Two dimensional Gabor Analysis: Numerical Challenges and Applications

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OFFICIAL ABSTRACT I

The natural setting for time-frequency analysis and Gabor Analysis is the world of functions/signals or distributions over LCA (locally compact Abelian) groups G. A substantial part if the theory (the existence of a Janssen representation for the frame operator, etc.) has been developed in this context, making use of appropriate function spaces, in particular the Banach Gelfand Triple based on the Segal algebra $SO(G)$. In this setting only occasionally a distinction is made between the one-dimensional or the multidimensional (e.g. Euclidean) setting.
In contrast, when it comes to implementation the situation changes. Not because Gabor analysis wasn't interesting for the multi-dimensional setting. In contrast, the first important papers in the field made the connection between Gabor expansions of images and the analogy with the visual system of humans.

OFFICIAL ABSTRACT II

But in terms of available code the situation is rather satisfactory for the case of 1D-signals: one can compute dual or tight Gabor atoms, construct Gabor multipliers, and has cheap algorithms for a cheap (and efficient) determination of approximate versions of these objects.

The talk will discuss the obstacles and additional problem which arise from the large dimensions (the number of pixels of the involved images), the computational costs and above all the huge storage requirements. One possible way out (or at least a special family of Gabor expansions and Gabor multipliers which can be realized) is the use of separable Gabor families. We will also try to explain the relevance of the double preconditioning approach in the 2D-setting.

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Complete discrete 2-D Gabor transforms by neural networks for image analysis and compression.
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The generalized Gabor scheme of image representation in biological and machine vision.
IEEE Tras. Pattern Anal. Mach. Intell., 10(4):452–468, Jul 1988.

I. Gertner and Y. Y. Zeevi.
Image representation with position-frequency localization.
Acoustics, Speech, and Signal Processing, 1991. ICASSP-91., International Conference on, 4:2353–2356, April 1991.

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Comments on 'Complete discrete 2-D Gabor transforms by neural networks for image analysis and compression'.
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R. Braithwaite and M. Beddoes.
Iterative methods for solving the Gabor expansion: considerations of convergence.
IEEE Trans. Image Process., 1(2):243–244, Apr 1992.

What are the Challenges?

We have quite a list of challenges (of different nature and different complexity) making Gabor Analysis for the $2D$ (and hence even more for the $3D$ and higher dimensional) setting quite challenging.

1. Phase space is now 4-dimensional, hence difficult to visualize, and of course we suffer from the curse of dimensionality (data structures);
2. We can handle the separalbe case relatively well, but we would like to understand better the non-separable case (!order of variables, etc.)
3. For the irregular case we cannot store anymore the full dual frame, leaving aside that it takes a long time to compute;
4. Even in the regular case one has memory constraints, approximate dual Gabor atoms might be a way out, etc.

SOME MATLAB CODE

```
GABFLT2.M hgfei , 22.01.2019, Prague
INPUT: PIC = input picture, perc: default perc = 1
USAGE:
[PICFLT,GFLGPICG,GFLT,G1,G2,GD1,GD2]=gabflt2(PIC);
[PICFLT,GFLGPICG,GFLT,G1,G2,GD1,GD2]=
                          = gabflt2(PIC,perc,G1,G2);
[PICFLT,GFLGPICG,GFLT,G1,G2,GD1,GD2] =
                                 = gabflt2(PIC,perc);
```

Classical References

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A group-theoretical approach to Gabor analysis.
Opt. Eng., 34:1697–1704, 1995.

H. G. Feichtinger, W. Kozek, P. Prinz, and T. Strohmer.
On multidimensional non-separable Gabor expansions.
In *Proc. SPIE: Wavelet Applications in Signal and Image Processing IV*, August 1996.

S. Paukner.
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Master's thesis, 2007.

O. Christensen, H. G. Feichtinger, and S. Paukner.
Gabor analysis for imaging.
In *Handbook of mathematical methods in imaging. In 3 volumes*, pages 1717–1757. New York, NY: Springer, 2nd edition edition, 2015.

The dual group and Pontrjagin

Abstractly speaking one has a natural Fourier transform for any LCA group G.

In the setting of **finite Abelian groups** or order $\# = N$ it is easy to find out that there is a collection of N vectors which are eigenvectors to the whole family of translation operators acting on $\mathbb{C}^N = \ell^2(G)$. Each of them takes values in the torus (actually within the group $\mathbb{Z}_N$ of unit roots of order N), and are called the *characters* of G.

They constitute a group, the so-called dual group $\widehat{G}$. Clearly this group (with *pointwise multiplication*) is of the same order N.

There is a natural embedding (hence isomorphism) from G into $\widehat{\widehat{G}}$ (see Pontrjagin for the general LCA case).

Recall who the FFT2 works

It is a good excercise for students to check that one has a natural identification of $\widehat{G_1 \times G_2}$ with $\widehat{G_1} \times \widehat{G_2}$.

This is the basis for the FFT algorithm for images, named `FFT2` in MATLAB[1].

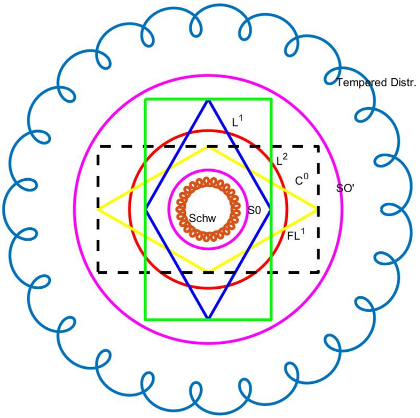
As it is well known the `FFT2`-command realizes the FFT in a row and then in a columnwise manner. The order does not matter, as we easily illustrate for the case of a square image $\boldsymbol{A}$ of size $N \times N$:

$$fft2(A) = F * (\boldsymbol{A} * F) = (F * \boldsymbol{A}) * F, \tag{1}$$

making use of the fact that the Fourier matrix is a symmetric (in the real sense, i.e. $F = F^t$)!

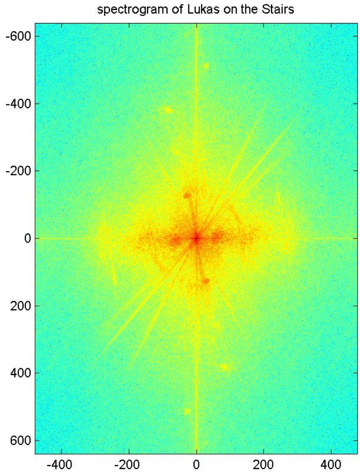
[1] Recall that `fft(A)` for a matrix is just the FFT applied to the collection of column vectors of the matrix A. Hence $F = fft(eye(N))$ is gives the matrix which realizing the FT: `fft(x) = F * x`.

Summarizing the landscape of spaces used

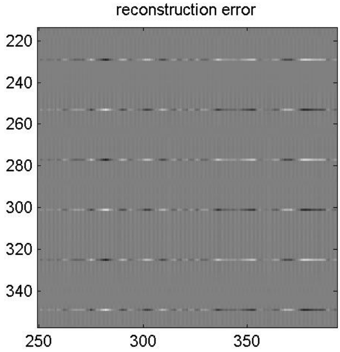


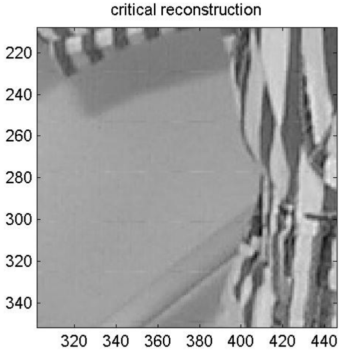
Test-Image with Stripes

FFT2 Spectrum of Test-Image

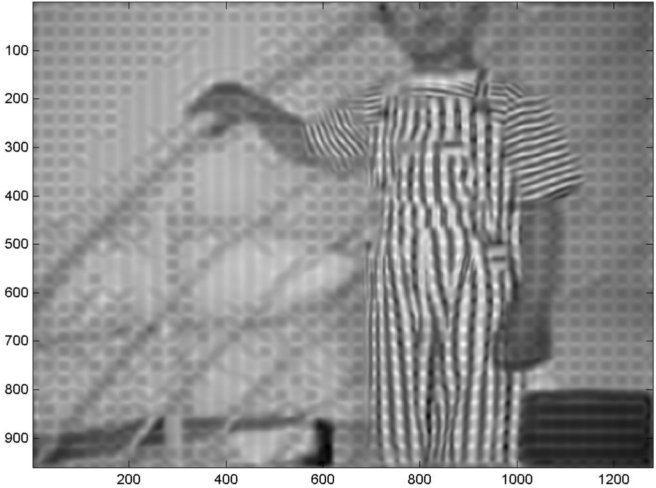


Critical Gabor Case for Images

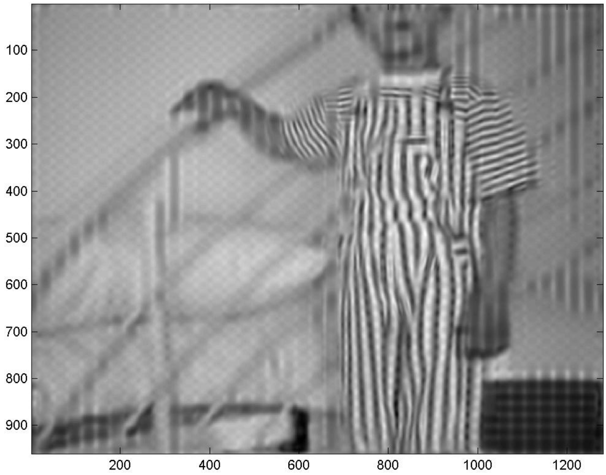




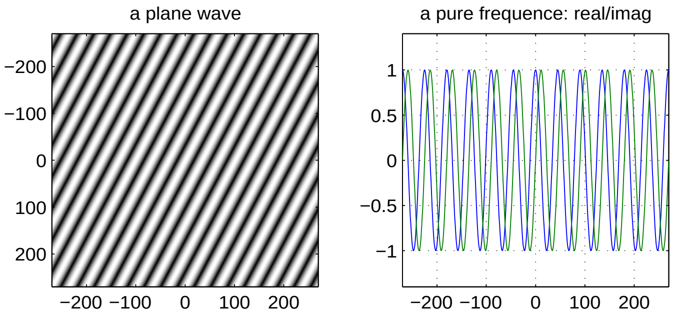
Compression using Gabor Expansions, 5%



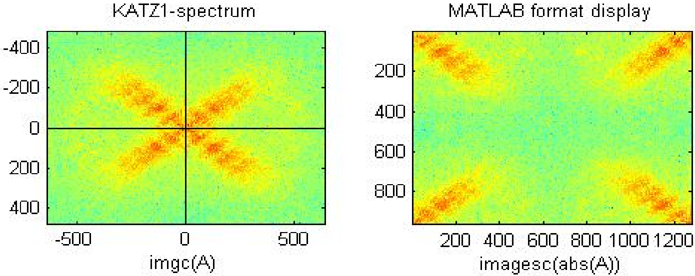
Compression using Gabor Expansions, 25%



2D-Gabor Transform: Plane Waves

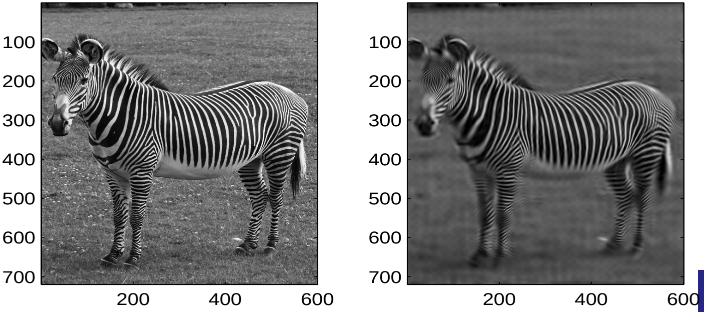


2D-Gabor Transform: Test-Images 2

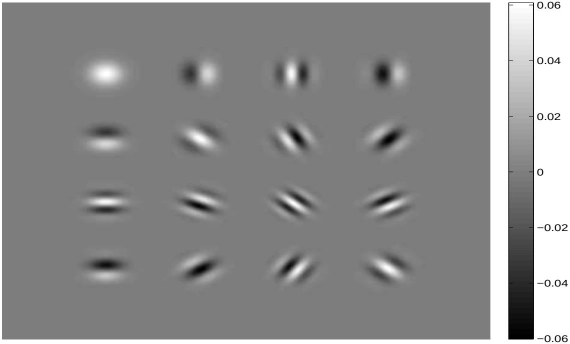


2D-Gabor Analysis: Test Images

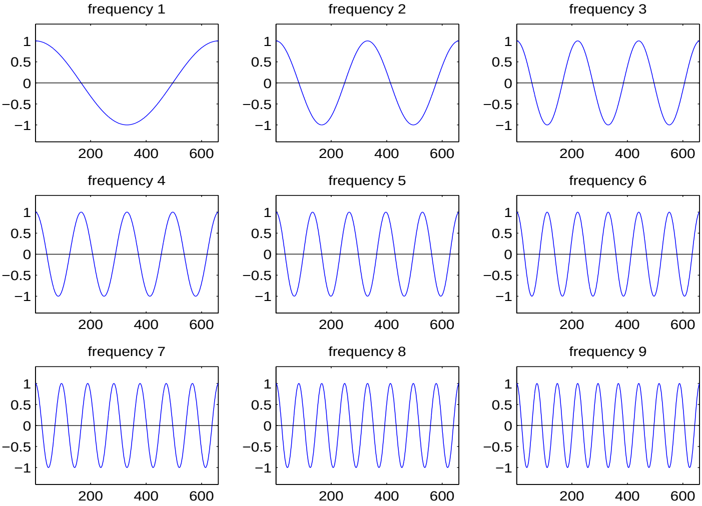
Image Compression: a Test Image



Showing the Elementary 2D-Building Blocks



Building blocks for Discrete Cosine Transform DCT

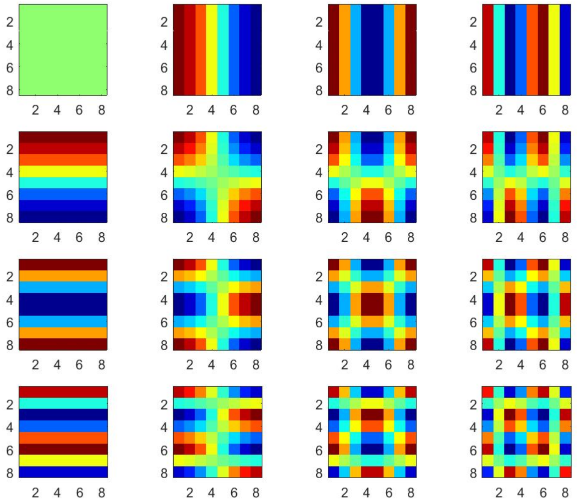


The JPEG compression

The widely used JPEG standard, established by the "Joint Photographic Experts Group" is based on the discrete cosine transform, a real version of the Fourier transform (real images give real coefficients).

- First a general image is decomposed into blocks of 8×8 pixels, (each of them in fact in the range of 0 to $255 = 2^8 - 1$, so one Byte or 8 Bits worth);
- Then depending on the chosen compression rate a fixed number of coefficients, from upper left to lower right corner (figure below) is stored and transmitted;
- Resynthesis from this set of coefficients provides the decoded image.

The building blocks for the Discrete Cosine Transform DCT2



https://en.wikipedia.org/wiki/JPEG with video demo!

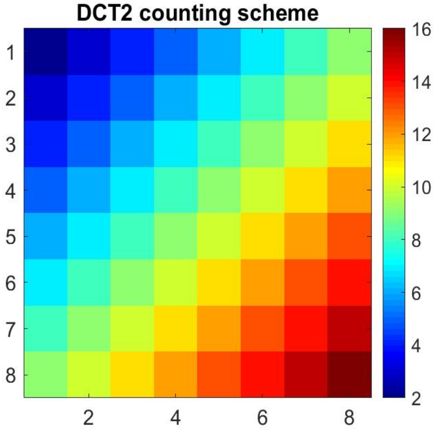


Figure: DCT2count.jpg

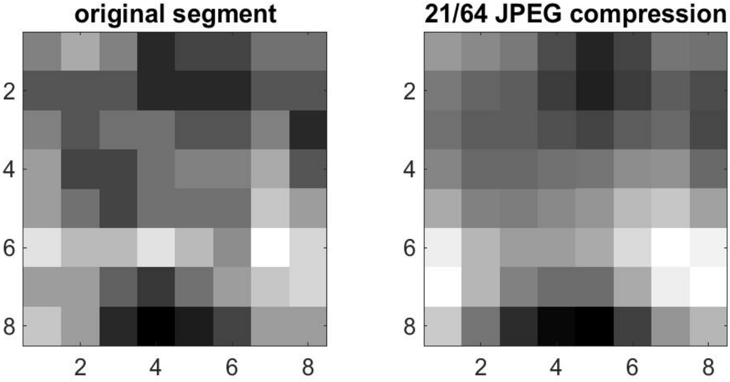


Figure: JPEG2164compr1.jpg

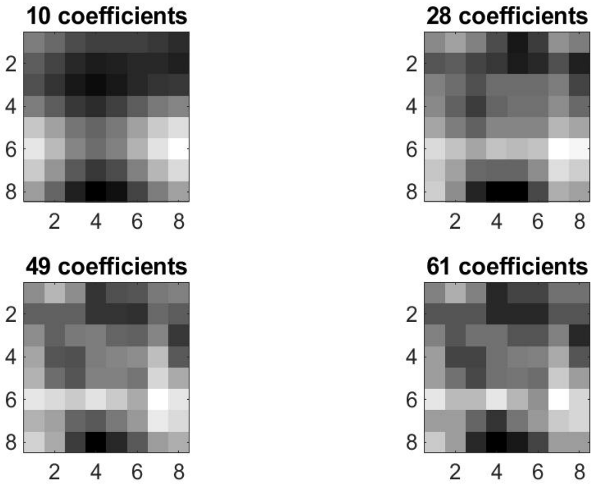


Figure: JPEG2164compr2.jpg

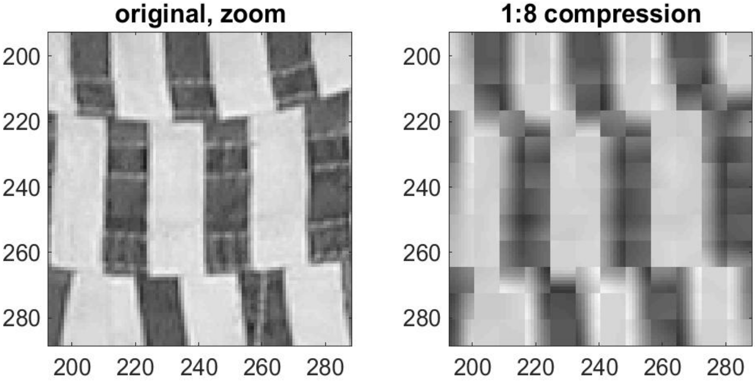


Figure: dctcompr18a.jpg

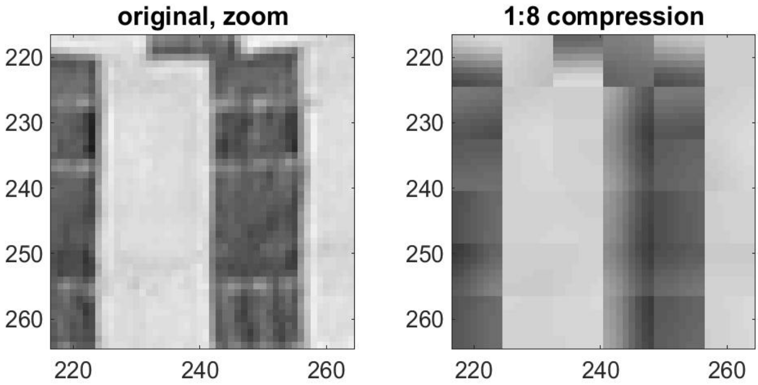


Figure: dctcompr18b.jpg

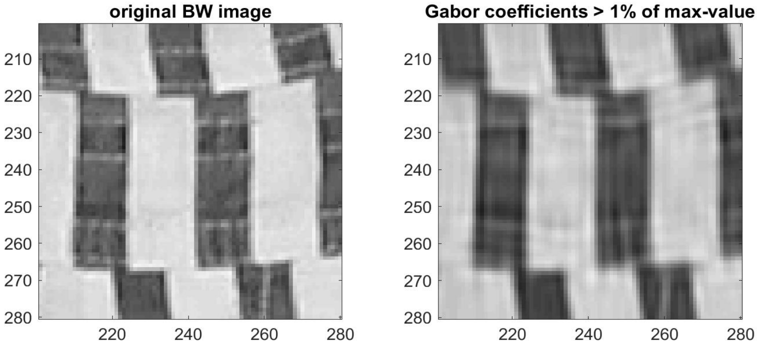


Figure: gabcompr18b.jpg

Further Variations of the Theme

- Variable atoms, or one out of K windows, randomly; (uniform estimates for good enough lattices);
- Slowly changing lattices and atoms, deformations
- Approximation by separable ones, with the chance of having better criteria or computing approximate dual Gabor atoms.

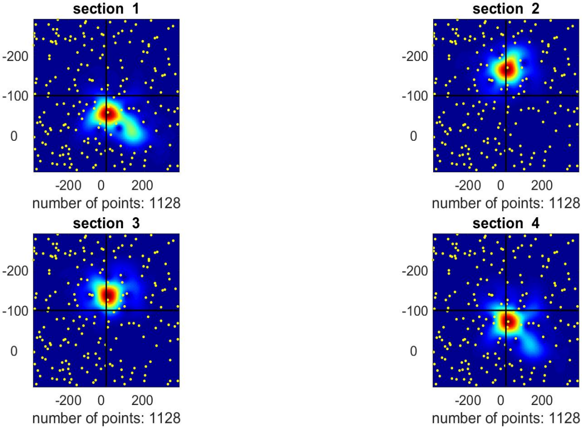


Figure: gabirrdu01.jpg

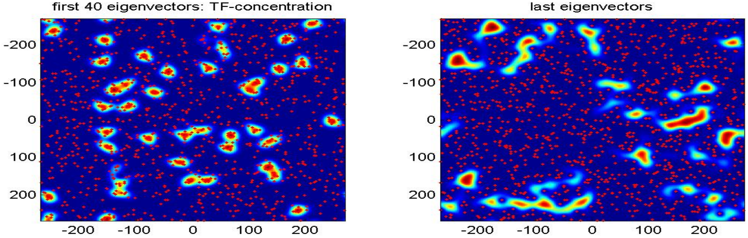


Figure: gabirrlocW.jpg

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