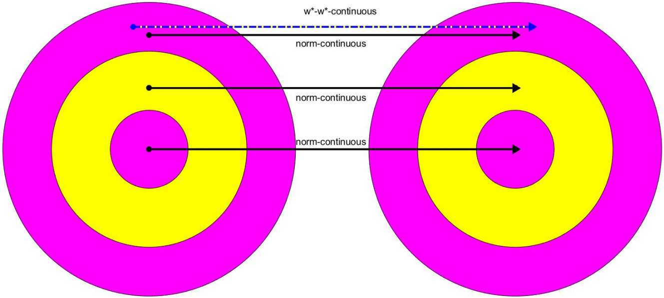


Figure: The description of a Banach space morphism.

Banach Gelfand Triples, the prototype

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Segal Algebra $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, 1979

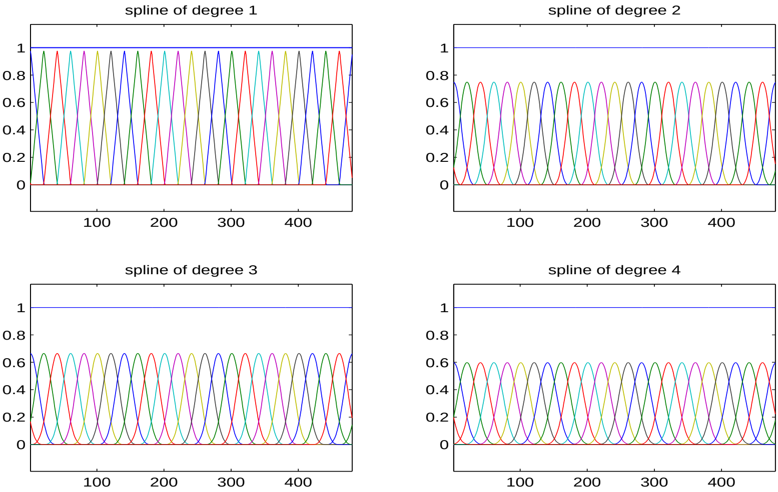
In the last 2-3 decades the Segal algebra $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ (equal to the modulation space $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$) and its dual, $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ have gained importance for many questions of Gabor analysis or time-frequency analysis in general.

It can be characterized as the smallest (non-trivial) Banach space of (continuous and integrable) functions with the property, that time-frequency shifts acts isometrically on its elements, i.e. with

$$\|T_x f\|_B = \|f\|_B, \quad \text{and} \quad \|M_s f\|_B = \|f\|_B, \quad \forall f \in B,$$

where T_x is the usual translation operator, and M_s is the *frequency shift* operator, i.e. $M_s f(t) = e^{2\pi i s \cdot t} f(t), t \in \mathbb{R}^d$. This description implies that $S_0(\mathbb{R}^d)$ is also Fourier invariant!

Illustration of the B-splines providing BUPUs



The Segal Algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$: description

There are many different ways to describe $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Originally it has been introduced as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$, but the standard approach is to describe it via the STFT (short-time Fourier transform) using a Gaussian window given by $g_0(t) = e^{-\pi|t|^2}$.

A short description of the Wiener Amalgam space for $d = 1$ is as follows: Starting from the basis of B-splines of order ≥ 2 (e.g. triangular functions or cubic B-splines), which form a (smooth and uniform) partition of the form $(\varphi_n) := (T_n\varphi)_{n\in\mathbb{Z}}$ we can say that $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ belongs to $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ if and only if

$$\|f\| := \sum_{n\in\mathbb{Z}} \|\widehat{f\cdot\varphi_n}\|_{\boldsymbol{L}^1} < \infty.$$

Using tensor products the definition extends to $d \geq 2$.

Banach Gelfand Triple (S_0, L^2, S_0'): BASICS

Let us collect a few facts concerning this Banach Gelfand Triple (BGTr), based on the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$:

- $S_0(\mathbb{R}^d)$ is dense in $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$, in fact within any $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p < \infty$ (or in $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$);
- Any of the L^p-spaces, with $1 \leq p \leq \infty$ is continuously embedded into $S_0'(\mathbb{R}^d)$;
- Any translation bounded measure belongs to $S_0'(\mathbb{R}^d)$, in particular any Dirac-comb $⊔⊔_\Lambda := \sum_{\lambda\in\Lambda} \delta_\lambda$, for $\Lambda \lhd \mathbb{R}^d$.
- $S_0(\mathbb{R}^d)$ is w^*-dense in $S_0'(\mathbb{R}^d)$, i.e. for any $\sigma \in S_0'(\mathbb{R}^d)$ there exists a sequence of test functions s_n in $S_0(\mathbb{R}^d)$ such that

$$(1) \quad \int_{\mathbb{R}^d} f(x)s_n(x)dx \to \sigma(f), \quad \forall f \in S_0(\mathbb{R}^d). \qquad (6)$$

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

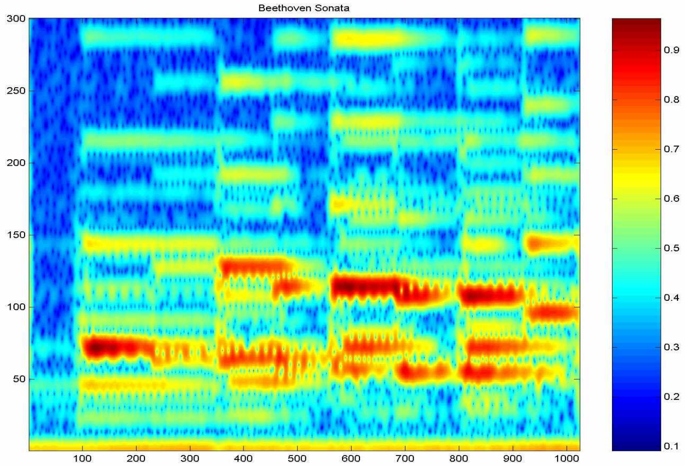
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

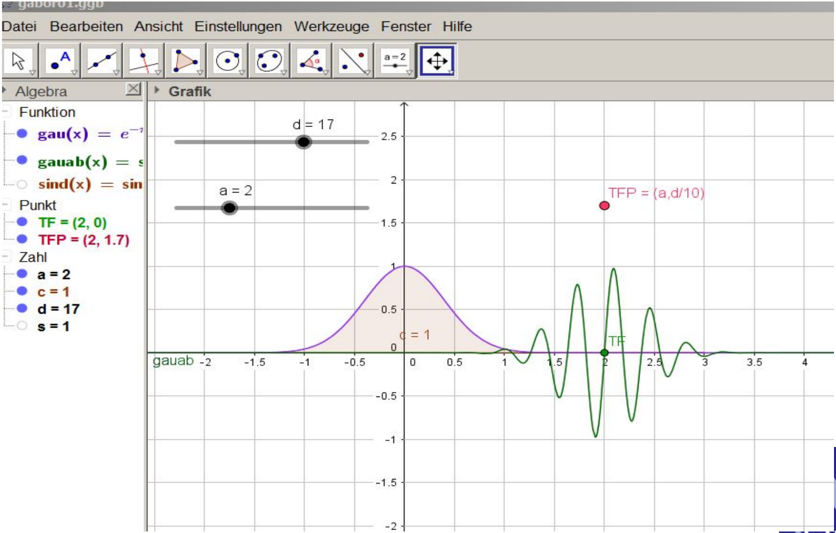
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a practical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2\|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. We have the **Moyal identity**

$$\|V_g(f)\|_2 = \|g\|_2\|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!)\, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \ \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$

In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \tag{7}$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \tag{8}$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h \rangle = \langle V_g^*(V_g(f)), h \rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)} \, d\lambda. \quad (9)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda) g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle \, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (10)$$

This is quite analogous to the situation of the Fourier transform

$$(6) \quad f = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle \, \chi_s \, ds, = \int_{\mathbb{R}^d} \hat{f}(x) e^{2\pi i s \cdot} ds \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (11)$$

with $\chi_s(t) = exp(2\pi i \langle s, t \rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Introducing the space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

The Banach space (and actually Segal algebra) $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ has been introduced by the speaker in 1979, in a paper in Monatshefte f. Math., entitled "On a New Segal Algebra". Most of the basic properties if this space of test functions, including is minimality among Banach spaces of functions which are isometrically invariant under time-frequency shifts, and the Fourier invariance have been demonstrated already in that first paper. Modern approaches to this space, called $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$ in the book of Gröchenig, can be found it his book. It is now common practice to define $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ via the membership of the STFT (short time Fourier transform) with respect to a Gaussian window $g_0(t) = e^{-\pi|t|^2}$ and choose as a norm

$$\|f\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_g(\lambda)| d\lambda = \|V_g(f)\|_1. \qquad (12)$$

Characterization of $S_0'(\mathbb{R}^d)$ and w^*-convergence

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_0'(\mathbb{R}^d)$ if and only if its (continuous) STFT is a *bounded* function. Furthermore convergence corresponds to uniform convergence of the spectrogram (different windows give equivalent norms!).
We can also extend the Fourier transform form $S_0(\mathbb{R}^d)$ to $S_0'(\mathbb{R}^d)$ via the usual formula $\hat{\sigma}(f) := \sigma(\hat{f})$.
The weaker convergence, arising from the functional analytic concept of w^*-convergence has the following very natural characterization: A (bounded) sequence σ_n is w^*- convergence to σ_0 if and only if for one (resp. every) $S_0(\mathbb{R}^d)$-window g one has

$$V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda) \quad \text{for} n \to \infty,$$

uniformly over compact subsets of phase-space.

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (11) (Fourier inversion) and the new formula formula (10). While the building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see for example [6]).

Gabor Analysis is the theory describing how one can get exact recovery while still using a not too dense lattice Λ.

Applications to Translation Invariant Systems

Engineers like to describe "translation invariant systems" as convolution operators by some *impulse response*, or equivalently by the pointwise multiplication of $\hat{f}$ (the input signal) by some *transfer function*. In sloppy terms:

$$T(f) = \mu * f, \quad \text{with} \quad \mu = T(\delta_0)$$

$$\widehat{Tf} = h \cdot \hat{f}.$$

Here we refer to the engineering terminology: A TILS is linear operator (often the domain is left undefined!) with the property that

$$T_x \circ T = T \circ T_x, \quad x \in \mathbb{R}^d.$$

Translation Invariant Systems II, TILS2

Theorem

$$\mathcal{H}_G(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d)) = \mathbf{S}_0'(\mathbb{R}^d)$$

i.e. for every $T \in \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$ *commuting with translation there is a unique* $\sigma = \sigma_T$ *such that* $Tf(x) = \sigma(T_x f^{\checkmark})$ *where* $f^{\checkmark}(x) = f(-x)$. *Also the converse is true, and the operator norm of* T *is equivalent to the* $\mathbf{S}_0'$*-norm of* σ.

Corollary

Any translation invariant operator from $(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ *to* $(\mathbf{L}^q(\mathbb{R}^d), \|\cdot\|_q)$, $1 \leq p, q < \infty$ *can be represented (on* $\mathbf{S}_0(\mathbb{R}^d)$*) as a convolution operator by* $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$ *or with the transfer "function"* $h = \widehat{\sigma}$ *(Fourier multipliers).*

Classical Analysis and Summability

Among the functions typically used in Fourier analysis only the so-called BOX-function ($\mathbf{1}_Q$) (being discontinuous) and its Fourier transform, the SINC-function (not belonging to $\boldsymbol{L}^1(\mathbb{R}^d)$) are NOT elements of $\boldsymbol{S}_0(\mathbb{R})$, while (according to F. Weisz)

all the classical summability kernels belong to $\boldsymbol{S}_0(\mathbb{R}^d)$

.

The space $\boldsymbol{S}_0(\mathbb{R}^d)$ is also the natural domain for the *Poisson summation formula*, another important tool in Fourier analysis:

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{13}$$

There are counter-examples, but they all work only when one is using function spaces not contained in $\boldsymbol{S}_0(\mathbb{R}^d)$.

A First Application to the Fourier Transform

When it comes to the approximate realization of a Fourier related task we can point to joint work with N. Kaiblinger:

Theorem

Given $f \in \boldsymbol{S}_0(\mathbb{R})$ and $\varepsilon > 0$ it is enough to apply the FFT to a sequence of equi-distant samples, taken sufficiently fine and over a sufficiently long interval, then apply the FFT *to this sequence and regain a continuous (and compactly supported) function $\hat{f}_a$ in $\boldsymbol{S}_0(\mathbb{R})$ via (linear or) quasi-inteprolation, with*

$$\|\hat{f} - \hat{f}_a\|_{\boldsymbol{S}_0} < \varepsilon.$$

The Metaplectic Invariance of $S_0(\mathbb{R}^d)$

Aside from the convenient properties of $S_0(\mathbb{R}^d)$ (including the possibility to use such a space over general LCA groups) the first important application of $S_0(\mathbb{R}^d)$ is in the Lecture Notes of Hans Reiter, entitled *Metaplectic Groups and Segal Algebras* which appeared in the Springer Lect. Notes in Mathematics in 1989, and which provides a detailed description of the use of $S_0(\mathbb{R}^d)$ for the analysis of the **metaplectic** group, which among others includes the *Fractional Fourier transforms*.

The invariance of $S_0(\mathbb{R}^d)$ under general automorphisms of the group $\mathbb{R}^d$ as well as under all the metaplectic operator, e.g. convolution or pointwise multiplication by the chirp functions $t \to e^{i\alpha t^2}, 0 \neq \alpha \in \mathbb{R}$ is crucial in this setting.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis I

ONE of the key questions in Gabor analysis is the question, when a Gabor family $(G,\Lambda) = (g_\lambda)_{\lambda\in\Lambda}$, with some Gabor atom $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is a Gabor frame, where $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is some lattice. Standard frame theory tells us the following things:

1. (g,Λ) defines a Gabor frame if and only if the frame operator
$$S_{g,\Lambda} : f \to \sum_{\lambda\in\Lambda} \langle f, g_\lambda \rangle g_\lambda$$
is invertible on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$;
2. In such a case $\widetilde{g} := S^{-1}(g) \in \boldsymbol{L}^2(\mathbb{R}^d)$ generates the dual frame, i.e. the dual frame is of the form $(\widetilde{g}_\lambda)_{\lambda\in\Lambda}$.
3. This allows two kinds of representations of any $f \in \boldsymbol{L}^2$:
$$f = \sum_{\lambda\in\Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda\in\Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda. \qquad (14)$$

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis II

The fact that it is impossible to find Gaborian Riesz bases with "good generators" (by the Balian-Low Theorem, i.e. for $g \in \mathbf{S}_0(\mathbb{R}^d)$ (g, Λ) never gives a Riesz basis for $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$!) makes it important to control the window g as well as the dual window in terms of "of good quality".

The first step is already the boundedness of the frame operator $S_{g,\Lambda}$, which is relatively easy to show for $g \in \mathbf{S}_0(\mathbb{R}^d)$ (and most of the time not available for $g \notin \mathbf{S}_0(\mathbb{R}^d)$). More important is the following observation: Whenever $g \in \mathbf{S}_0(\mathbb{R}^d)$ the operator $S_{g,\Lambda}$ is bounded on $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$. Gröchenig/Leinert have shown:

Theorem

Whenever $S_{g,\Lambda}$ is invertible on $\mathbf{L}^2(\mathbb{R}^d)$ for some $g \in \mathbf{S}_0(\mathbb{R}^d)$, it is also invertible on $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$, hence $\widetilde{g} = S^{-1}(g) \in \mathbf{S}_0(\mathbb{R}^d)$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis III

This can be used to check that the representation formula (14) is also valid in the $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$-sense for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and can be extended (now with w^*-convergence) to general $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$.

For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ it is true that a small jitter error, i.e. using instead of $V_g(\lambda) = \langle f, g_\lambda \rangle$ some nearby sampling value $V_g(\lambda + \gamma_\lambda)$ with $|\gamma_\lambda| \leq \gamma_0$ for some small constant γ_0. Then, e.g., the reconstruction of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ from these slightly perturbed samples will show error (in the $\boldsymbol{S}_0$-norm sense!).

Also for the computation of *approximate dual Gabor windows h* it is important to ensure a small error in the $\boldsymbol{S}_0$-norm sense, because otherwise it is *not possible* to control the error of the computable operator $\sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle h_\lambda$ in the operator norm sense (even on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$).

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis IV

In finite dimensions, e.g. over the group $\mathbb{Z}_N$, a Gabor family is a frame if and only if it is a generating system for $\mathbb{C}^N$, or in other words, if and only if every $\mathbf{x} \in \mathbb{C}^N$ can be represented as linear combination of elements from the Gabor family. Writing `GAB` for the Gabor family with atom $g \in \mathbb{C}^N$ it is clear that we need $n \geq N$ such vectors for the spanning property, resp. we need that GAB is of maximal rank N.

The "optimal representation" for a redundant system is then of course the **MNLSQ** solution $\mathbf{y}_0$, i.e. the choice of those coefficients which represent the given signal as $\mathbf{x} = GAB * \mathbf{y}_0$ which minimize $\|\mathbf{y}\|_{\mathbb{C}^n}$ among all coefficient sequences with $\mathbf{x} = GAB * \mathbf{y}$. This sequence can be obtained via the pseudo-inverse matrix `pinv(GAB)` via $\mathbf{y}_0 = \text{pinv}(GAB) * \mathbf{x}$. The collection of (conjugate) rows of $\text{pinv}(GAB)$ or columns of $\text{pinv}(GAB)' = \text{pinv}(GAB)'$ in MATLAB notation is just the dual frame!

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis V

The important formula which applies in this situation (it can be derived easily from the SVD decomposition of a matrix $\boldsymbol{A}$)

$$\mathrm{pinv}(\boldsymbol{A}') = inv(\boldsymbol{A} * \boldsymbol{A}') * \boldsymbol{A}$$

shows that the *dual frame* can be obtained by applying the inverse of the frame matrix $\mathbf{S} = \boldsymbol{A} * \boldsymbol{A}'$ to the elements of the original frame (columns of $\boldsymbol{A}$).
But it is better to use a commutative diagram for this, showing that and how the signal $\mathbf{x}$ can be reconstructed from the set of scalar products with the frame elements, i.e. from $\boldsymbol{A}' * \mathbf{x}$ by multiplying from the left with $(\boldsymbol{A} * \boldsymbol{A}')^{-1} * \boldsymbol{A}$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis VI

In the continuous setting and for Gabor frames with $S_0(\mathbb{R}^d)$-atoms g we have the following situation:

Theorem

Given a Gabor frame (g, Λ) with $g \in S_0(\mathbb{R}^d)$ one has:

- *the coefficient mapping $\mathbf{C} : f \to V_g(f)|_\Lambda$ is an BGTr homomorphism from $(S_0, L^2, S_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.*
- *For $\widetilde{g} = S^{-1}(g) \in S_0(\mathbb{R}^d)$ the Gabor synthesis mapping*

$$\mathbf{R} : (\mathbf{c}_\lambda) \to \sum_{\lambda \in \mathbf{\Lambda}} \mathbf{c}_\lambda \widetilde{\mathbf{g}}_\lambda$$

  is a BGTr homomorphism $(\ell^1, \ell^2, \ell^\infty)(\Lambda) \to (S_0, L^2, S_0')(\mathbb{R}^d)$;
- *$\mathbf{R}$ is a left inverse to $\mathbf{C}$: $\mathbf{R} \circ \mathbf{C} = \mathbf{Id}$ on (S_0, L^2, S_0').*

RELEVANT APPLICATIONS

After a quick general description of Banach Gelfand Triples (BGTr) in an abstract setting and the foundations of the concrete BGTr, based on the Segal Algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ we indicate some of the many applications, e.g.

1. Fourier Transform as unitary BGTr Automorphism
2. The Kernel Theorem
3. The Spreading representation of Operators
4. The Kohn-Nirenberg Symbol of Operators
5. Gabor Analysis and Janssen Representation
6. Robustness Considerations in Gabor Analysis
7. Generalized Stochastic Processes

The Kernel Theorem

It is clear that such operators between functions on $\mathbb{R}^d$ cannot all be represented by integral kernels using locally integrable $K(x,y)$ in the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x,y \in \mathbb{R}^d, \tag{15}$$

because clearly multiplication operators should have their support on the main diagonal, but $\{(x,x) \,|\, x \in \mathbb{R}^d\}$ is just a set of measure zero in $\mathbb{R}^d \times \mathbb{R}^d$!
Also the expected "rule" to find the kernel, namely

$$K(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y)) \tag{16}$$

might not be meaningful at all.

The Hilbert Schmidt Version

There are two ways out of this problem

- restrict the class of operators
- enlarge the class of possible kernels

The first one is a classical result, i.e. the characterization of the class $\mathcal{HS}$ of Hilbert Schmidt operators.

Theorem

A linear operator T on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ is a Hilbert-Schmidt operator, i.e. is a compact operator with the sequence of singular values in ℓ^2 if and only if it is an integral operator of the form (15) with $K \in \boldsymbol{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$. In fact, we have a unitary mapping $T \to K(x,y)$, where $\mathcal{HS}$ is endowed with the Hilbert-Schmidt scalar product $\langle T, S \rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*)$.

The Schwartz Kernel Theorem

The other well known version of the kernel theorem makes use of the *nuclearity* of the *Frechet space* $\mathcal{S}(\mathbb{R}^d)$ (so to say the complicated topological properties of the system of seminorms defining the topology on $\mathcal{S}(\mathbb{R}^d)$).
Note that the description cannot be given anymore in the form (15) but has to replaced by a "weak description". This is part of the following well-known result due to L. Schwartz.

Theorem

There is a natural isomorphism between the vector space of all linear operators from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, i.e. $\mathcal{L}(\mathcal{S}, \mathcal{S}')$, and the elements of $\mathcal{S}'(\mathbb{R}^{2d})$, via $\langle Tf, g \rangle = \langle K, f \otimes g \rangle$, for $f, g \in \mathcal{S}(\mathbb{R}^d)$.

The S_0-KERNEL THEOREM

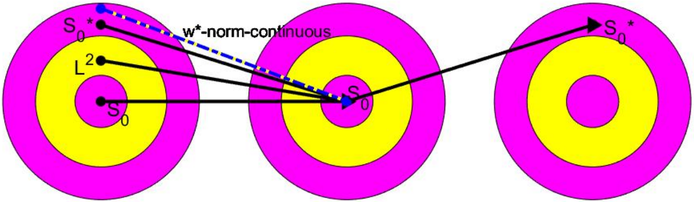
In the current setting we can describe the kernel theorem as a unitary Banach Gelfand Triple isomorphism, between operator and their (distributional) kernels, extending the classical Hilbert Schmidt version.

First we observe that $\boldsymbol{S}_0$-kernels can be identified with $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, i.e. the *regularizing operators* from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\mathbb{R}^d)$, even mapping bounded and w^*- convergent nets into norm convergent sets. For those kernels also the recovery formula (16) is valid.

Theorem

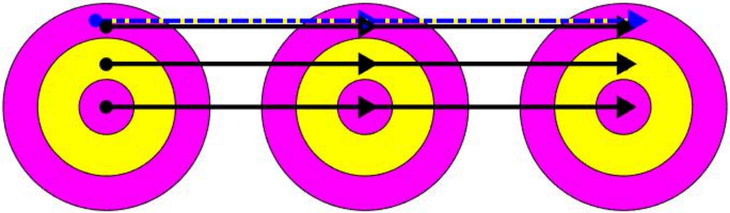
The unitary Hilbert-Schmidt kernel isomorphisms extends in a unique way to a Banach Gelfand Triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \mathbb{R}^d)$.

;



;

Figure: Composing a regularizing operator with the most general type of



Applications to Gabor Multipliers

This last property can be used to e.g. describe the best approximation of a given operator by a Gabor multiplier. The most important Gabor multipliers arise from tight regular Gabor frames, i.e. families of the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$, with Λ being any lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, with the property (writing g_λ for $\pi(\lambda)g$) with the following reconstruction property:

$$f = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda, \quad f \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0'). \tag{17}$$

We also can write $P_\lambda : f \to \langle f, g_\lambda\rangle g_\lambda$. Given a numerical sequence over the lattice (m_λ) the Gabor multiplier $G_m := \sum_{\lambda\in\Lambda} m(\lambda)P_\lambda$. The problem of best approximation of some $\mathcal{HS}$ operator by Gabor multipliers can be reformulated as an approximation problem using spline-type spaces via the Kohn-Nirenberg connection.

There is just one Fourier transform

As a colleague (Jens Fischer) at the German DLR (in Oberpfaffenhausen) puts it in his writing: "There is just one Fourier Transform!" And I may add: and it is enough to know about $(\boldsymbol{S}_0'(\mathbb{R}^d), \| \cdot \|_{\boldsymbol{S}_0'})$ in order to understand this principle and to make it mathematically meaningful.

In **engineering courses** students learn about discrete and continuous, about periodic and non-periodic signals (typically on $\mathbb{R}$ or $\mathbb{R}^2$), and they are treated separately with different formulas. Finally comes the DFT/FFT for finite signals, when it comes to computations. The all look similar.

Mathematics students learning Abstract Harmonic Analysis learn that one has to work with different LCA groups and their dual groups. Gianfranco Cariolaro (Padua) combines the view-points somehow in his book Unified Signal Theory (2011).

w^*-convergence justifies the various transitions!

Periodicity and Fourier Support Properties

The world of distributions allows to deal with continuous and discrete, periodic and non-periodic *signals* at equal footing. Let us discuss how they are connected.
The general Poisson Formula, expressed as

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} \tag{18}$$

can be used to prove

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda * f) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} \cdot \mathcal{F}(f), \tag{19}$$

or interchanging convolution with pointwise multiplication:

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda \cdot f) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} * \mathcal{F}(f). \tag{20}$$

I.e.: Convolution by $\sqcup\!\sqcup$ (corresponding to *periodization*) corresponds to pointwise multiplication (i.e. *sampling*) on the Fourier transform domain and *vice versa*.

Approximation by discrete and periodic signals

The combination of two such operators, just with the assumption that the sampling lattice Λ_1 is a subgroup (of finite index N) of the periodization lattice Λ_2 implies that

$$\sqcup\!\sqcup_{\Lambda_2} * [\sqcup\!\sqcup_{\Lambda_1} \cdot f] = \sqcup\!\sqcup_{\Lambda_1} \cdot [\sqcup\!\sqcup_{\Lambda_2} * f], \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \qquad (21)$$

For illustration let us take $d = 1$ and $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = N\alpha\mathbb{Z}$ and hence $\Lambda_1^\perp = (1/\alpha)\mathbb{Z}$. Then the periodic and sampled signal arising from equ. 21 corresponds to a vector $\mathbf{a} \in \mathbb{C}^N$ and the distributional Fourier transform of the periodic, discrete signal is completely characterized is again discrete and periodic and its generating sequence $\mathbf{b} \in \mathbb{C}^N$ can be obtained via the DFT (FT of quotient group), e.g. $N = k^2, \alpha = 1/k$, and period k.

Approximation by discrete and periodic signals 2

It is not difficult to verify that in this way, by making the sampling lattice more and more refined and periodization lattice coarser and coarser the resulting discrete and periodic versions of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, viewed as elements within $\boldsymbol{S}_0'(\mathbb{R}^d)$, are approximated in a bounded and w^*-sense by discrete and periodic functions.
This view-point can be used as a justification of the fact used in books describing heuristically the continuous Fourier transform, as a limit of Fourier series expansions, with the *period going to infinity*.

Mutual w^*-approximations

The density of test functions in the dual space can be obtained in many ways, using so-called *regularizing operators*, e.g. combined approximated units for convolution and on the other hand for pointwise convolution, based on the fact that we have

$$(\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d), \quad \text{and} \tag{22}$$

$$(\boldsymbol{S}_0(\mathbb{R}^d) \cdot \boldsymbol{S}_0'(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d). \tag{23}$$

Alternatively one can take finite partial sums of the Gabor expansion of a distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ which approximate σ in the w^*-sense (boundedly), for Gabor windows in $\boldsymbol{S}_0(\mathbb{R}^d)$.

On the other hand one can approximate test functions (in the w^*-sense) by discrete and periodic signals!

Approximation of Distributions by Test Functions

These properties of *product-convolution operators* or *convolution-product operators* can be used to obtain a w^*-approximation of general elements $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ by test functions in $\boldsymbol{S}_0(\mathbb{R}^d)$. For example, one can take a Dirac family obtained by applying the compression operator

$$\mathrm{St}_\rho(g) := \rho^{-d} g(x/\rho), \quad \rho \to 0$$

in order to approximate σ by bounded and continuous functions of the form $\mathrm{St}_\rho(g_0) * \sigma$.
For the localization one can use the dilation operator

$$\mathrm{D}_\rho(h)(z) = h(\rho z), \quad \rho \to 0,$$

so altogether

$$\sigma = w^* - \lim_{\rho \to 0} \mathrm{D}_\rho g_0 \cdot [(\mathrm{St}_\rho g_0) * \sigma]$$

where all the functions on the right hand side belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.

CONCLUSION

1. I wanted to promote the use of Banach Gelfand Triples;
2. Specifically emphasize that $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ is particularly useful for classical Fourier analysis, but in particular for TFA and Gabor Analysis;
3. Convey the idea that there are not many pratical ways to introduce the pair $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ and its dual $\left(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}\right)$ without making use of Lebesgue integration or Schwartz theory;
4. That these spaces are closely related to Wiener Amalgam Spaces and modulation spaces introduced in the 80th.

So let us summarize the key facts which I wanted to most of our relevant papers at NuHAG are found at

`www.nuhag.eu/bibtex www.nuhag.eu/talks`

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