NuHAG Numerical Harmonic Analysis Group

200 Years of Fourier Analysis: From Mathematical Theory to Modern Digital Signal Processing

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Personal Background

A glimpse on my personal background/path:

1. First exposed to Lebesgue integration and Fourier Series as a 18/19 year old student of mathematics;
2. PhD and habilitation thesis in Harmonic Analysis;
3. Turning to applications, image processing, mobile communications; working with engineers;
4. European projects, e.g. EUCETIFA (European Center for Time-Frequency Analysis);
5. Editor to J. Fourier Analysis and Applications, since 2000;
6. Officially retired but still teaching, at European TUs.

Various Aspects

Addressing various topics of a quite general nature:

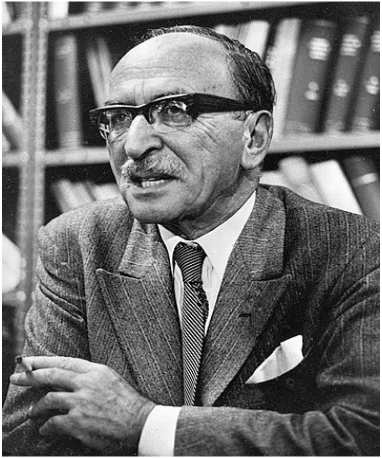
- Fourier Analysis is a mature (200 year old) subject of mathematics which is still going strong $>$ a suitable subject for a discussion of the development of mathematical ideas;
- Modern digital technologies depend heavily on the principles developed in connection with Fourier Analysis;
- Not always is the theory developed by mathematicians ready for applications, nor do applied scientists go deep enough in them mathematical analysis of their problems;
- Mathematics is not just the abstract and complicated subject as it is usually seen, and not just a collection of formulas.

Jean Baptiste Josef Fourier: 1768 - 1830

https://en.wikipedia.org/wiki/Joseph_Fourier

Dennis Gabor, born as Günszberg Denes, 1900-1979

https://en.wikipedia.org/wiki/Dennis_Gabor



Classical Fourier Series

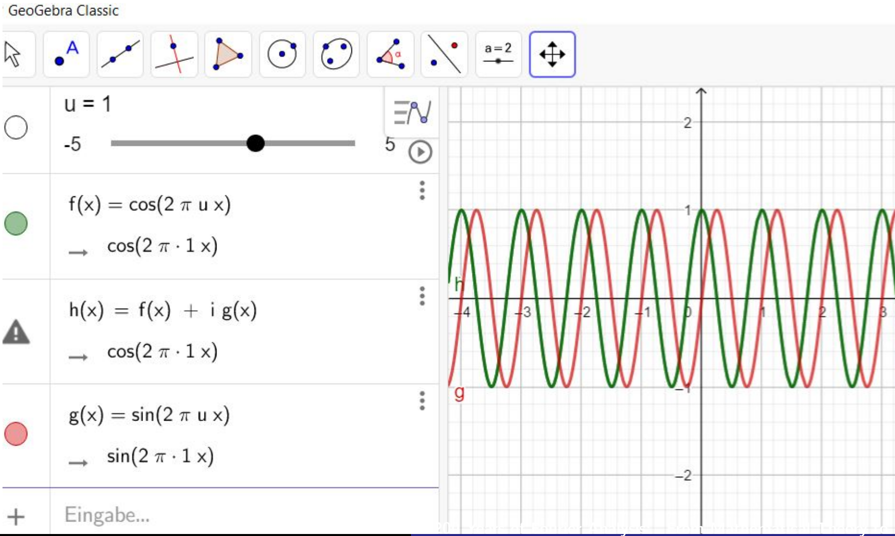
The classical approach (going back to 1822) to the theory of *FOURIER SERIES* appears in the following form: Looking at the partial sums of the (formally then infinite) Fourier series we expect them to approximate "any periodic function" in **some sense**:

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(2\pi nx) + b_n \sin(2\pi nx)]. \quad (1)$$

Assuming this is possible it is not so hard to find out, using the properties of the building blocks ($\cos(x), \sin(x)$, addition rules, derivatives, integration) that one can expect for any $z \in \mathbb{R}$:

$$a_n = \int_z^{z+1} f(x)\cos(2\pi nx)dx, \;\; b_n = \int_z^{z+1} f(x)\sin(2\pi nx)dx. \quad (2)$$

Illustrating the Building Blocks



Classical Fourier Series II

In my course on classical Fourier series I was taught that the representation

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi nx) + b_n \sin(2\pi nx)]. \qquad (3)$$

should be taken only as a "formal expression", which has to be justified using complicated arguments and using a variety of strange tricks/methods!

But was does this mean?

What kind of concrete, mathematical questions should be asked?
How are *summability methods* saving the situation?

Until now Fourier series are often seen as a mystery!

Ingredient 1: the Complex Numbers

First of all we have to note that by the time (1822!!, which was at the life-time of Carl Friedrich Gauss! [1777-1855]) the modern **concept of a function** was not available yet. Thanks to Leonhard Euler ([1707 - 1783]) the complex numbers and their connection to trigonometric functions had been known

$$e^{ix} = \cos(x) + i \sin(x), \quad i = \sqrt{-1}. \tag{4}$$

It was known what *polynomials* are and how to compute with them, and even to take "polynomials of infinite degree" (power series, with well defined regions of uniform convergence), hence **Taylor expansions** were known (going back to the English mathematician Brook Taylor [1685-1731]). Both methods constitute important parts of approximation theory.

Exponential Functions

Complex numbers allow to use polynomials with complex coefficients $a_0, \cdots a_n$ and complex argument $z \in \mathbb{C}$:

$$p_a(z) = a_0 + a_1 z + \cdot + a_n z^n = \sum_{k=0}^{n} a_k z^k. \quad (5)$$

A *power series* is an infinite sum $\sum_{k=0}^{\infty} a_k z^k$, such as the series defining the exponential function which is convergent for any $z \in \mathbb{C}$

$$e^z := 1 + z/1! + z^2/2! + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (6)$$

and satisfies the exponential law

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2} \quad z_1, z_2 \in \mathbb{C}, \quad (7)$$

one of the basic identities for (modern) Fourier Analysis.

Ingredient 2: Integrals

Of course the determination of the coefficients using **integrals** (over the period of the involved functions) is one of the corner-stones of the classical theory, raising some questions:

- What is the meaning of an integral *in the most general case*?
- What kind of functions can be integrated (over $[a, b]$?
- What can be said about the Fourier coefficients $(a_n)_{n\geq 0}$ or $(b_n)_{n\geq 1}$? (decay for $n \to \infty$, summability).

While the foundations of "calculus" had been laid down by Isaac Newton [1642 - 1726] and Gottfried Wilhelm Leibniz [1646 - 1716] long before Fourier it was **Bernhard Riemann** [1826 - 1866] who gave a clean definition and showed that e.g. every *continuous function* can be integrated over any interval $[a, b]$. He showed that the Fourier coefficients tend to zero $(n \to \infty)$.

A Timeline

Another non-trivial part of the reasoning is the justification for the formula 2. In fact, it is only a necessary condition on the coefficients which can be easily obtained, using integrals, telling us nothing about convergence.

Isaac Newton [1642 - 1726]
Gottfried Wilhelm Leibniz [1646 - 1716]

AFTER FOURIER

Bernhard Riemann [1826 - 1866]
Karl Weierstrass [1815-1897]
Henri Leon Lebesgue [1875 - 1941]
Norbert Wiener [1894 - 1964]
Andre Weil [1906 1998]

Convergence Issues: Pointwise?

So let us return to the question of convergence: The key question being: **In which sense do the partial sums converge?**

In fact, it turned out that a more general problem appeared: What does convergence mean, and can one form classes of functions (nowadays Banach spaces or even topological vector spaces of such objects) such that one can guarantee convergence in those space in the corresponding norm (or topology).
The classical view-point was of course: Can one establish pointwise convergence (Dirichlet-conditions, J.P. Lejeune-Dirichlet [1805-1859])? Or uniform convergence at least for continuous functions (no, according to A.N.Kolmogorov [1903-1987], already in 1923 a found a counter-example and in 1926 he was able to prove that the Fourier series of an L^1-function can diverge **everywhere!**).

Convergence Issues: The Idea of Summability

Of course one has to mention **Lipolt Fejer** [1880 - 1959] (born Leopold Weiss) and a long list of names pursuing the problems related to **summability**, which allows to describe the limit of a series not just by looking at the partial sums.

The idea is to change the question from the question of convergence of the (partial sum) of the Fourier series to *the question of recovering a function from its Fourier coefficients*. For example, Fejer was suggesting to take (as a replacement for the ordinary partial sums) the *arithmetic means of the partial sums*.

Fejer's Theorem of 1900 states that for every continuous periodic function f the (now known as) **Fejer means of the Fourier series converge uniformly to f.**

Classical References

J. Fourier.
Theorie analytique de la chaleur, par M. Fourier.
Chez Firmin Didot, pere et fils, 1822.

P. Lejeune Dirichlet.
Sur la convergence des séries trigonométriques qui servent représenter une fonction arbitraire entre des limites données.
J. Reine Angew. Math., 4:157–169, 1829.

H. Lebesgue.
Recherches sur la convergence des series de Fourier.
Math. Ann., 61:251–280, 1905.

A. Kolmogoroff and G. Seliverstoff.
Sur la convergence des séries de Fourier.
Atti Accad. Naz. Lincei, Rend., VI. Ser., 3:307–310, 1926.

K. Kolmogoroff.
Une serie de Fourier-Lebesgue divergente partout.
C. R. Acad. Sci., Paris, 183:1327–1328, 1926.

Early Hungarian References to Fourier Series

L. Fejer.
Untersuchungen über Fouriersche Reihen.
Math. Ann., 58(1-2):51–69, 1903.

F. Riesz.
Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung.
Math. Z., 2(3-4):312–315, 1918.

M. Riesz.
Über summierbare trigonometrische Reihen.
Math. Ann., 71(1):54–75, 1911.

L. Fejer and F. Riesz.
Über einige funktionentheoretische Ungleichungen.
Math. Z., 11(3-4):305–314, 1921.

Fourier History of in a Nutshell

1. 1822: J.B.Fourier proposes: Every periodic function can be expanded into a Fourier series using only pure frequencies;
2. up to 1922: concept of functions developed, set theory, Lebesgue integration, $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$;
3. first half of 20th century: Fourier transform for $\mathbb{R}^d$;
4. A. Weil: Fourier Analysis on Locally Compact Abelian Groups;
5. L. Schwartz: Theory of Tempered Distributions
6. Cooley-Tukey (1965): FFT, the Fast Fourier Transform
7. L. Hörmander: Fourier Analytic methods for PDE (Partial Differential Equations);

The Perfect Integral

By the beginning of the 20th century **Henri Leon Lebesgue** had developed his integral, and also given lectures on the application of this new techniques to trigonometric series. He published a number of important papers between 1904 and 1907.

From a modern (functional analytic) view-point his integral, which included the definition of the so-called *Lebesgue spaces* such as $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ or $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ (and of course later the $\boldsymbol{L}^p$-theory, duality etc.) opened the way to the field of (linear) **functional analysis**, which developed rapidly, the foundations being lead by e.g. David Hilbert [1862 - 1943], Friedrich Riesz [1880 - 1956] and Stefan Banach [1892 - 1945].

The Beginning of Functional Analysis

J. Dieudonne.
History of Functional Analysis, volume 49 of *North-Holland Mathematics Studies*.
North-Holland Publishing Co., Amsterdam-New York, 1981.

E. Kreyszig.
Zur Entwicklung der zentralen Ideen in der Funktionalanalysis.
Elemente der Mathematik, 41:25–35, 1986,.

H. Heuser.
Zur Ideengeschichte der Funktionalanalysis.
Math. Semesterber., 35(1):38–63, 1988.

E. Kreyszig.
Friedrich Riesz als Wegbereiter der Funktionalanalysis.
Elemente der Mathematik, 45(5):117–130, 1990.

Fourier Transform over the Real Line

The work of H.L. Lebesgue paved the way to a clean definition of the Fourier transform for "functions of a continuous variables" as an *integral transform* naturally defined on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$

$$\|f\|_1 := \int_{\mathbb{R}} |f(x)| dx, \quad f \in \boldsymbol{L}^1(\mathbb{R}). \tag{8}$$

The (continuous) Fourier transform for $f \in \boldsymbol{L}^1(\mathbb{R})$ is given by:

$$\hat{f}(s) := \int_{\mathbb{R}} f(x) e^{-2\pi i s x} dx, \quad s \in \mathbb{R}. \tag{9}$$

With this normalization the inverse Fourier transform looks similar, just with the conjugate exponent, and thus, *under the assumption that f is continuous and $\hat{f} \in \boldsymbol{L}^1(\mathbb{R})$* we have pointwise

$$f(t) = \int_{\mathbb{R}} \hat{f}(s) e^{2\pi i s t} ds. \tag{10}$$

Plancherel's Theorem: Unitarity Property of FT

Using the density of $\boldsymbol{L}^1(\mathbb{R}) \cap \boldsymbol{L}^2(\mathbb{R})$ in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ it can be shown that the Fourier transform extends an a natural and unique way to $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$:

Theorem

The Fourier (-Plancherel) transform establishes a unitary automorphism of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, i.e. one has

$$\|f\|_2 = \|\hat{f}\|_2, \quad f \in \boldsymbol{L}^2(\mathbb{R}),$$

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}).$$

In some sense *unitary* transformations of a Hilbert transform is like a change form one ONB to another ONB in $\mathbb{R}^n$.

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the "continuous representation" formula, using $\chi_s(x) = e^{2\pi isx}$, $x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{11}$$

This looks like a perfect orthogonal expansion, but unfortunately the "building blocks" $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

Convolution and the Fourier Transform

Another important fact about the Fourier transform is the so-called **convolution theorem**, i.e. the Fourier transform converts convolution into pointwise multiplication.
Again it is natural to define convolution on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g(x-y)f(y)dy \quad xa.e.; \quad (12)$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}).$$

For positive functions f, g one even has equality. This is relevant for the determination of probability distributions of a sum of *independent* random variables. Assume X has density f and Y has density g then the random variable $X + Y$ has probability density distribution $f * g = g * f$.

Banach Algebras

Theorem

Endowed with the bilinear mapping $(f, g) \to f * g$ the Banach space $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ becomes a commutative Banach algebra with respect to convolution.

The **convolution theorem**, usually formulated as the identity

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}), \tag{13}$$

implies

Theorem

The Fourier algebra, defined as $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}) := \{\hat{f} \mid f \in \boldsymbol{L}^1(\mathbb{R})\}$, with the norm $\|\hat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ is a Banach algebra, closed under conjugation, and dense in $\left(\boldsymbol{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$ (continuous functions, vanishing at infinity).

Mathematics of 20th Century

Jumping into the 40th of the last century one can say that **Abstract Harmonic Analysis** was created, with $\mathbb{R}$ replaced by a general a general LCA (locally compact Abelian) group.
In engineering terminology this allows to discuss *continuous and discrete variables*, but also *periodic or non-periodic* functions as functions on different groups, such as $\mathcal{G} = \mathbb{R}^d, \mathbb{Z}^d, \mathbb{Z}_N, \mathbb{T}^k$ etc., their product being called *elementary groups*.
The fundamental fact in all these cases is the existence of an translation for functions, defined as

$$[T_z f](x) = f(x - z), x, z \in \mathcal{G},$$

and the existence of an invariant integral, the so-called *Haar measure* (Alfred Haar, [1885 - 1933]).

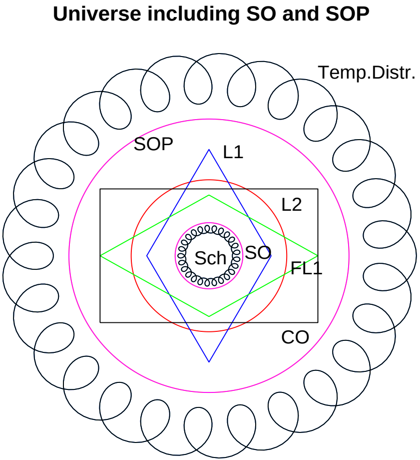
Laurent Schwartz Theory of Tempered Distributions

Laurent Schwartz is mostly known for having introduced the space of tempered distributions, a vector space of **generalized functions** or **distributions**, invariant under the Fourier transform. defined by the simple relation

$$\hat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

Just in order not to leave out an important mathematical application area of tempered distributions (and their generalizations) let us mention the work of *Lars Hörmander* [1931-2012] exploring Fourier methods for PDEs (partial differential operators). He as well as *Elias Stein* [1931-2019] have developed Fourier Methods for the multi-dimensional setting.

The Classical Setting of Test Functions & Distributions



Fourier Transforms of Tempered Distributions

His construction *vastly extends the domain of the Fourier transform* and allows even polynomials to have a Fourier tranform. Among the objects which can now be treated are also the Dirac measures δ_x, as well as **Dirac combs** $⧢ = \sum_{k\in\mathbb{Z}^d} \delta_k$.
Poisson's formula, which expresses that one has for $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \qquad (14)$$

can now be recast in the form

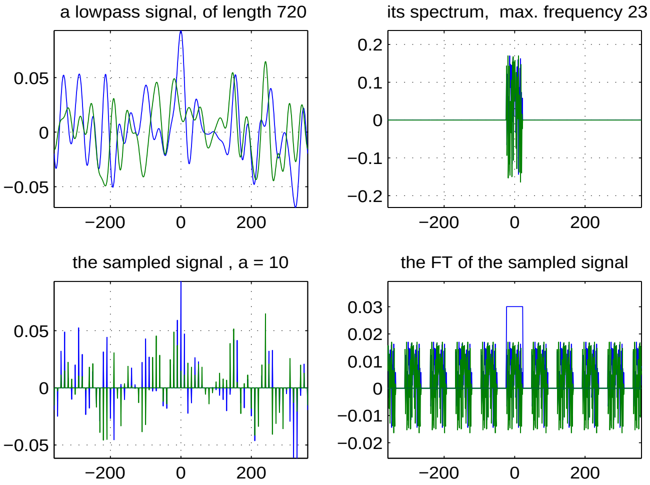
$$\widehat{⧢} = ⧢.$$

Sampling and Periodization on the FT side

This result is the key to prove **Shannon's Sampling Theorem** which is usually considered as the fundamental fact of digital signal processing (Claude Shannon: 1916 - 2001).
Shannon's Theorem says that one can have **perfect reconstruction** for band-limited functions.
If the so-called *Nyquist criterion* is satisfied (sampling distance small enough), i.e. $\text{supp}(\hat{f}) \subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) g(x - \alpha k), \quad x \in \mathbb{R}^d. \tag{15}$$

A Visual Proof of Shannon's Theorem



FFT: Fast Fourier Transform

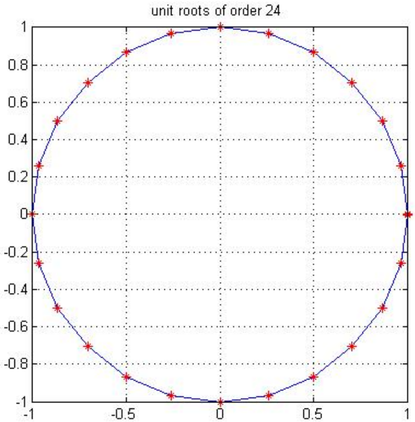
Originally introduced as a tool that should allow to approximately compute Fourier integrals based on suitable discretization of the continuous function ($f \in \boldsymbol{L}^1(\mathbb{R})$) in 1965 (by Cooley and Tuckey at IMB), the FFT has become the backbone of *digital signal processing*.

Instead of providing a lot of formulas let us mention that one possible interpretation of the (linear) mapping $\mathbf{a} \mapsto \mathbf{b} := \mathrm{fft}(\mathbf{a})$, from $\mathbb{C}^N$ to $\mathbb{C}^N$.

The most useful interpretation of the *usual formula* is:

Convert the set of coefficients $\mathbf{a} = (a_k)_{k=0}^{N-1}$ to the sequence of *values of the polynomial* $p_{\mathbf{a}}(z)$ over the unit roots of order N.

Unit roots of order 24



Basic DFT/FFT Properties

This interpretation explains various aspects of the DFT/FFT:

1. The matrix representing the DFT/FFT is (up to scaling) just a *unitary Vandermonde matrix*; hence inversion is easy;
2. The fast algorithm is based on group theoretical properties of the unit roots of order N if N is even, e.g. unit roots of order 24 are just two copies of unit roots of order 12;
3. Clearly pointwise multiplication of the values of the polynomials corresponds to the Cauchy product of the coefficients ("multiplying out rule for polynomials");
4. *Sampling* corresponds to *periodization* on the other side;
5. The FFT allows to compute the FT of discrete and periodic signals *exactly*.

Where do we use Fourier Analysis in our Daily Life?

Perhaps you have to think a bit? But there are MANY opportunities, and few activities do not involve the use of FFT-based technology.

1. You use your mobile phone to communicate?
2. You listen to music? (MP3 or WAV-files);
3. You download images? (JPEG format);
4. Your computer communicates with your printer;
5. You watch digital videos (streaming)?
6. So how do the data reach your device?

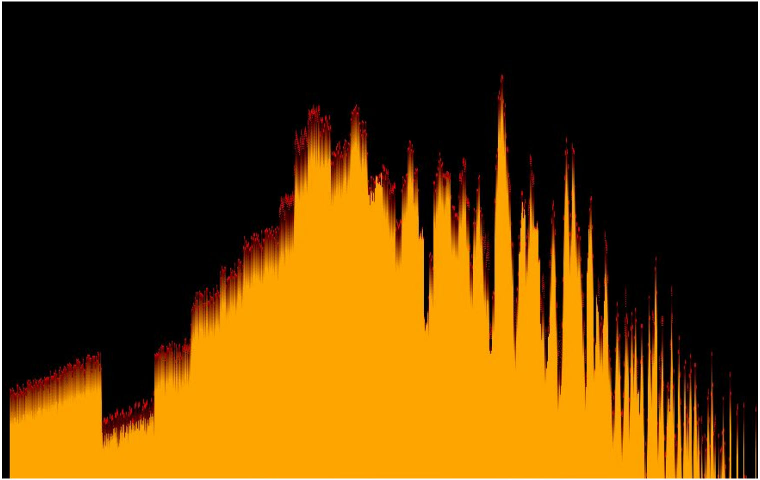
The answer is: There is a lot if digital signal processing going on in the background, using the FFT (Fast Fourier Transform).

CD Players with 44100 Samples per Second

A direct consequence of Shannon's Sampling Theorem (combined with laser techniques) is the availibility of CD players (and digital communication), using also *coding theory*:

Gabor Analysis in our kid's daily live (MP3)

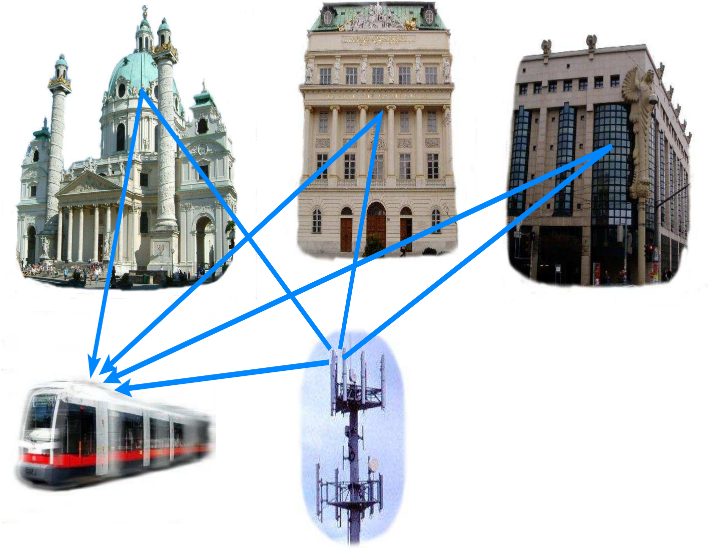
The Windows Media Player allows to **visualize music**, e.g. like



Mobile Communication using Fourier Methods

We exchange `WAV`, `MP3`, `JPG` data and digital voice:

Mobile Communication



Medical Imaging using Tomographs

Medical Imaging using the Radon Transform

Tomography and the Radon Transform

Mathematical key idea behind tomography

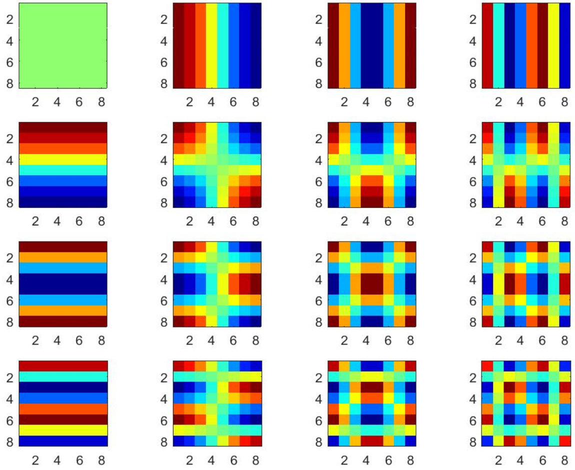
- The tomographic device measures the attenuation of of X-rays through the tissue along many-many straight lines, between (rotating) X-ray source and sensor array;
- Different tissues have known absorption behaviour, thus attenuation indicates integrated density along lines;
- Mathematically speaking the task is the invert a **sampled Radon transform** which can be obtained from these data
- After regridding the data arising on a polar grid an IFFT2 provides one possible way to produce images (slices),
- Modern **Compressed Sensing** methods improve further

The JPEG compression

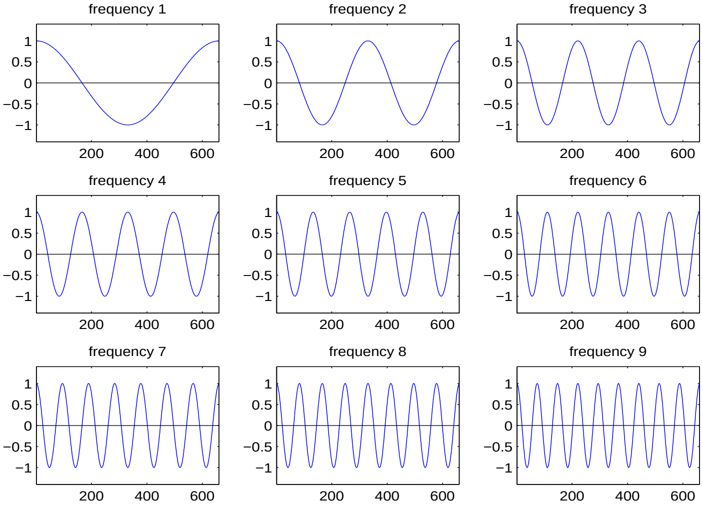
The widely used JPEG standard, established by the "Joint Photographic Experts Group" is based on the discrete cosine transform, a real version of the Fourier transform (real images give real coefficients).

- First a general image is decomposed into blocks of 8×8 pixels, (each of them in fact in the range of 0 to $255 = 2^8 - 1$, so one Byte or 8 Bits worth);
- Then depending on the chosen compression rate a fixed number of coefficients, from upper left to lower right corner (figure below) is stored and transmitted;
- Resynthesis from this set of coefficients provides the decoded image.

The building blocks for Discrete Cosine Transform DCT2



Building blocks for Discrete Cosine Transform DCT



The Key-players for Time-Frequency Analysis (TFA)

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

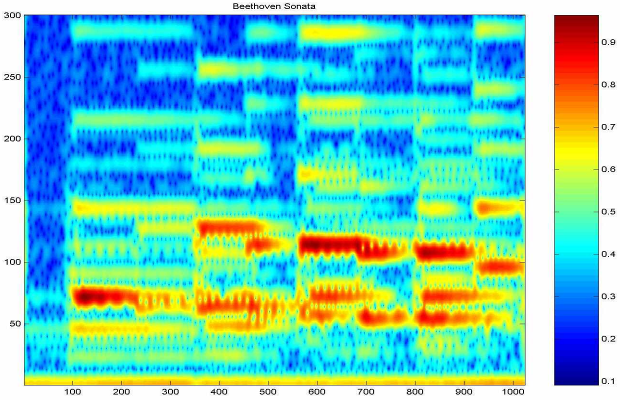
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \ \ \lambda = (t,\omega);$$

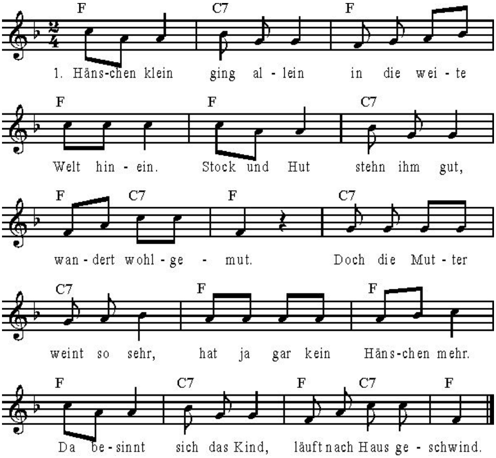
A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:

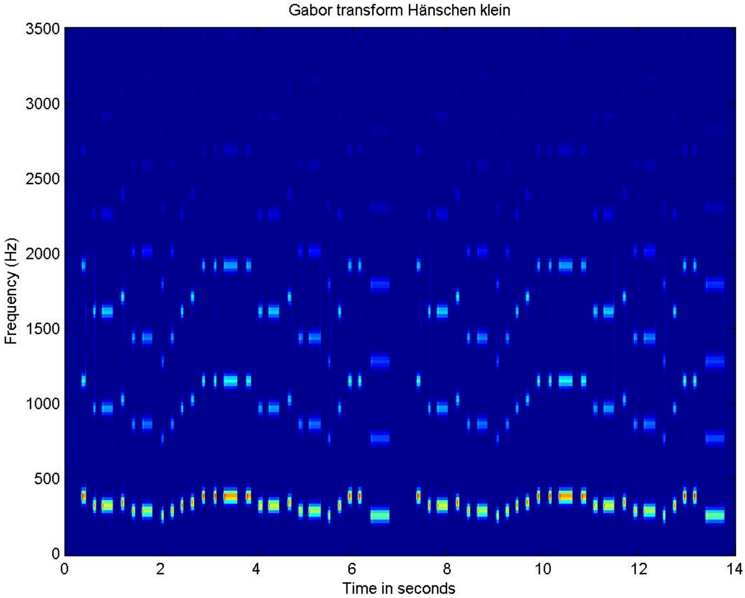


Time-Frequency and Musical Score

Time-Frequency Analysis and Music

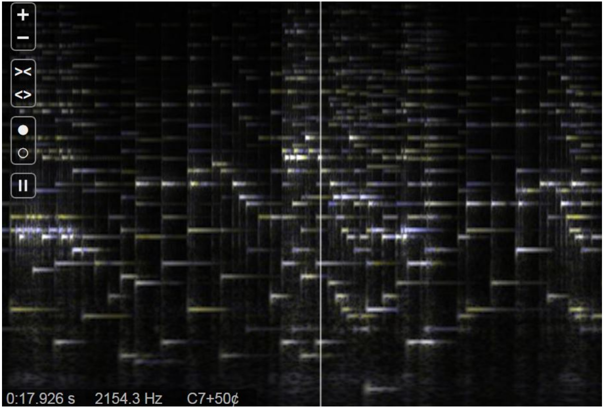


The Short-Time Fourier Transform of this Song



The Gaborator at www.gaborator.com

This software is based on work of my former students (at ARI). An almost professional version allows to upload WAV files:



Downloadable program STX, at ARI

Austrian Acoustic Research Institute (P. Balasz):

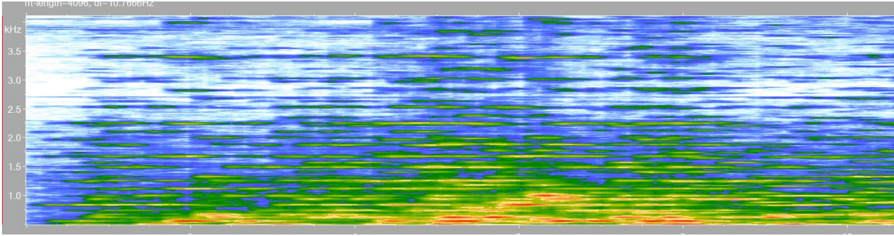


Figure: Spectrogram of Kosic playing Bartok 3rd Piano Concerto

Downloadable program STX, at ARI

Austrian Acoustic Research Institute (P. Balasz):

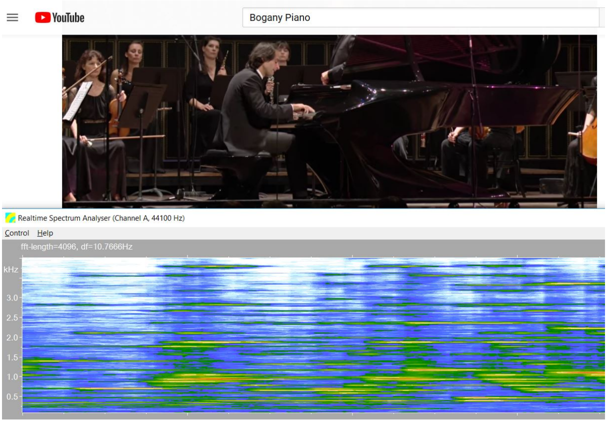
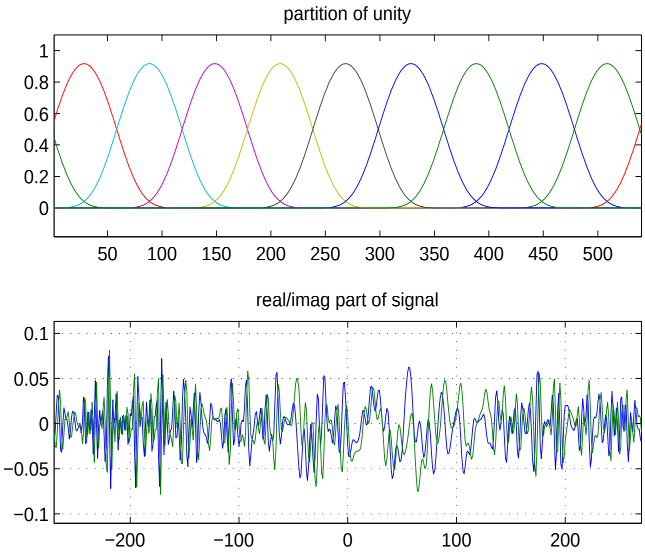
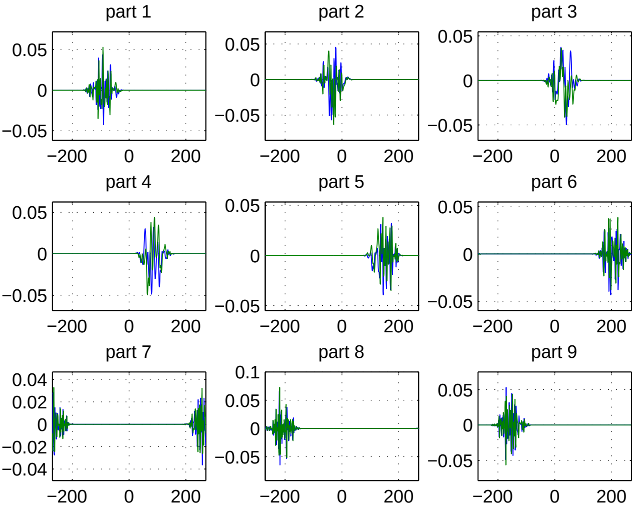


Figure: T. Bogany playing on "his" piano, Mozart piano concert

Motivated by Musical Score: Sliding FFT

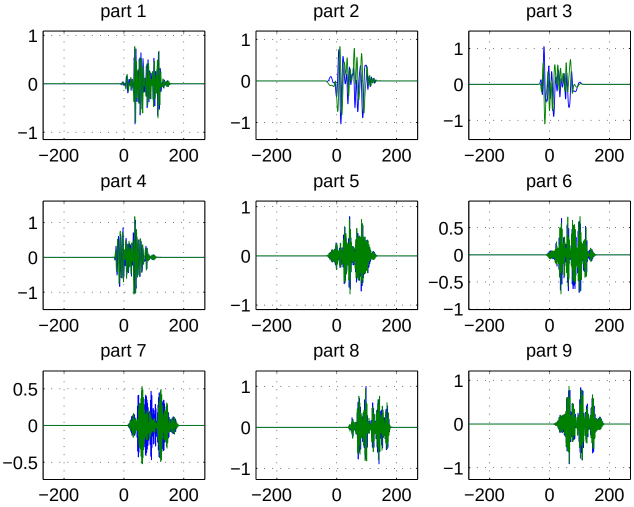


… and cut the signal into pieces



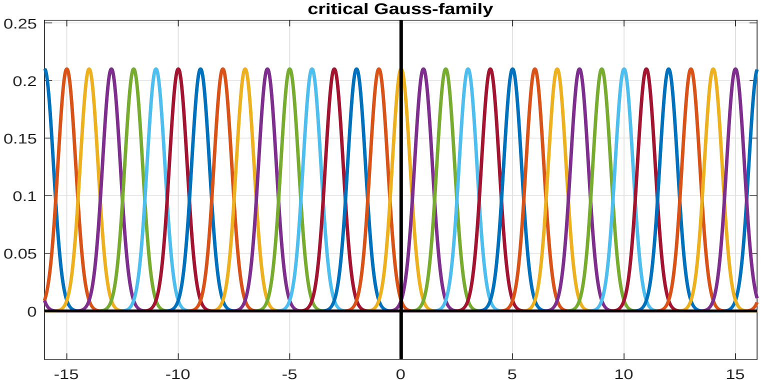
… and do localized spectra

MP3 is using the masking effect on those spectra!

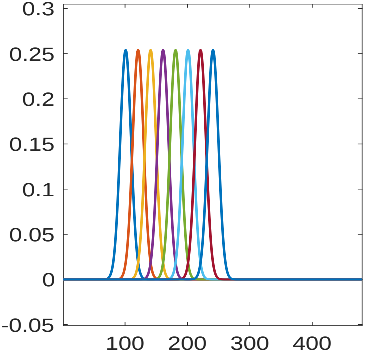


D.Gabor's Suggestion of 1946

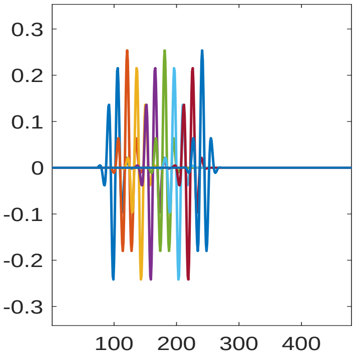
Choose the Gauss-function, because it is the unique minimizer to the *Heisenberg Uncertainty Relation* and choose the critical, so-called von-Neumann lattice, which is simply $\mathbb{Z}^2$.



D.Gabor's Suggestion of 1946, II

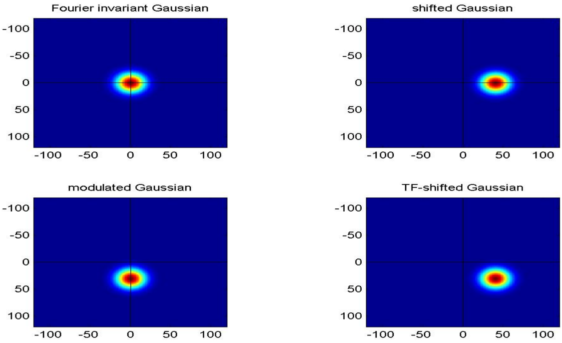


plot(real(G(:,151:30:364)));



plot(real(G(:,153:30:364)));

D.Gabor's Suggestion of 1946, III



Justification and Shortcoming

D. Gabor proposed to use integer time and frequency shifts (which commute!) of the Gauss function and the TF-lattice $a\mathbb{Z} \times b\mathbb{Z}$, with $a = 1 = b$, based on the following arguments:

1. The Gauss function is optimally concentrated in the time-frequency sense;
2. If $ab > 1$ then the collection of (Gabor) atoms does not span $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$;
3. If $ab < 1$ then there is a kind of redundancy and consequently linear dependency (hence non-uniqueness of the coefficients);

From a modern point of view the case $ab < 1$ is suitable, one has to use minimal norm coefficients for uniqueness. On the other hand the case $ab > 1$ provides Riesz basic sequences which are useful for *mobile communication*.

Mathematical Development of Gabor Analysis

It is remarkable that the (now seen as highly relevant) study of the expansions suggested in the seminal paper of D. Gabor in 1946 started only in the 1980s with work of A.J.E.M. Janssen (and HGFei).

The first books on the subject appeared in 1998, 2001 and 2003 (all related to NuHAG).

Function spaces related to Gabor expansions have been introduced already earlier, the so-called *modulation spaces* $(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}})$. They relate to Gabor expansions (weighted summability conditions on the Gabor coefficients) in a similar way as classical Besov-Triebel-Lizorkin spaces $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$ and $(\boldsymbol{F}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{F}^s_{p,q}})$ relate to *wavelet expansions*.

Nowadays this setting turns out to be quite fruitful for the treatment of *pseudo-differential operators*.

Modern Viewpoint I

Todays Rules of the Game

Choose a good *window* or *Gabor atom* (any $g \in \mathcal{S}(\mathbb{R}^d)$ will do) and try to find out, for which lattices $\Lambda \in \mathbb{R}^{2d}$ the signal f resp. its STFT (with that window) can recovered in a STABLE way from the samples, i.e. from the values $\langle f, \pi(\lambda)g \rangle$.
We speak of *tight Gabor frames* (g_λ) if we can even have the expansion (for some constant $A > 0$)

$$f = A \cdot \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad \forall \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Note that in general *tight frames* can be characterized as orthogonal projections of orthonormal bases of larger spaces!!!

Modern Viewpoint II

Another basic fact is that for each $g \in \mathcal{S}(\mathbb{R}^d)$ one can find, if Λ is dense enough (e.g. $a\mathbb{Z} \times b\mathbb{Z} \subset \mathbb{R}^d$ for $ab < 1$ in the Gaussian case) a *dual Gabor window* $\widetilde{g}$ such that one has at least

$$f = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda \tag{16}$$

$\widetilde{g}$ can be found as the solution of the (positive definite) linear system $S\widetilde{g} = g$, where $Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$, so using $\widetilde{g}$ instead of g for analysis or synthesis corrects for the deviation from the identity operator. An important fact is the commutation relation $S \circ \pi(\lambda) = \pi(\lambda) \circ S$, for all $\lambda \in \Lambda$.
Thus (16) is just $S \circ S^{-1} = Id = S^{-1} \circ S$ in disguise!).

Modern Viewpoint III

The possibility of having such *tight Gabor frames* is resulting from the continuous reconstruction formula, valid for arbitrary $\boldsymbol{L}^2$-atoms g. Writing again for $\lambda = (t, \omega)$ and $\pi(\lambda) = M_\omega T_t$, and furthermore $g_\lambda = \pi(\lambda) g$ we have in fact for any $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle g_\lambda \, d\lambda.$$

It follows from **Moyal's formula** (energy preservation):

$$\|V_g(f)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} = \|g\|_2 \|f\|_2, \quad f, g \in \boldsymbol{L}^2. \tag{17}$$

This setting is well known under the name of coherent frames when $g = g_0$, the Gauss function. Its range is the *Fock space*.

Modern Viewpoint IV

There is a similar representation formula at the level of operators, where we also have a continuous representation formula, valid in a strict sense for *regularizing operators*, which map w^*-convergent sequences in $\boldsymbol{S}_0'(\mathbb{R}^d)$ into norm convergent sequences in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle T, \pi(\lambda)\rangle_{\mathcal{HS}} \pi(\lambda) d\lambda. \tag{18}$$

It establishes an isometry for Hilbert-Schmidt operators:

$$\|T\|_{\mathcal{HS}} = \|\eta(T)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}, \quad T \in \mathcal{HS},$$

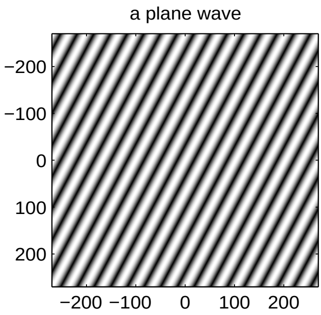
where $\eta T = \langle T, \pi(\lambda)\rangle_{\mathcal{HS}}$ is the *spreading function* of the operator T. The proof is similar to the proof of Plancherel's theorem.

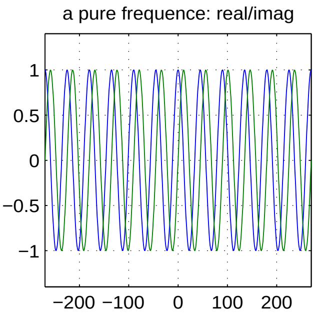
Gabor Riesz bases and Mobile Communication

Another usefulness of "sparsely distributed" Gabor systems comes from mobile communication:

1. Mobile channels can be modelled as slowly varying, or underspread operators (small support in spreading domain);
2. TF-shifted Gaussians are joint approximate eigenvectors to such systems, i.e. pass through was some attenuation only;
3. underspread operators can also be identified from transmitted pilot tones;
4. Communication should allow large capacity at high reliability.

2D-Gabor Transform: Plane Waves





2D-Gabor Analysis: Test Images

2D-Gabor Transform: Test-Images 2

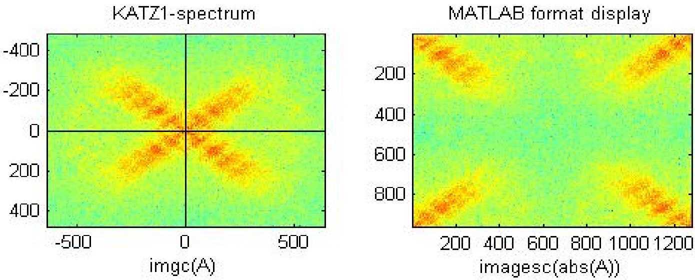
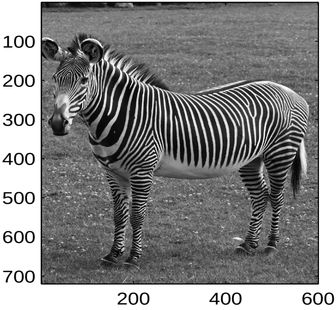
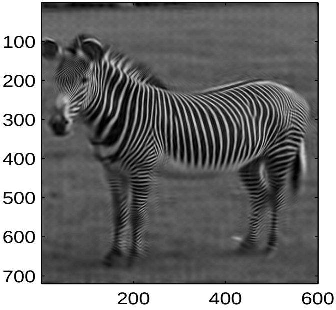
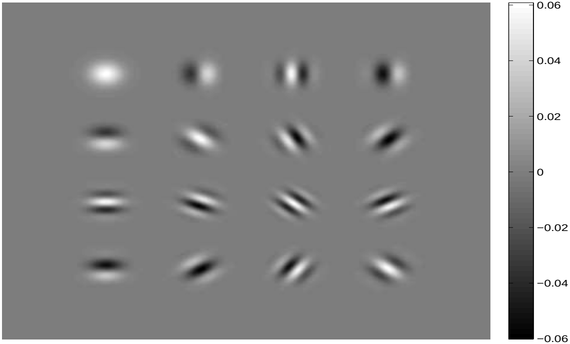


Image Compression: a Test Image





Showing the Elementary $2D$-Building Blocks



Usefulness and Applications of Gabor Frames:

The question of Gabor frames is of interest, when a signal (say some audio signal, or some image, cf. introduction) is to be *decomposed* into meaningful elementary building blocks, somehow like *transcription*. Ideally the distribution of *energy* in the signal goes over into an equivalent energy distribution. AND WHAT can we do with this:

a) contributions may be irrelevant (or disturbing) and can be eliminated (the bird contributing to the open air classical concert): denoising of signals

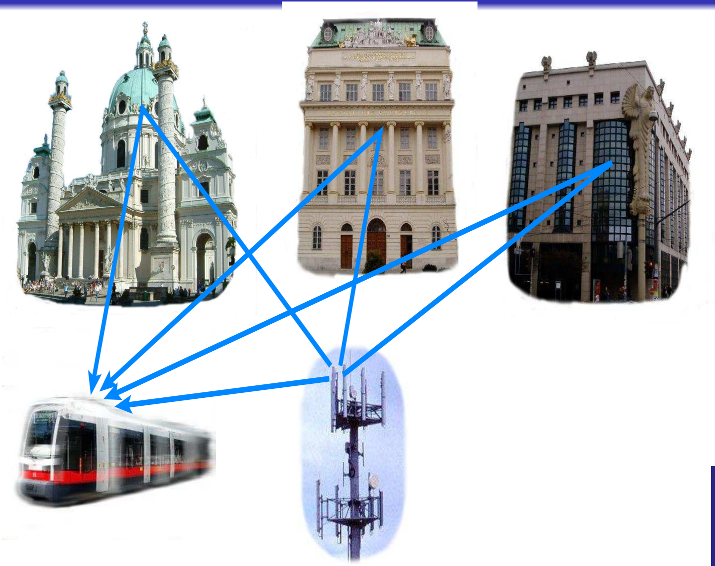
b) signals can be separated in a TF-situation;

c) unimportant, small contributions can be omitted (+ masking effect): allows for efficient lossy compression schemes >> **MP3**.

Applications of Gabor Riesz Bases:

Of course Gabor Riesz bases (for subspaces) will correspond to lattices Λ with at most N points. Ideally, the Gram matrix of the corresponding system is diagonal dominant (there is the so-called *piano-reconstruction theorem* $>$ transcription?).
They are very useful in mobile communication. The fact, that smooth envelopes (as used for Gabor frames), multiplied with pure frequencies are at least approximate eigenvectors for so-called *slowly varying channels* makes them useful for mobile communication. The physical assumption of limited multi-path propagation (variable kernels over time) and Doppler (due to movement) related to underspread operators, i.e. to matrices whose spreading function is supported on a given rectangular domain.

Mobile Communication



Applications of Gabor Riesz Bases:

The information, encoded as a collection of coefficients which we will call (c_{λ°) are used to form a linear combination of the elements of our Gaborian Riesz basis. I.e. the sender *plays slowly a melody on the piano*.
Assume we are able to estimate the approximate eigenvalues $(d_{\lambda^\circ}$ of the involved building blocks $(g_{\lambda^\circ}$, the approximate eigenvector property of these building blocks implies that the receiver obtains $\sum_{\lambda^\circ} c_{\lambda^\circ} d_{\lambda^\circ} g_{\lambda^\circ}$. Knowing the factors (d_{λ°) (by sending so-called pilot tones) and the biorthogonal basis the receiver can then (approximately) recover the set of coefficients (c_{λ°) sent by the sender.
In other words, *the receiver listens to the music behind a wall, knowing e.g. that higher frequencies are absorbed more (or less) then others and figures out, what has been played.*

Operating on the Audio Signal: Filter Banks

Finally let us operate on the Gabor Coefficients

Definition

Let g_1, g_2 be two L^2-functions, Λ a TF-lattice for $\mathbb{R}^d$, i.e. a discrete subgroup of the phase space $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Furthermore let $\mathbf{m} = (\mathbf{m}(\lambda))_{\lambda \in \Lambda}$ be a complex-valued sequence on Λ. Then the **Gabor multiplier** associated to the triple (g_1, g_2, Λ) with (*strong* or) **upper symbol m** is given as

$$G_{\mathbf{m}}(f) = G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda \in \Lambda} m(\lambda) \langle f, \pi(\lambda) g_1 \rangle \, \pi(\lambda) g_2. \qquad (19)$$

g_1 is called the *analysis* window, and g_2 is the synthesis window. If $g_1 = g_2$ and $\mathbf{m}$ is real-valued, then the Gabor multiplier is self-adjoint. Since the constant function $\mathbf{m} \equiv 1$ is mapped into the Identity operator if $g_1 = g_2$ is a Λ-tight Gabor atom this is often the preferred choice.

A few relevant references

K. Gröchenig: Foundations of Time-Frequency Analysis, Birkhäuser, 2001.
H.G. Feichtinger and T. Strohmer: Gabor Analysis, Birkhäuser, 1998.
H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, Birkhäuser, 2003.
G. Folland: Harmonic Analysis in Phase Space. Princeton University Press, 1989.
I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.
Some further books in the field are in preparation, e.g. on modulation spaces and pseudo-differential operators.
See also www.nuhag.eu/talks.

Astronomial Insight

Time-Frequency Analysis and Black Holes

Breaking News of Oct. 2017

Today, Oct. 3rd, 2017, the **Nobel Prize in Physics** was awarded to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!

The Prize-Winners are

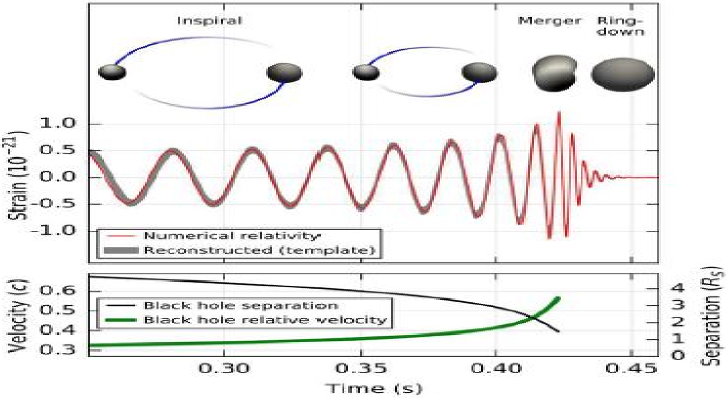
Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

https://en.wikipedia.org/wiki/Gravitational-wave_observator

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".

There had been two strategies:

- Searching for 2500 explicitely determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.

NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).

Further Topics

Obviously there is a long list of topics which can be connected with the topic of Fourier and Gabor Analysis:

1. Despite the fact that Fourier Analysis is well established mathematically we should change the perspective in teaching it, making it more application oriented, but still mathematically correct.
2. Make use of modern tools (media, mathematical Software, like MATLAB or GEOGEBRA) and encourage mathematical experiments, already to high-school students.
3. Mathematicians and applied scientists should talk more to each other and explain their work.
4. A new, simplified theory of distributions ("**mild distributions**", based on the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$) is on the way.

THANK YOU

Thank you for your attention!!

at University of PECS!

More at www.nuhag.eu

Access to the collection of all Talks by HGFei can be obtain by request at hans.feichtinger@univie.ac.at
as well as to all of the NuHAG publications.