Computational Aspects of Time-Frequency and Gabor Analysis

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39th Meeting of Prague computer science seminar

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Spectrogram of a Piece of ASONANCE

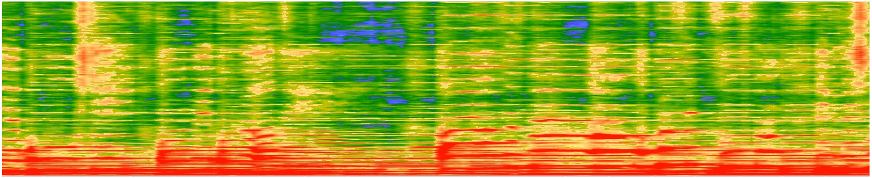


Figure: Asonance02.jpg:
Spectrogram based on a YOUTUBE recording of ASONANCE
horizontal axis: time, vertical axis: frequency

TEST

The End of a Piece of Music

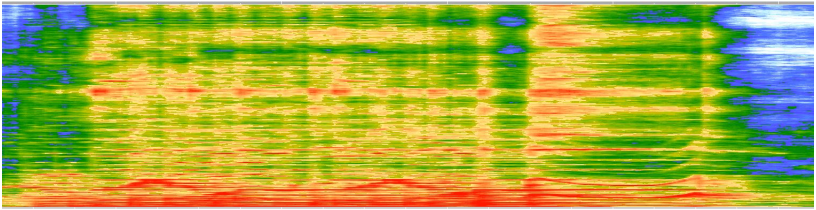


Figure: Asonance06.jpg

This is another piece of music of ASONANCE which can be viewed at YOUTUBE.

ChromatOrg1

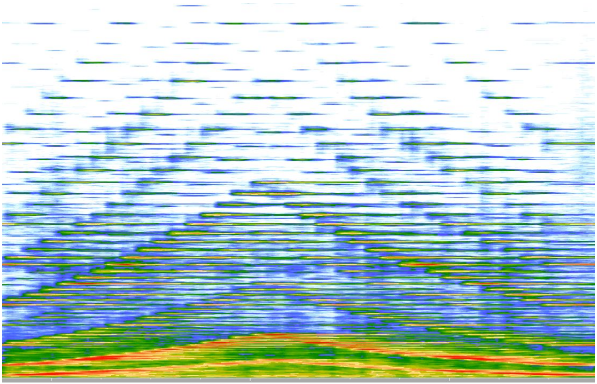


Figure: ChromatOrg1.jpg

Synthetic Sound via MATLAB

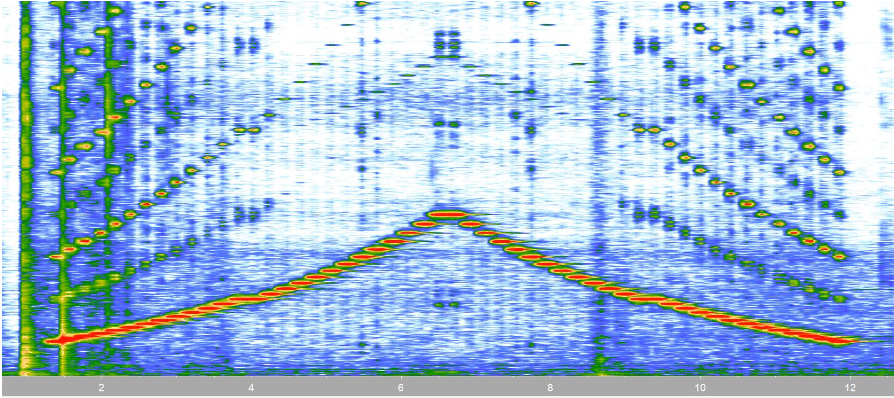


Figure: MATLsound1.jpg

Put together using Hanning windows of length $11025 = 44100/4$.

Figure: MIKMEamaz.jpg

WavFeuerAdel16

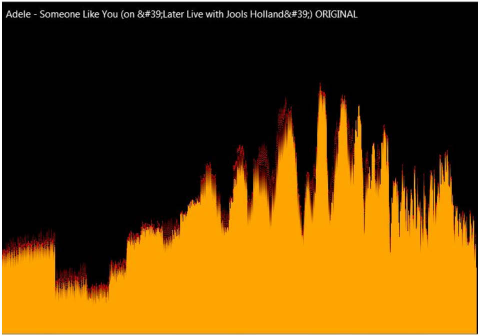


Figure: WavFeuerAdel16.jpg

Structure of the talk

Fourier Analysis started in 1822 with a publication of **J.B.Fourier**, presenting a solution of the *heat equation*. A solid foundation of this *integral transform*, based on the Lebesgue integral (using $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ and $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$). In 1965 the FFT, the so-called Fast Fourier Transform (i.e. an efficient implementation of the DFT, the discrete Fourier transform) was suggested.

1. Examples and Demonstrations
2. Background on Fourier Transforms
3. Short Time FT and Gabor Analysis
4. Algorithmic Questions
5. Applications and Challenges

Jarnik Lecture in Prag, 2017

In my Jarnik Lecture in Prag, October 4th, 2017, I have presented the following aspects of Fourier Analysis:

1. History of Fourier Analysis
2. From Complex Numbers to Digital Signal Processing
3. Mathematical Developments, up to Distribution Theory
4. Real World Applications

In the current semester (also sommer 2017 and 2018) I have been teaching a pair of courses, one starting from Linear Algebra, using MATLAB, the other presenting a simplified theory of *mild distributions*, see `www.nuhag.eu/prag18`.

Hans G. Feichtinger: Jarnik Lecture: Fourier Analysis in the 21st Century. Talk at Charles University. Oct. 4th, 2017, see `www.nuhag.eu/talks`, search Feichtinger.

Modern Applications of Fourier Analysis

It is even hard to say where Fourier Analysis does not play a role.

1. Digital Signal and Image Processing
2. Mobile Communication
3. CD-player, HiFi standard: $44100 = (2 \cdot 3 \cdot 5 \cdot 7)^2$ s/sec;
4. MP3 Audio Compression
5. JPEG Image Compression
6. Tomography
7. Modern Solvers for PDE
8. Spectroscopy, Crystallography
9. Discovery of Gravitational Waves

Different Flavours of the Fourier Transform

Most books on the Fourier Transform, including those for engineers or physicists, distinguish between different types of signals:

- continuous variable, periodic functions (Fourier series);
- continuous variables, non-periodic (Fourier transform);
- discrete variables, non-periodic (time series);
- discrete and periodic signals (DFT, FFT);

Abstract Harmonic Analysis views them as versions of *one principle*, providing a *decomposition of functions* over locally compact Abelian groups, such as $\mathbb{U}, \mathbb{R}^n, \mathbb{Z}^k, \mathbb{Z}_N$, into "basic building blocks", called *pure frequencies* or *plane waves*.

Gianfranco Cariolaro Unified Signal Theory, Springer, London, (2011) p.xx+927.

Jens Fischer: Four Particular Cases of the Fourier Transform Mathematics (MDPI), Vol.12 No.6, (2018), p.335.

The Classical Integral Formula

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i s \cdot t} \, dt, \quad t, s \in \mathbb{R}^d, \tag{1}$$

where $s \cdot t$ denotes the *scalar product* of $s, t \in \mathbb{R}^d$.
The *inverse Fourier transform* then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{2}$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, which is *not* satifsied for arbitrary functions $f \in \boldsymbol{L}^1(\mathbb{R}^d)$. In the general case ($f \in \boldsymbol{L}^1(\mathbb{R}^d)$) one can obtain f from $\hat{f}$ using classical summability methods, convergent in the $\boldsymbol{L}^1$-norm.

The Idea of Conceptual Harmonic Analysis

In the last years I have been involved in the study of questions in

1. Abstract Harmonic Analysis;
2. Function Spaces and Distributions;
3. Numerical Fourier Analysis (**NuHAG**!), such as scattered data approximation or Gabor Analysis;
4. Real World Applications, e.g. Mobile Communication;

What I found was a strong *discrepancy* between the fine details developed by mathematicians, claiming to do something useful, and the interesting application areas, where often lack of precision was prevalent.
Aside from having fast algorithm, how does the FFT relate to the integral transform, can we guarantee precision, in which sense?

Convolution powers, starting from random vector

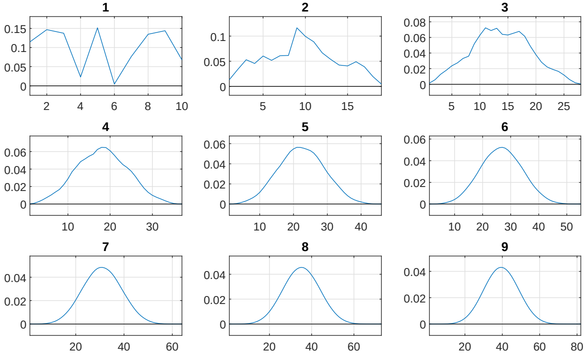


Figure: convpowers02.eps

Experimental evidence for the **central limit theorem**

The DFT: Discrete Fourier Transform

Instead of describing it through formulas, let us describe it by a simple reinterpreation, and by what one can say about it:

1. The DFT can be described as the linear mapping which starts from a sequence of coefficients, defining a polynomial

$$p_{\mathbf{a}}(x) := a_1 + a_2 x + \cdots a_n x^{n-1},$$

which is then evaluated at the unit roots of order N, starting at $1 = \omega^0$, in the clockwise sense.
Hence the *matrix realizing the DFT* is just a Vandermonde-matrix, properly arranged (and thus invertible!).

2. This explains immediately the so-called Convolution Theorem: The *Cauchy-Product* providing the coefficients of a product polynomial is the same as the coefficients of the polynomial obtained by multiplying their values at unit roots properly!).

1. But looking at the entries of the Fourier matrix one easily can verify that they are orthogonal vectors, thus (up to the scaling factor $\sqrt{N}$) the Fourier transform is a *unitary mapping*. Consequently it is just a change of basis to another orthonormal basis, consisting of *pure frequencies*.
2. Also it is easy to understand that regular sampling (leaving only e.g. every second or third sample value, if 2 or 3 divide N) corresponds to periodization on the Fourier transform side. This is the basis for the understanding of the **Shannon Sampling Theorem** which allows to recover *band-limited signals* from (dense and regular) sampling sequence (which is in turn the basis for digital audio, using just 44100 samples per second)!

uroots12num.pdf

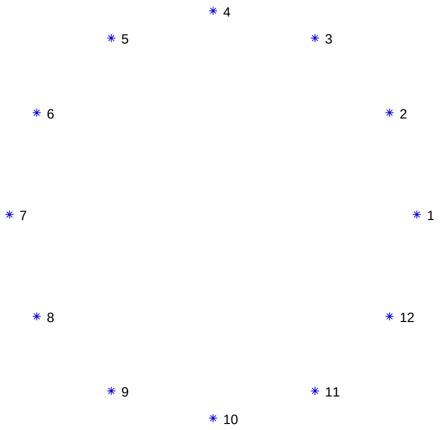


Figure: uroots12num.pdf

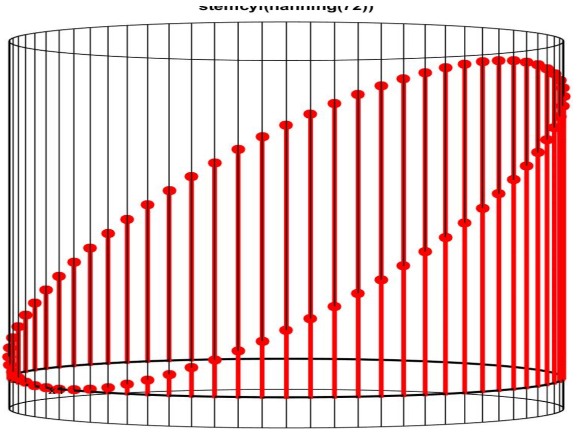


Figure: stemcyl72.jpg

shannon14415.pdf

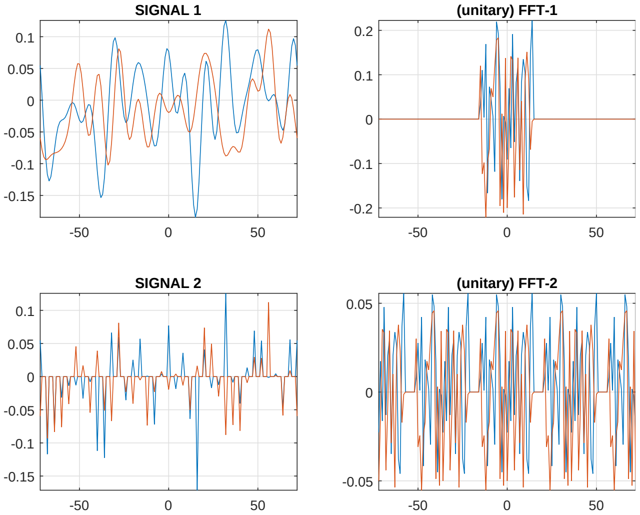


Figure: shannon14415.pdf

Image Processing and FFT2

Convolution and the Fourier transform are crucial in order to characterize *time-invariant linear systems* as convolution operators (convolution by a so-called *impulse response*) or as a multiplication operator (by the so-called *transfer function*, which is simply the Fourier transform of the impulse response.

Similar facts are valid in higher dimensions. For $d = 2$, i.e. for applications in image processing we have operators which commute with ordinary $2d$-translation, such as a blurring operator or an edge-detection operator.

The (invariant) functions, i.e. joint (complex-valued) eigenvectors of all these translation operators turn out to be the *plane waves*, or the (tensor) products of pure frequencies. This separation argument allows to implement the FFT2 by iteration of ordinary FFTs (row-wise/column-wise). Impulse responses are now called *point-spread functions*.

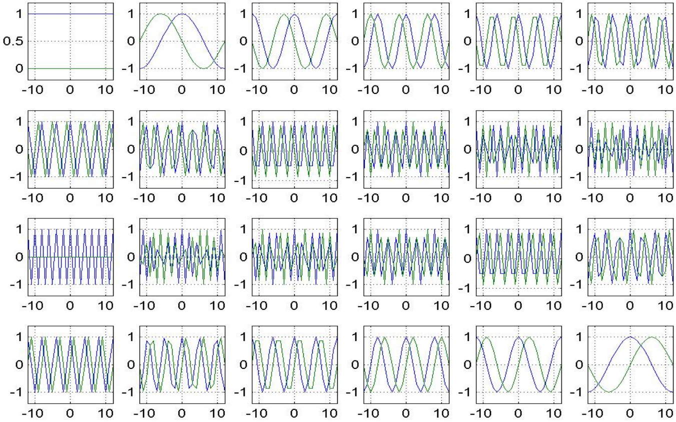
Some MATLAB Code

```
>> n = 512; PIC = rand(n); x = rand(n,1);
F = fft(eye(n));
norm(F*x - fft(x)),
ans = 1.9386e-13
>> norm(F-F.','fro') (showing real symmetry)
ans = 6.3377e-14
norm(F*PIC*F - fft2(PIC),'fro')
ans = 1.1192e-10
```

This experiments also indicates why the order (FFT row-wise or column-wise) does not matter! (associativity law)

Hans G. Feichtinger: presentation at MATLAB Expo, Munich, 2016
http://univie.ac.at/nuhag-php/dateien/talks/3609_EXPOFeiE6.pdf

The rows of the Fourier matrix (real/imag)



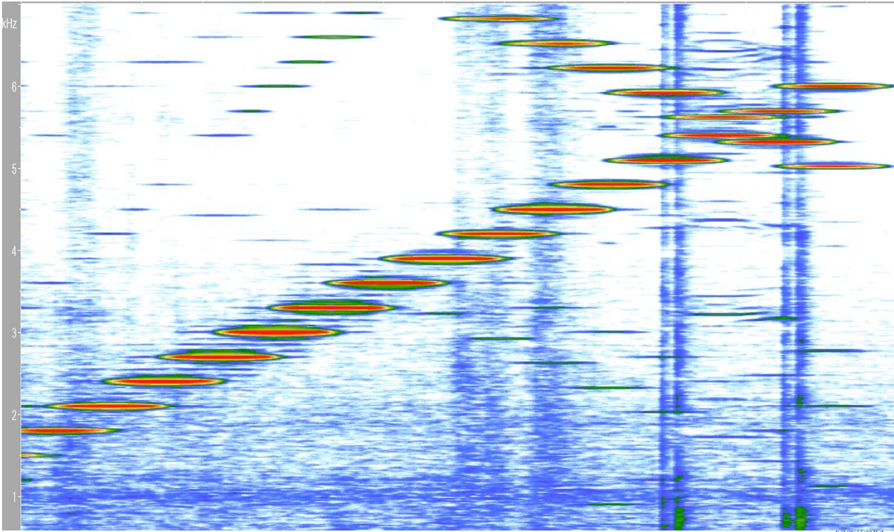


Figure: GaborAtoms02.jpg

MATLupdown1

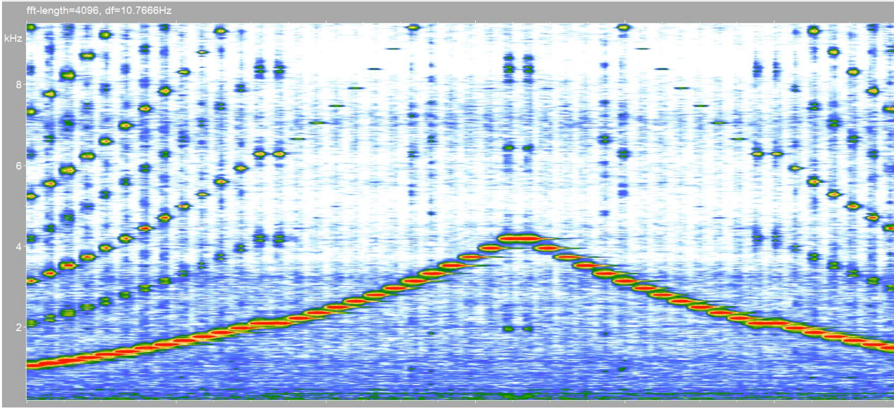


Figure: MATLupdown1.jpg

52	C5	c^2	523,251
51	B4	h^1	493,883
50	A#4/Bb4	ais^1/b^1	466,164
49	A4 [2]	a^1 Kammerton	440,000
48	G#4/Ab4	gis^1/as^1	415,305
47	G4	g^1	391,995
46	F#4/Gb4	fis^1/ges^1	369,994
45	F4	f^1	349,228
44	E4	e^1	329,628
43	D#4/Eb4	dis^1/es^1	311,127
42	D4	d^1	293,665
41	C#4/Db4	cis^1/des^1	277,183
40	C4[3]	c^1	261,626

Figure: TonleiterCC1.jpg

Cosine-Transforms: DCT, ICDT, DCT2, ICDT2

Sometimes the use of complex-valued building block fro real-valued signals is not appreciated, and then one is calling for real variants of the Fourier transform. This can be obtained by using orthogonal matrices with are based on the cosine functions and form a (real) basis for real-valued signals. We will display some of the building blocks below.
Again we can reduce the two-dimensional version by interation of the one-dimensional version. So if one has a fast code for the DCT/IDCT one has automatically a (well parallelizable) version for the 2d-situation!

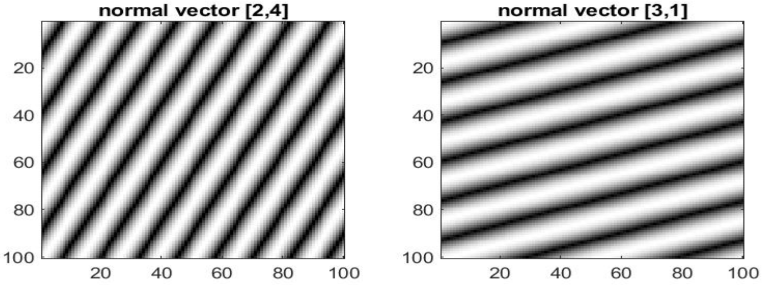
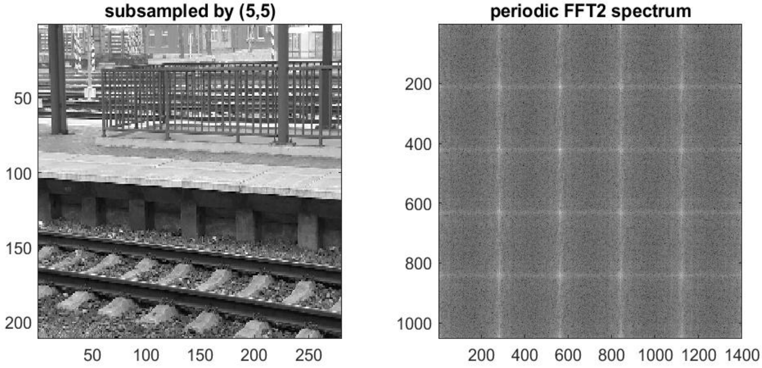
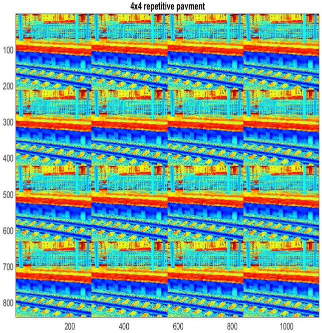


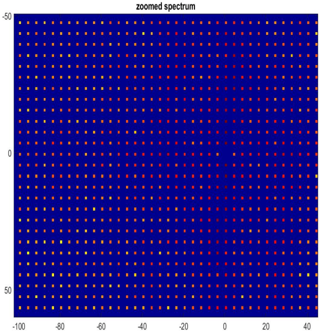
Figure: planewav2431.jpg

subsamp55.jpg



repfourbfour4.jpg





bsplindat03b.pdf

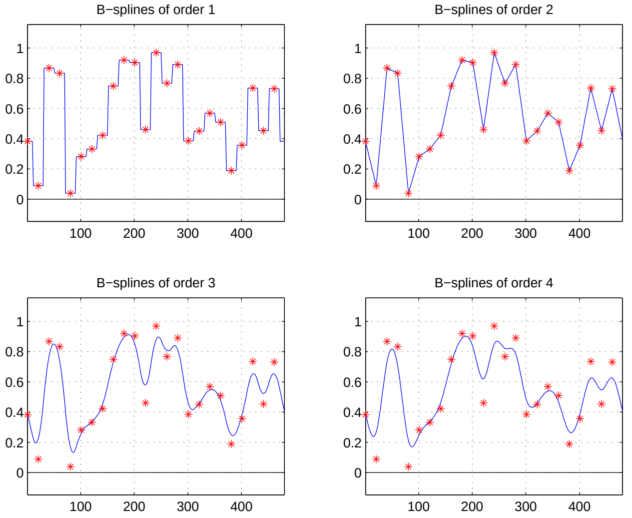


Figure: bsplindat03b.pdf

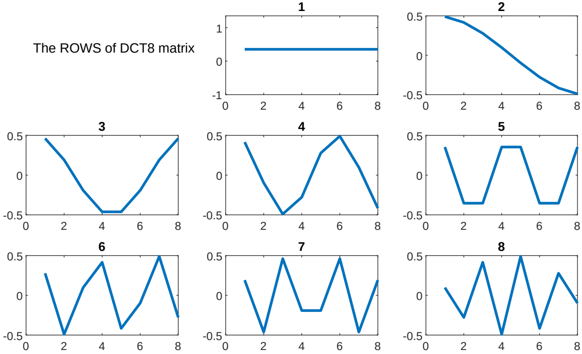


Figure: DC88demo2.eps

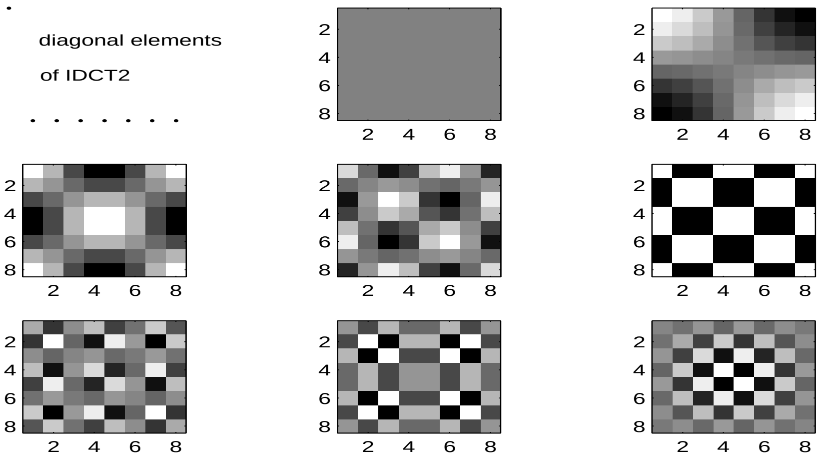


Figure: DCT88diag.pdf

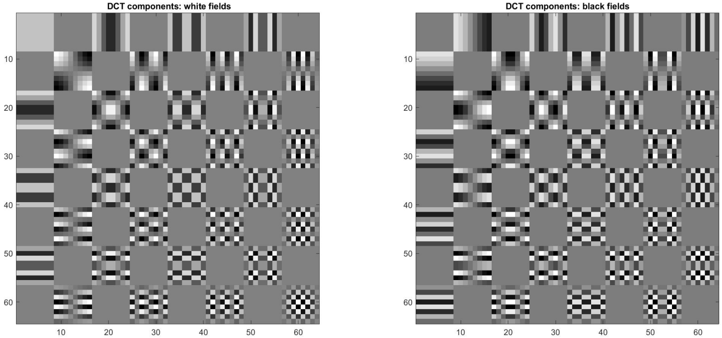


Figure: demoDCT202.jpg

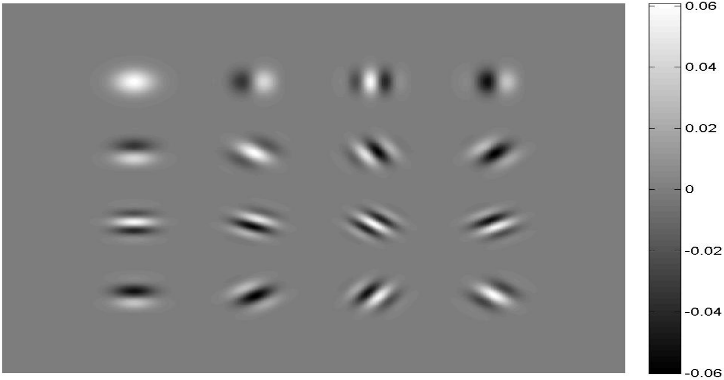


Figure: gabatom2b.jpg

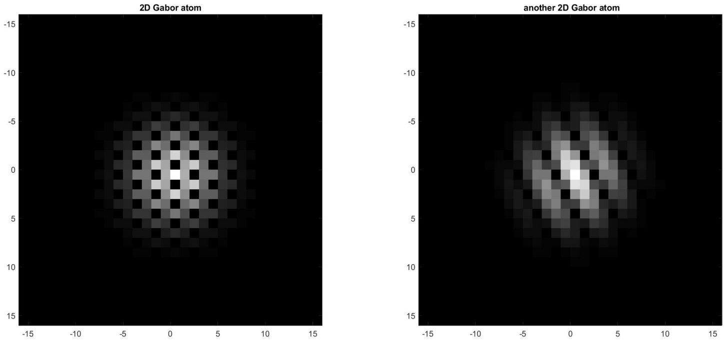


Figure: demogab202b.jpg

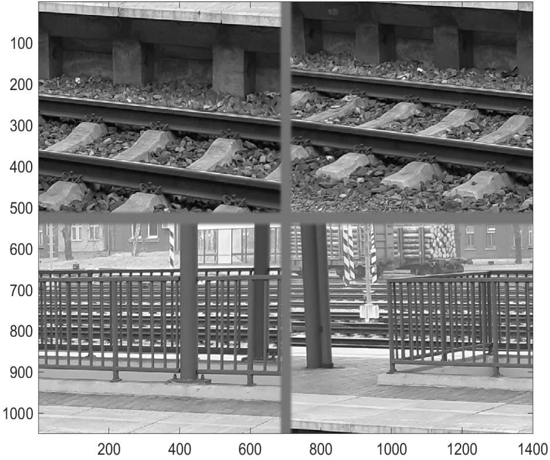


Figure: breclavsh1.jpg

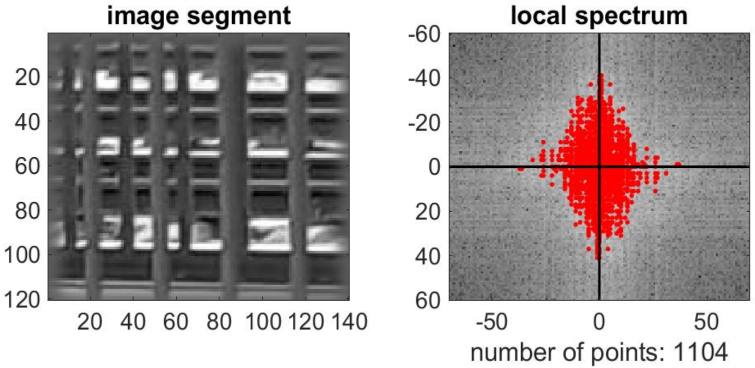
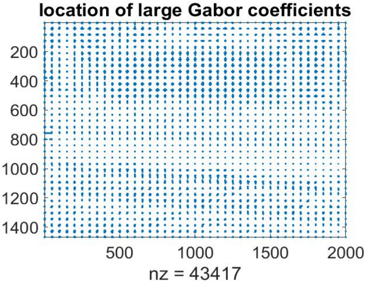


Figure: brecloc12.jpg



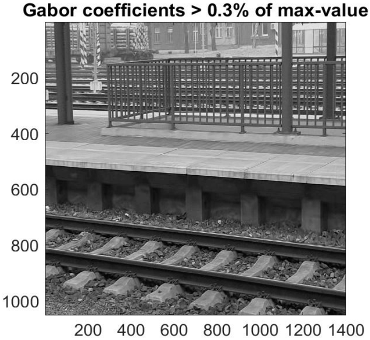
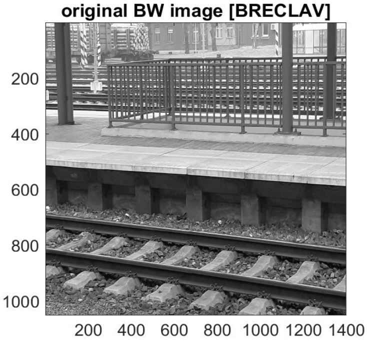


Figure: breclav002b.jpg



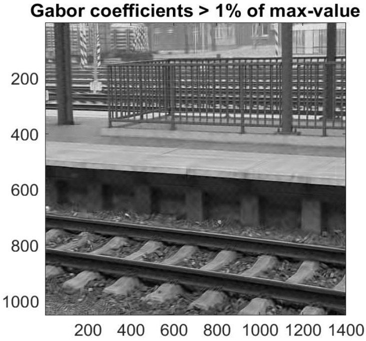
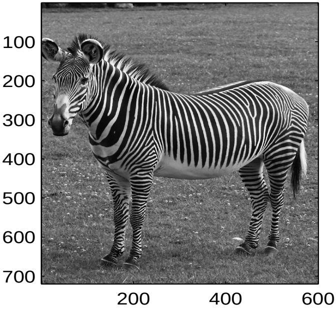
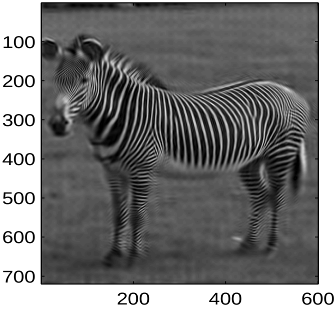


Figure: breclav002.jpg

Image Compression





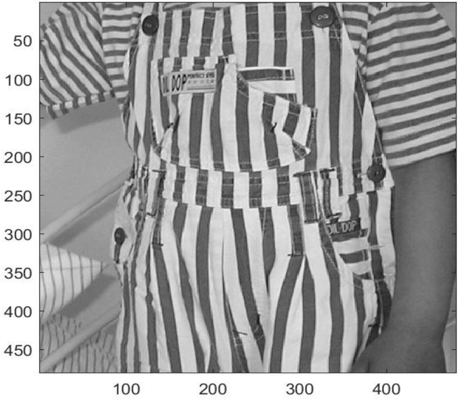


Figure: stripes480A.jpg: A test image, with stripes of different kind in different directions

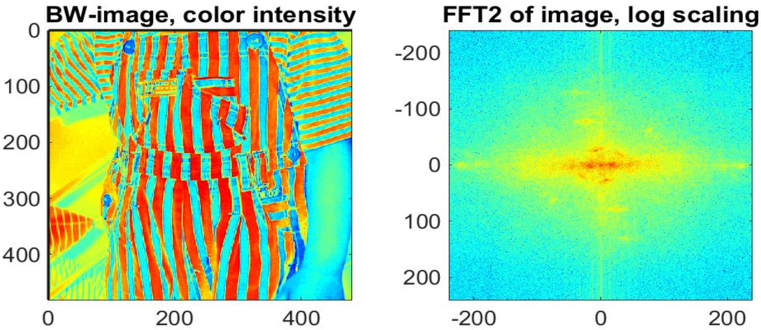


Figure: stripesFT2A.jpg

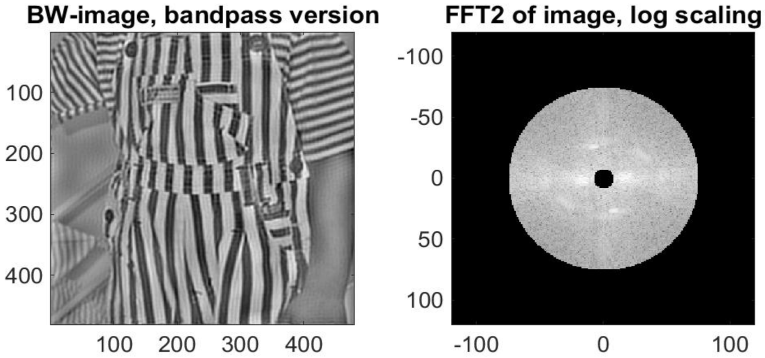


Figure: stripesFT2Ebw.jpg

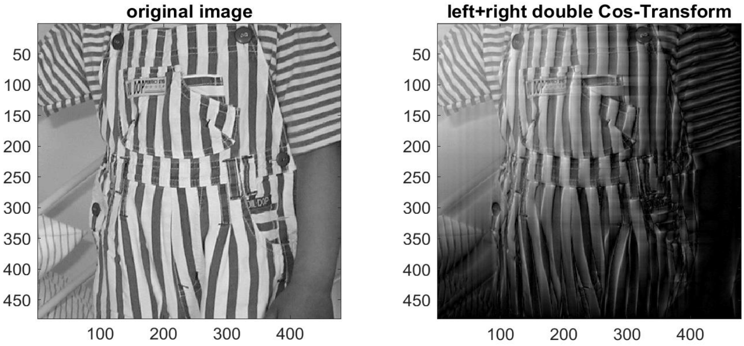


Figure: stripesFT2Fdoub.jpg

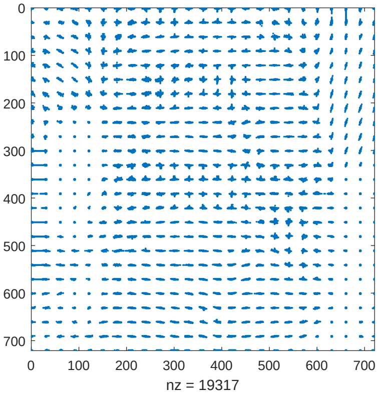


Figure: stripGab720a.eps

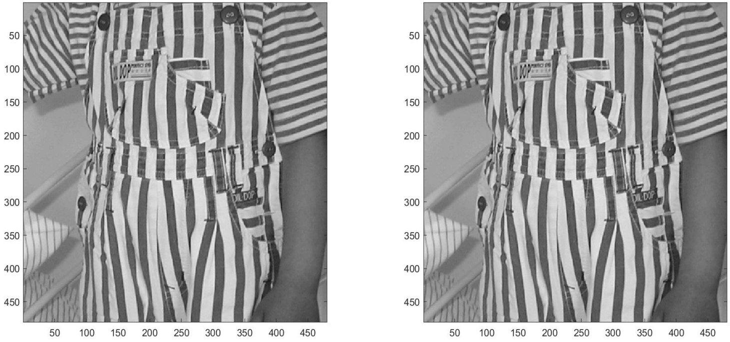
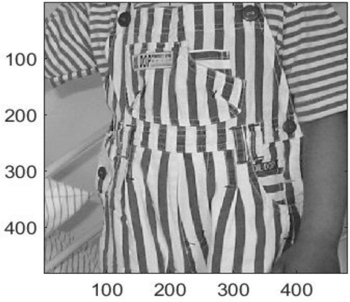


Figure: stripesFILTZA.jpg: ca. 3-4 percent of Gabor coefficients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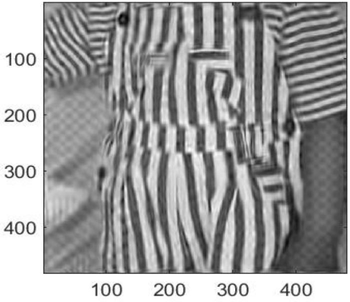
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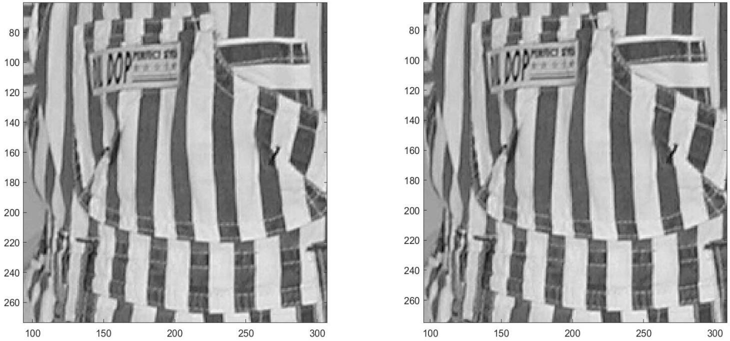


Figure: stripesFILTZ1.jpg

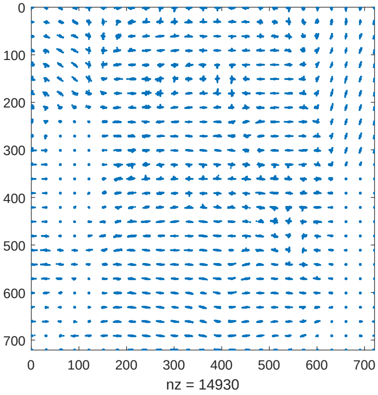


Figure: stripGab720b.eps: 1/75 of max coordinate

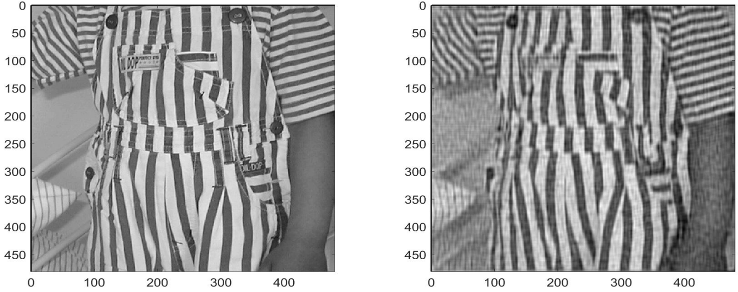


Figure: compGabDC01.jpg

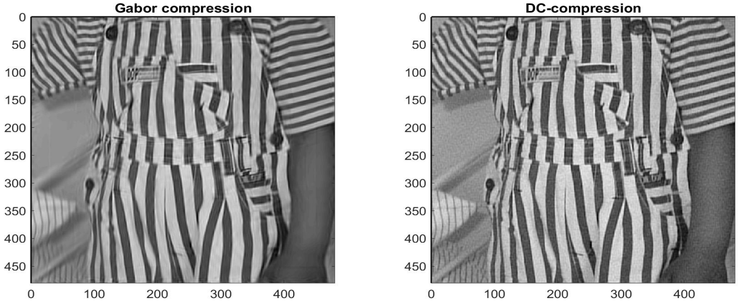


Figure: compGabDC02.jpg: almost exactly the same number of coefficients, ca. 8 percent, or 19200 coefficients for both schemes.

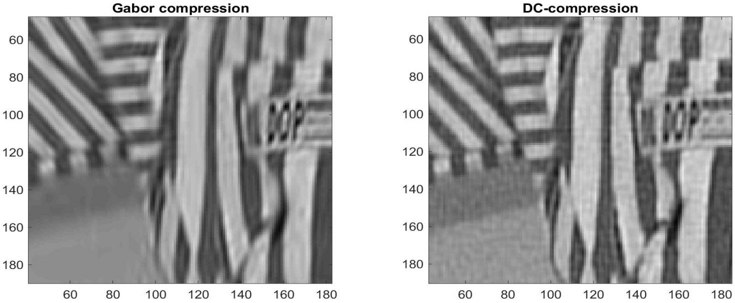


Figure: compGabDC02b.jpg

Fourier Transform over the Real Line

The work of H.L. Lebesgue paved the way to a clean definition of the Fourier transform for “functions of a continuous variables” as an *integral transform* naturally defined on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$

$$\|f\|_1 := \int_{\mathbb{R}} |f(x)| dx, \quad f \in \boldsymbol{L}^1(\mathbb{R}). \tag{3}$$

The (continuous) Fourier transform for $f \in \boldsymbol{L}^1(\mathbb{R})$ is given by:

$$\hat{f}(s) := \int_{\mathbb{R}} f(x) e^{-2\pi i s x} dx, \quad s \in \mathbb{R}. \tag{4}$$

With this normalization the inverse Fourier transform looks similar, just with the conjugate exponent, and thus, *under the assumption that f is continuous and $\hat{f} \in \boldsymbol{L}^1(\mathbb{R})$* we have pointwise

$$f(t) = \int_{\mathbb{R}} \hat{f}(s) e^{2\pi i s t} ds. \tag{5}$$

Plancherel's Theorem: Unitarity Property of FT

Using the density of $\boldsymbol{L}^1(\mathbb{R}) \cap \boldsymbol{L}^2(\mathbb{R})$ in $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$ it can be shown that the Fourier transform extends an a natural and unique way to $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$:

Theorem

The Fourier (-Plancherel) transform establishes a unitary automorphism of $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$*, i.e. one has*

$$\|f\|_2 = \|\hat{f}\|_2, \quad f \in \boldsymbol{L}^2(\mathbb{R}),$$

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}).$$

In some sense *unitary* transformations of a Hilbert transform is like a change form one ONB to another ONB in $\mathbb{R}^n$.

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the "continuous representation" formula, using $\chi_s(x) = e^{2\pi isx}$, $x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{6}$$

This looks like a perfect orthogonal expansion, but unfortunately the "building blocks" $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

Convolution and the Fourier Transform

Another important fact about the Fourier transform is the so-called **convolution theorem**, i.e. the Fourier transform converts convolution into pointwise multiplication.
Again it is natural to define convolution on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g(x-y)f(y)dy \quad xa.e.; \quad (7)$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}).$$

For positive functions f, g one even has equality. This is relevant for the determination of probability distributions of a sum of *independent* random variables. Assume X has density f and Y has density g then the random variable $X + Y$ has probability density distribution $f * g = g * f$.

Banach algebras

Theorem

Endowed with the bilinear mapping $(f,g) \to f * g$ *the Banach space* $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ *becomes a commutative Banach algebra with respect to convolution.*

The **convolution theorem**, usually formulated as the identity

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}), \tag{8}$$

implies

Theorem

The Fourier algebra, defined as $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}) := \{\hat{f} \mid f \in \boldsymbol{L}^1(\mathbb{R})\}$, *with the norm* $\|\widehat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ *is a Banach algebra, closed under conjugation, and dense in* $\left(\boldsymbol{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$ *(continuous functions, vanishing at infinity).*

Function space norms

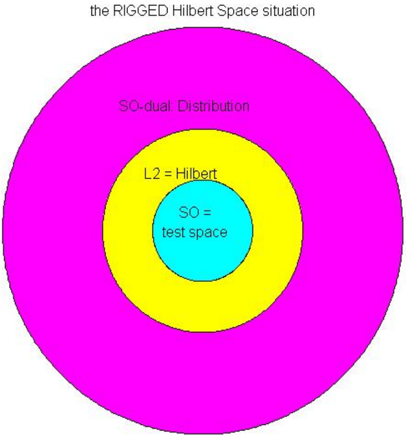
Function spaces are typically infinite-dimensional, therefore we are interested to allow convergent series. In order to check on them we need norms and completeness (in the metric sense), i.e. Banach spaces!

The classical function space norms are

- $\|f\|_\infty := \sup_{t\in\mathbb{R}^d} |f(t)|$;
- $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$;
- $\|f\|_2 := (\int_{\mathbb{R}^d} |f(x)|^2 dx)^{1/2}$;
- $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} := \sup_{\|f\|_\infty \leq 1} |\mu(f)|$, or $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} = \int_{\mathbb{R}^d} 1 d|\mu|$.
- $\|h\|_{\mathcal{F}\boldsymbol{L}^1} = \|f\|_1$, for $h = \hat{f}$.

A schematic description: the simplified setting

Testfunctions ⊂ Hilbert space ⊂ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

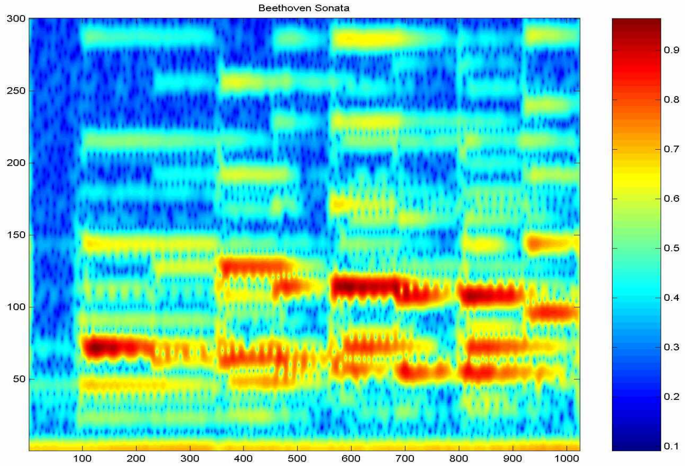
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

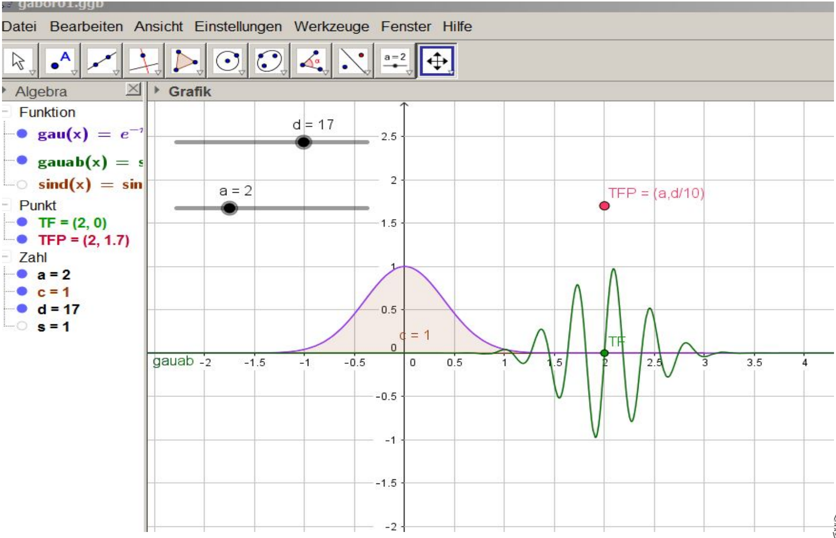
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2 = 1$ makes $f \mapsto V_g(f)$ isometric, hence we request this from now on. Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^*: \mathcal{H}_2 \to \mathcal{H}_1$, simply because $\forall h \in \mathcal{H}_1$

$$\langle h, h\rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!)\, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th\rangle_{\mathcal{H}_2} = \langle h, T^*Th\rangle_{\mathcal{H}_1}, \tag{9}$$

and thus by the *polarization principle* $T^*T = Id$.

In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \tag{10}$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \tag{11}$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h \rangle = \langle V_g^*(V_g(f)), h \rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)} \, d\lambda. \quad (12)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda) g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle \, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (13)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle \, \chi_s \, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (14)$$

with $\chi_s(t) = exp(2\pi i \langle s, t \rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

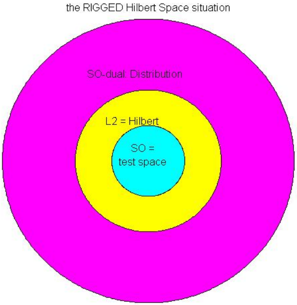
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$ for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
Moreover $\mathcal{F}$ is *uniquely determined* (as BGTr isomorphism) by the property that $\widehat{\chi_s} = \delta_s$.

A Banach Space of Test Functions (Fei 1979)

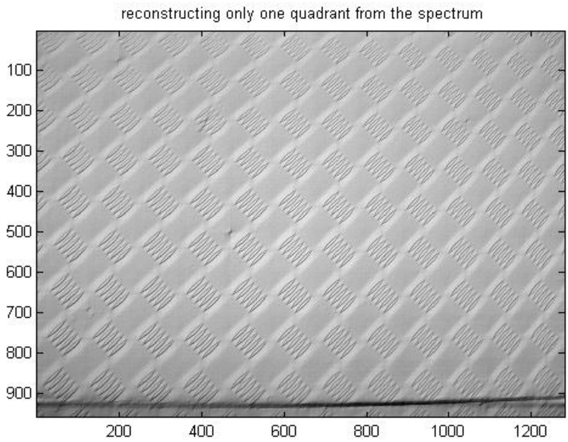
A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

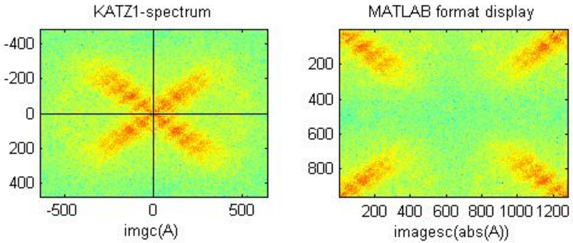
The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

The audio-engineer's work: Gabor multipliers

2D-Gabor Transform



Ordinary FFT2 of test image



Space Variant Blurring Operators

Ordinary convolution can be turned into multiplication operators on the Fourier transform side. Consequently an invertible convolution operator can be inverted by carrying out a division on the Fourier domain side (assuming that the pointwise inverse of the transfer function is well known and behaves nicely).
Now assume that we have a space-variant blurring function (or a *variable point-spread function*). But even if one has a perfect description it is not at all trivial to invert such an operator. As a first approximation one may view such an operator as Gabor multiplier with hopefully non-zero weights. Then one can hope that for regular weights one has good control of the difference between the inverse of such a Gabor multiplier and the Gabor multiplier with inverse symbol (bad and good audio engineer!).

Frame and Riesz Basis

I do not want to give a formal definition of Frames or Riesz bases here, a good source is the book(s) of Ole Christensen.
One should think of **frames** as the correct generalization (to the setting of infinite dimensional Hilbert spaces) of the concept of *generating systems* (plus some stability). Each frame has a (canonical, and many non-canonical) dual frames.
In Linear Algebra frames appear (under a different name) as matrices whose columns span the target space, i.e. $\mathbb{C}^m$ or $\mathbb{R}^m$. Since for each $\mathbf{x} \in \mathbb{C}^m$ one has many representations on chooses the minimal norm representation. The (row-)vectors generating the corresponding coefficients are those which appear in the solution of the MNLSQ-problem (get those coefficients which have *minimal Euclidean norm*). This is closely related to the *Moore-Penrose inverse* of a (rectangular) $\boldsymbol{A}$ of full rank, or in MATLAB notation $pinv(\boldsymbol{A}') = pinv(\boldsymbol{A})'$.

Frame Properties

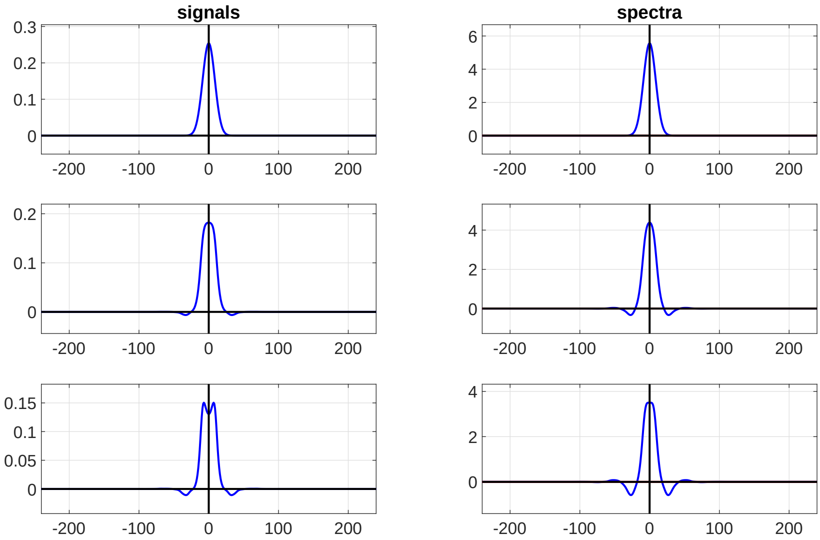
Frames are countable families in a Hilbert space which have the nice property that any $\mathbf{x} \in \mathcal{H}$ (e.g. $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$) can be represented in a stable way, i.e. as unconditionally convergent series with coefficients in ℓ^2 (and control of the norm).
Specifically so-called **tight frames** $(h_i)_{i\in I}$ have the following two properties which make them look like ONBs in $\mathcal{H}$:

1. $\|h\|_{\mathcal{H}} = \sqrt{\sum_{i\in I} |\langle h, h_i\rangle|^2}, \quad h \in \mathcal{H};$
2. $h = \sum_{i\in I} \langle h, h_i\rangle |h_i, \quad h \in \mathcal{H}.$

We will be mostly interested in tight Gabor frames in order to build *Gabor multipliers*.

Ole Christensen: An Introduction to Frames and Riesz Bases, 2016, in Book Series: Applied and Numerical Harmonic Analysis, Birkhäuser,

Dual Gabor Atoms, Tight Gabor atoms



Gabor vs. Wavelet Theory and Fourier Analysis

When comparing Gabor Analysis with Wavelet Theory or Fourier Analysis we observe the following differences and similarities:

- Wavelet theory and Gabor Analysis are *localized*, unlike the Fourier Analysis (pure frequences last forever);
- Wavelet systems can be orthogonal AND *nice*, i.e. smooth and well decaying. The Balian-Low Theorem tell us that this is *impossible* in the Gabor case, so we have to live with *redundancy* and non-orthogonality;
- Still, regular Gabor Analysis, in particular tight Gabor families (which work almost like ONBs) can be constructed in a numerically efficient way.

Some Challenges in Time-Frequency Analysis

Whereas the *regular Gabor Analysis* is well established, and we know how to find e.g. the best approximation (in the Frobenius norm, i.e. with respect to the Euclidean structure if $\mathbb{C}^{n^2}$) of a given matrix by a Gabor multiplier the question of describing *optimality*, we see still an extensive collection of open question, also with respect to numerically efficient realization of theoretical ideas.

Recent work of *Markus Faulhuber* indicates that at least for (generalized) Gaussian windows this is related to (sphere) packing problems, based on invariance properties (using symplectic and metaplectic groups), see [1],[3], and [2]. For $1D$-signals and ordinary (Fourier invariant) Gaussians appear to be optimal.

Irregular Gabor families

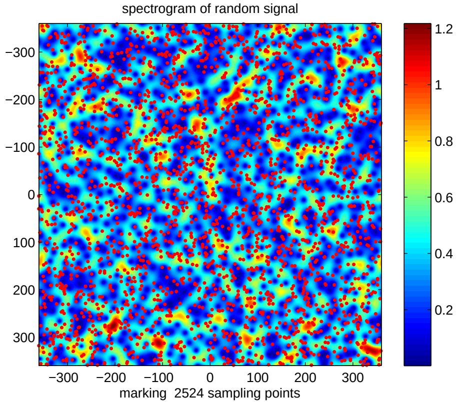


Figure: gabirrs1.pdf

Irregular Gabor families

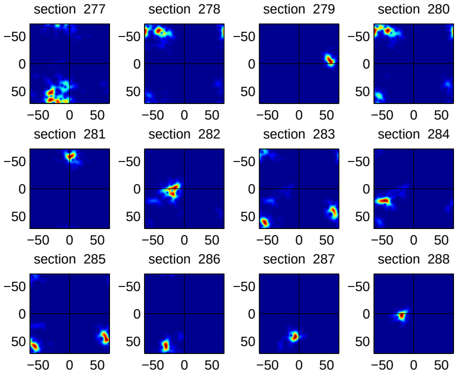


Figure: gabirrlast1.pdf

This picture shows that the dual frame for an irregular Gabor

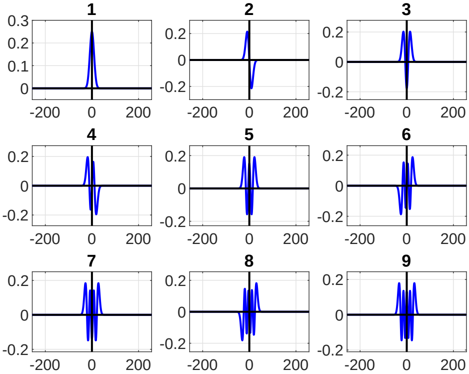


Figure: hermit9bold.eps

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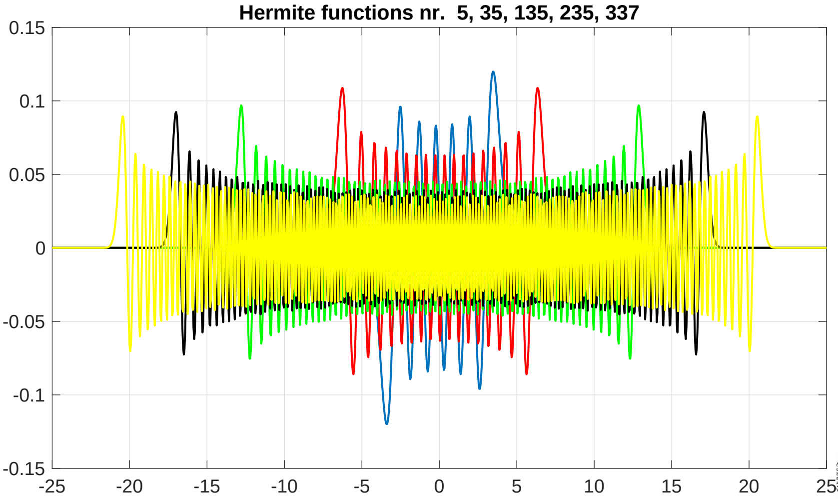

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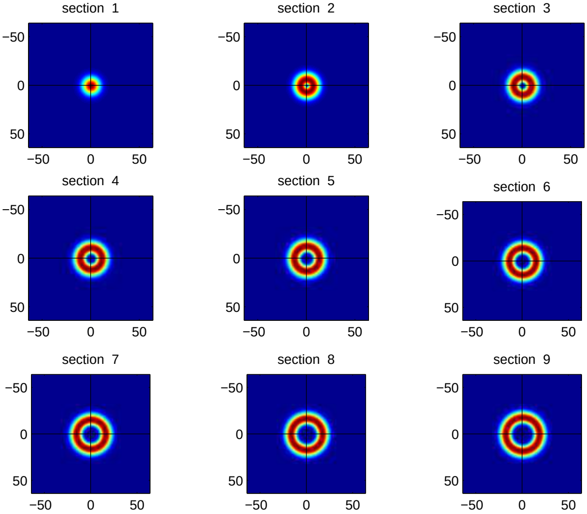

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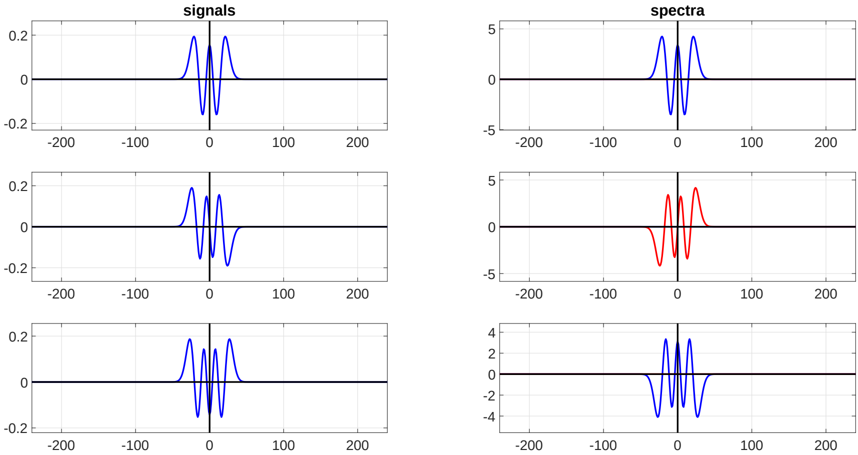

Figure: herm5to7A.eps

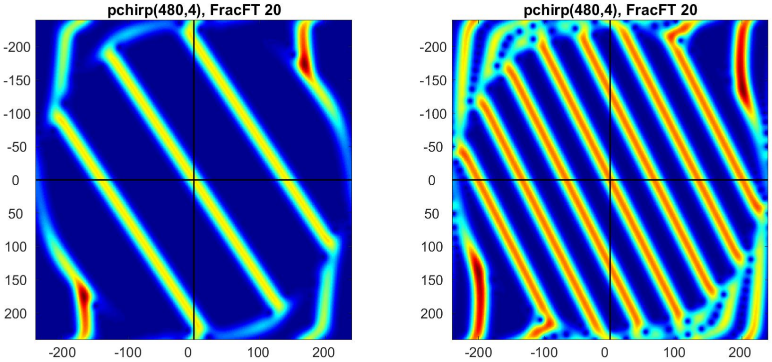

Figure: FracFTchirp01b.jpg

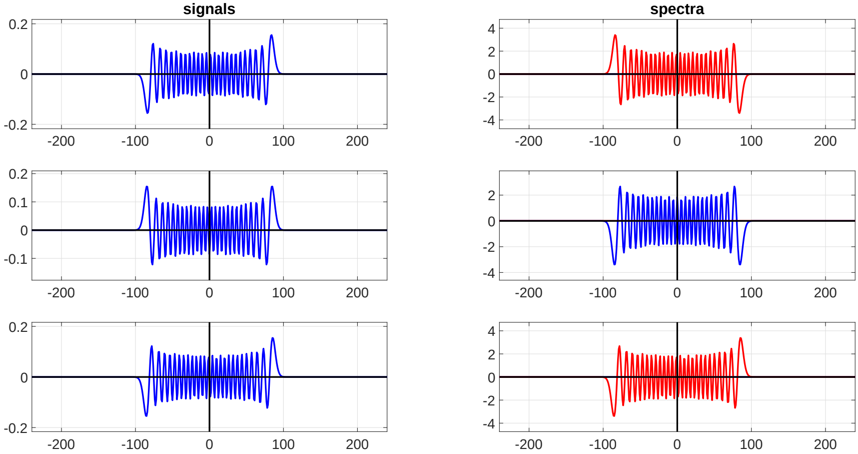

Figure: herm50to52.eps

Publications related to this context

K. Gröchenigs book [9] is a standard book on time-frequency analysis. The *Segal algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is described there as $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, the modulation space corresponding to the parameters $p = 1 = q$.

Feichtinger, H.G.: Choosing Function Spaces in Harmonic Analysis, in "Excursions in harmonic analysis. Vol. 4. [4]: Features some ideas concerning construction principles of function spaces.

Hans G. Feichtinger: Gabor expansions of signals: computational aspects and open questions in "Landscapes of Time-Frequency Analysis", Birkhäuser/Springer, Vol.ATFA17 (2018) p.1-31. [6] points to the role of Numerical Gabor Analysis.

Feichtinger, H. G.: Thoughts on Numerical and Conceptual Harmonic Analysis [5] describes ideas how finite discrete groups **can approximate the continuous case**.

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In Pammy Machanda et al., editor, *ISIAM Proceedings*, pages 1–42, 2019.

K. Gröchenig.
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Encore: last slides of Jarnik Lecture

A few relevant references

K. Gröchenig: Foudations of Time-Frequency Analysis, Birkhäuser, 2001.

H.G. Feichtinger and T. Strohmer: Gabor Analysis, Birkhäuser, 1998.

H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, Birkhäuser, 2003.

G. Folland: Harmonic Analysis in Phase Space. Princeton University Press, 1989.

I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.

Some further books in the field are in preparation, e.g. on modulation spaces and pseudo-differential operators.

See also www.nuhag.eu/talks.

Added in proof! last night

Time-Frequency Analysis and Black Holes

Breaking News

Today, Oct. 3rd, 2017, the **Nobel Prize in Physics** was awarded to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!

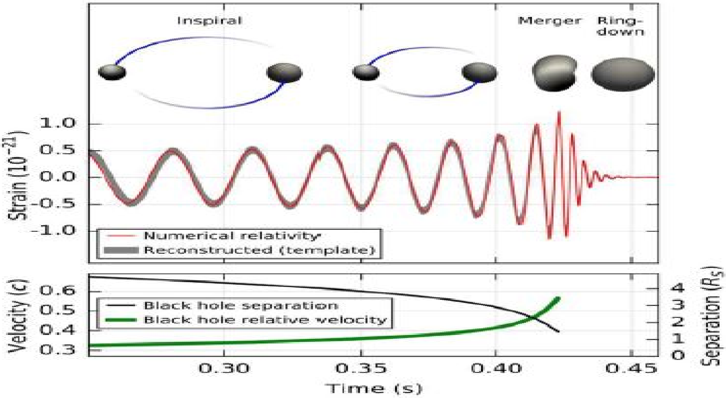
The Prize-Winners are

Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".

There had been two strategies:

- Searching for 2500 explicitely determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.

NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).

THANK YOU

Thank you for your attention

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