Numerical Harmonic Analysis Group

Coherent States, Gabor multipliers and the Banach Gelfand Triple

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Memory of Prof. Ahmed Intissar

On the style of my presentation

This slide is mostly for those who do not listen to my talk given during the conference.
I will say a few words about the situation of science in general, and mathematical analysis and physics or engineering specifically.
Recall that our academic system requires to publish new results in international journals, present our result at conferences. It is important, that the system is self-governing, and that there is no scientific goal, given my politics or so.
As a consequence we spend more time in developing sophisticated concepts than communicating on open problems which are of relevance outside our community of specialists or related fields.
Hence I will try to share a few ideas which one could describe as "driven by applications".

Mathematicians, Physicists, and Engineers

Fourier Analysis is a good meeting ground for mathematicians and physicists, and both communities make a lot of use of Fourier Analytic methods. The same is true for engineers, mostly when it comes to communication theory.
However, the approaches taken are quite different, and one can have long discussions about the different attitudes of these different communities.
Buzzwords that come to mind are mathematical rigor, modelling of natural phenomena or practical applicability.
Divergent Integrals do not bother physicists, if they allow to support a heuristic argument. Constructive Realiziation of certain linear mappings to not bother mathematicians, while engineers do not give much thought to the Transition between the Continuous and Discrete Domains.

The different Ages of Fourier Analysis

Fourier Analysis is soon (first paper of J. Fourier published in 1822) reach its 200 anniversary. We have the following big steps:

1. Fourier series, what are functions/integrals (19th century);
2. Fourier transforms, Lebesgue spaces $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$
3. Functional analytic methods (Hilbert/Banach spaces)
4. Abstract Harmonic Analysis (LCA groups)
5. Theory of distributions (L. Schwartz)
   with important applications to PDE (L. Hörmander)
6. Gelfand transforms and Gelfand triples
7. Time frequency analysis and Gabor analysis
8. also wavelet theory (ca. 35 years now).

Fourier Analysis in our Daily Life

Although Fourier Analysis appears to be a well established subject (at least within mathematics), where it seems clear how one should present the subject to students[1], this mathematical approach appears to be almost disconnected to the actual use of Fourier analysis in our daily life (a subject that is open for outreach activities of mathematics):

- Mobile phones
- MP3, WAV-files
- JPG images
- noise cancelling headphones ...

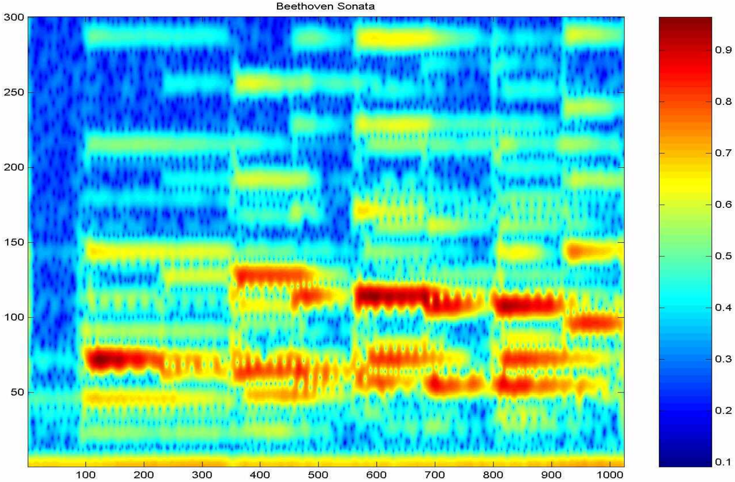
This has partially to do with the fact that real world signals are not periodic, nor well decaying, nor even pointwise well defined.

[1] Starting with Lebesgue integration, maybe reaching the FFT or distribution theory and Sobolev spaces

A Demo of a Spectrogram

Let us first take a look on a few spectrograms, as one can produce them using the STX program (from the ARI web-page, the Acoustic Research Institute of the Austrian Academy of Sciences, under Peter Balazs).
We take some piece of audio (the signal to be analyzed) and look at the spectrogram, or the STFT (Short time or Sliding Window Fourier transform).

Beethoven Spectrogram (Piano Sonata)



Morocco Chaabi Music

Morocco Chaabi Music

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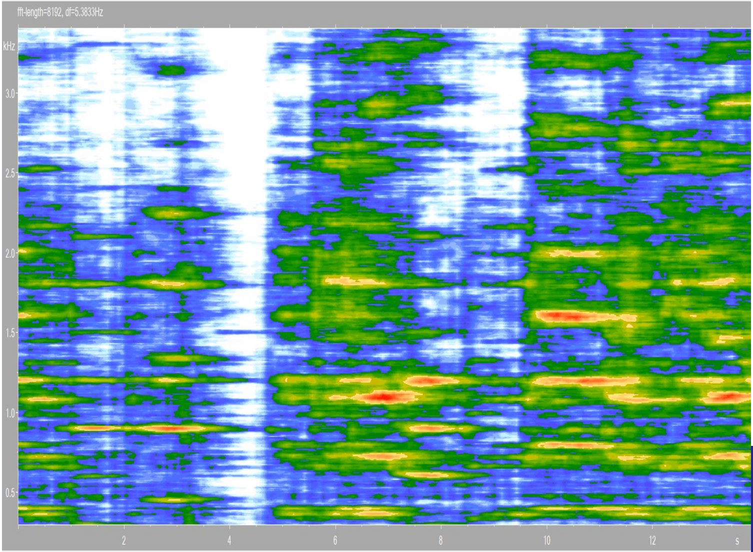
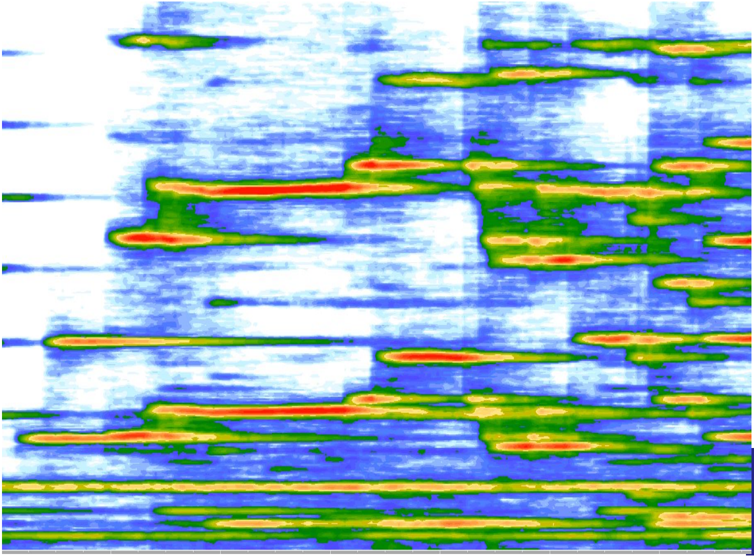
Morocco Chaabi Music



Morocco: Flute Music



A Maroccanian Sound Example

... if it can be played if *button* (below) is pressed...

marakflute.m4a

The spectrogram displayed is realized with the help of the STX program, which is downloadable from ARI homepage (Acoustic Research Institute, Austrian Academy of Sciences, head: Peter Balazs).

See also `www.gaborator.com`

Why are Gabor Multipliers useful?

Why are spectrograms so useful?
One of the important features of the STFT (or the Gabor expansion, which is just a sufficiently dense regular sampling analogue of the STFT) is the fact that the coefficients have a natural interpretation as the level of energy in the given "signal". It is quite comparable to a (blurred, graphical) score obtained from the given piece of music or audio-signal.
Hence it is natural to interpret certain disturbances as "noise" which one would like to *remove*. Here comes in the question how one could restore the signal from the spectrogram (other the original one, or the sampled one, or the modified one), i.e. the invertibility of the STFT (or Gabor coefficient) mapping.

The audio-engineer's work: Gabor multipliers

Gabor Multipliers: Motivation

Let us again look at this scenario in a mathematical way:

1. Each slider represents a particular *frequency range*;
2. The *position of each slider* then represents the amplification or damping of that frequency range;
3. A given (fixed) profile of the sliders describes roughly the *transfer function* of a time-invariant filter;
4. The positions of the sliders, labeled by the points of a uniform time-grid and their frequency bins, represent the *coefficients describing a Gabor multiplier*.

Thus a Gabor multiplier contains the description, in which way the audio-engineer is influencing the high, or low or medium range frequencies, at different times. So he may perform time-variant filtering. For 2D we talk about space-variant blurring.

Gabor Multipliers: Natural Questions

There are a few natural questions arising in this context:

1. What is the structure of the linear space of all Gabor multipliers?
2. When is a Gabor multiplier invertible, or a Hilbert-Schmidt operator?
3. Can one determine the best approximation of a HS-operator by a HS-Gabor-Multiplier?
4. Which operators can be well approximated or even represented as Gabor multipliers?
5. How can one invert a Gabor multiplier?
6. How do Gabor multipliers (or STFT multipliers, so-called *Anti-Wick operators* depend on the choice of parameters, the window used or the lattice in phase space $\mathbb{R} \times \widehat{\mathbb{R}}$)?

The Heuristic Approach

If we want to analyse "general signals", including any pure frequency, Dirac measures, but of course also any of the functions in any of the spaces $\boldsymbol{L}^p(\mathbb{R}^d)$ we should ensure that the short-time Fourier transform is a bounded function.
In fact, for any tempered distribution $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ one can define the STFT via

$$V_{g_0}(\sigma)(\lambda) = \sigma(\pi(\lambda)g_0), \qquad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

We will be interested in the subspace of all *mild distributions* arising as the subspace of tempered distributions[2] which have bounded spectrogram (STFT).
As we will point out this space can be introduced directly as the dual space of a relatively simple case, called the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

[2] we will finally do it without the theory of Schwartz!

Foundations of Time-Frequency Analysis

In *WORDS* (and for those who have already a vague idea what it could be) time-frequency analysis is the part of harmonic analysis resp. mathematical analysis at large, which makes use of both time and frequency variables, and thus makes use of the family of TF-shifts (first we apply a time-shift T_t and then a frequency shift or *modulation* operator M_s (a shift-operator on the frequency side. Naturally the Fourier transform plays an important role, since we have the crucial connection

$$M_s = \mathcal{F}^{-1} T_s \mathcal{F}, \quad s \in \mathbb{R}^d.$$

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \; \lambda = (t, \omega).$$

Inversion Theorem

The following inversion formula is known for the STFT. If $g, \gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ and $\langle \gamma, g \rangle \neq 0$ then for all $f \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$f = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma \, d\omega \, dx, \tag{1}$$

where the equality is understood in a vector-valued weak sense (see Gröchenig [7, p.44]). Moreover, if $K_n \subset \mathbb{R}^{2d}$ $(n \geq 1)$ is a nested exhausting sequence of compact sets and

$$f_n = \frac{1}{\langle \gamma, g \rangle} \int_{K_n} V_g f(x, \omega) M_\omega T_x \gamma \, d\omega \, dx$$

then $f_n \to f$ in $\boldsymbol{L}^2(\mathbb{R}^d)$ norm.

Coherent Expansions

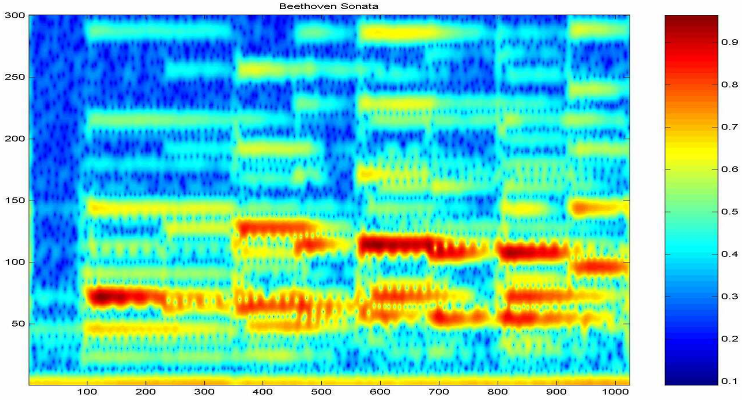
We have not time to go into the specifics of the coherent states representation which arises when one takes the Gauss function $g_0(t) = e^{-\pi t^2}$ as the moving *window*.

This choice can be motivated by the optimal concentration of this function in the TF-plane resp. *phase space* resp. the complex plane.
The range of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ under the (unitary) mapping $f \mapsto V_{g_0}(f)$ is known as the *Fock space*, a reproducing Hilbert space of analytic functions over the complex plane.

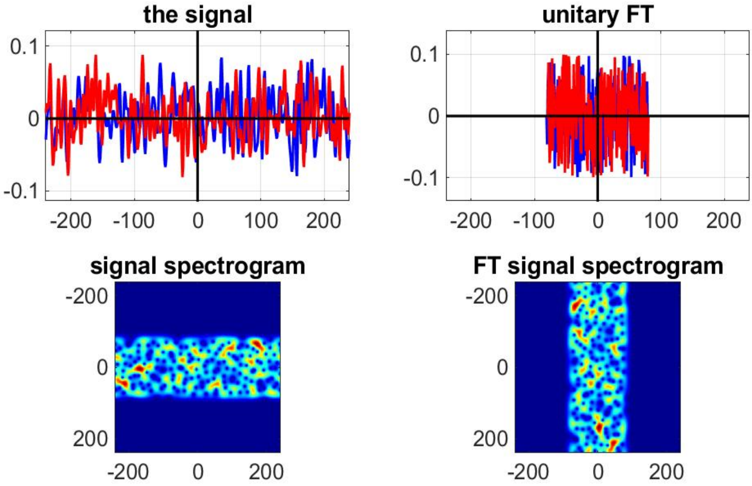
It is plausible that this has a lot of redundancy and since 1992 it is known that it is enough to know $V_{g_0}(f)$ over any lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$, with $ab < 1$.

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depicted using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



The effect of the Fourier Transform



Various Function Spaces

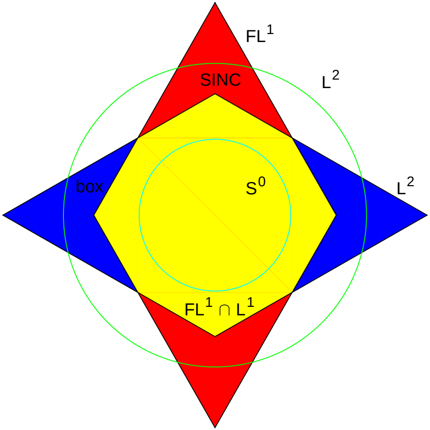


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

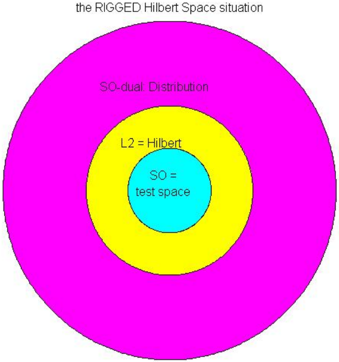
(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

A schematic description: the simplified setting

In our picture this simply means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \qquad (2)$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Atomic decompositions

One can characterize $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ as the smallest Banach space containing at least one non-zero Schwartz function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, which is isometrically invariant under TF-shifts (e.g. like $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$), in fact for any such $g \neq 0$ one has

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{f = \sum_{n\geq 1} c_n \pi(\lambda_n) g \mid \sum_{n\geq 1} |c_n| < \infty\}.$$

The natural inf-norm is then an equivalent norm on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ for any such g. Consequently the w^*-convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$ can be characterized via pointwise convergence (uniformly over compact regions) over compact subsets.

Different Perspectives

We can (and should) discuss Gabor Analysis from several different perspectives, because at the beginning of the development it was necessary to develop the necessary mathematical tools from different perspectives.
First let us recall, that (following Andre Weil, who of course never saw the problems of TF-analysis) Gabor analysis can be realized over any LCA (locally compact Abelian) group. The pure frequencies are a natural family of *characters*, i.e. functions of absolute value one, which form the *dual group* under pointwise multiplication.
So it can be carried out also over $\mathcal{G} = \mathbb{Z}_N$, the multiplicative group of unit roots of order N. Here the characters are simply the columns (or rows) of the DFT matrix, mapping $\mathbb{C}^N$ into $\mathbb{C}^N$ (up to scaling in a unitary way).

Prehistory of TF-Analysis

Before time-frequency has been established as a recognized (and meanwhile rather active field) within mathematical analysis methods in this direction have been used by people in digital audio or other applied scientists who wanted to analyze signals.
A significant contribution was D. Gabor's important paper [6] from 1946, where he suggested to use the normalized Gaussian as the window because it has optimal joint concentration in the TF-domain
He also suggested to use only TF-shifted such Gaussians along the integer (also called von-Neumann) lattice $\mathbb{Z}^2$.

Different Levels of Gabor Analysis

The best way to explain the essential feature of *Gabor Analysis* is to split the insight essentially into three steps:

1. The linear algebra view on Gabor Analysis;
2. The group theoretical background of GA;
3. The functional analytic tools relevant for GA.

Gabor Analysis and Linear Algebra I

At the *linear algebra* level we can view Gabor analysis over finite Abelian groups (such as $\mathbb{Z}_N$, the cyclic group of order N) as covering the following questions:

- When is a Gabor family generating $\mathbb{C}^N = \ell^2(\mathbb{Z}_N)$;
- When is a Gabor family linear independent;
- Is it possible to have an orthonormal Gaborian basis for $\mathbb{C}^N$?

Clearly a generating system has to have at least N elements, and a linear independent set cannot have more than N elements.

Gabor Analysis and Linear Algebra II

The SVD (Singular Value Decomposition) allows to describe these three situations in the following catalogue:

1. If a family of vectors forms a generating system for $\mathbb{C}^N$ then every vector $\mathbf{y} \in \mathbb{C}^N$ has a *minimal norm representation*. Putting such $M \geq N$ vectors into a $M \times N$-matrix $\boldsymbol{A}$ these coefficients can be obtained by taking scalar products[3] of the vector $\mathbf{y}$ against the M columns of the matrix $\mathrm{pinv}(\boldsymbol{A}') = \mathrm{pinv}(\boldsymbol{A})'$.
2. If $M \leq N$ vectors are linear independent one finds that the columns of $\mathrm{pinv}(\boldsymbol{A}')$ constitute the biorthogonal family to the given family of column vectors.

In a modern language we are speaking of the *dual frame* resp. the biorthogonal Riesz basic sequence.

[3] Here the prime indicates transpose conjugate of the matrix.

Gabor Analysis and Group Theory I

We will concentrate on *regular Gabor families*, i.e. Gabor families which arise in the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$, where $\pi(n,k) = M_k T_n, 0 \leq k, n \leq N-1$ and $\Lambda \lhd \mathbf{Z}_N \times \mathbf{Z}_N$ (i.e. additive subgroups of the finite phase space, which contains N^2 elements).

Typically Λ is a subgroup of the form $\mathbb{Z}_{N/a} \times \mathbb{Z}_{N/b}$, where a, b are both divisors of N, the so-called time-step a and the frequency-step b, which has obviously $N/a \cdot N/b = N^2/(ab)$ elements, which is compared to the dimension N of the signal space, the redundancy factor $\mathrm{red} = N/(ab)$.

Thus $\mathrm{red} > 1$ represents the case of *oversampling*, where a (linear dependent) family of Gabor atoms hopefully constitutes a Gabor frame, while for $\mathrm{red} < 1$ the expectation is to have a linear independent family. The case $\mathrm{red} = 1$ is also called the *critical case*, despite the chance to have a *basis* (only) in that case.

Gabor Analysis and Group Theory II

Group theory comes enters the scene in several ways.
First of all one can ask, whether the (canonical) dual of a Gabor frame is again a Gabor frame. The answer is a clear "YES" for the regular case (Λ is a subgroup of phase space).
The same is true for the Riesz basic sequence case.
Moreover, there is a very interesting and *unique* (to time-frequency analysis) duality principle, usually referred to A.Ron and Z.Shen (going back to the observation of Wexler and Raz, two engineers): The generator of a biorthogonal sequence of a (sparse) Gabor family is up to scaling the same as the generator of the dual frame of the corresponding (oversampled) *adjoint lattice*.

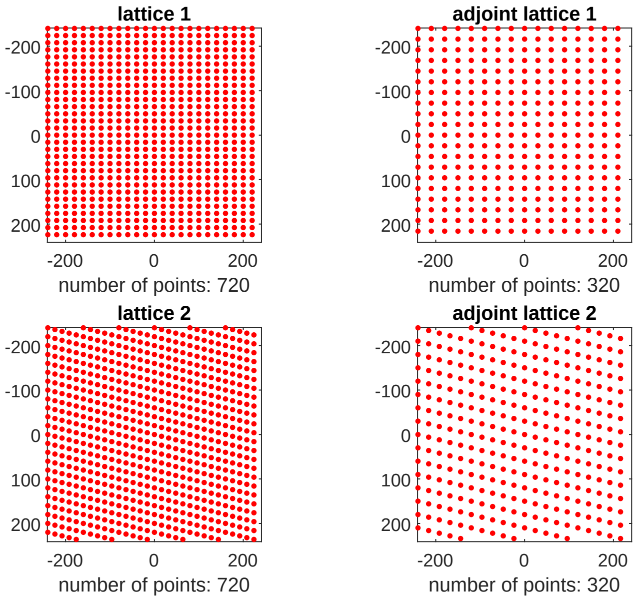


Figure: compadilat2f

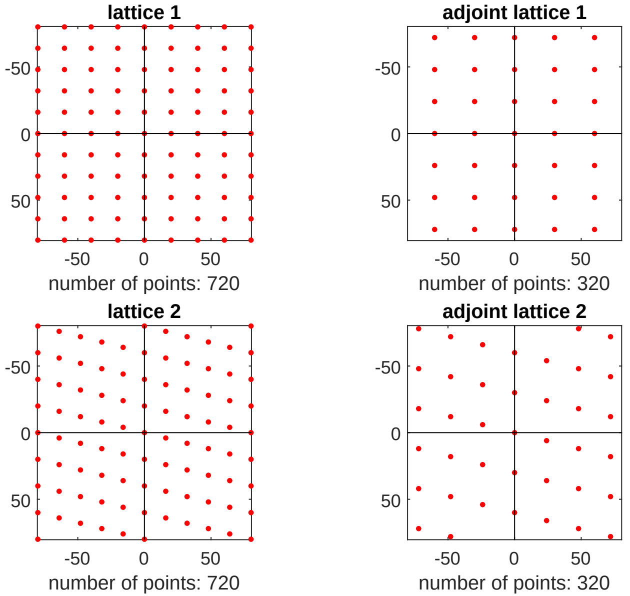
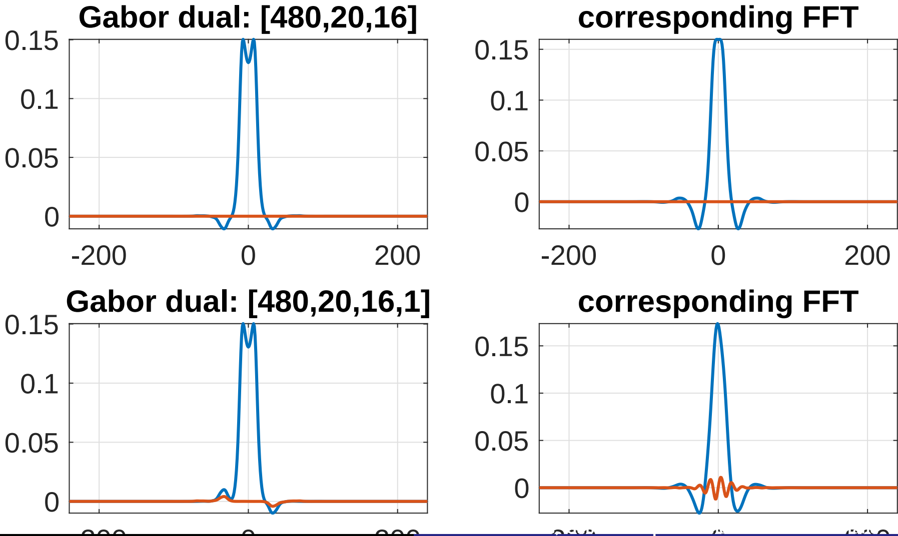
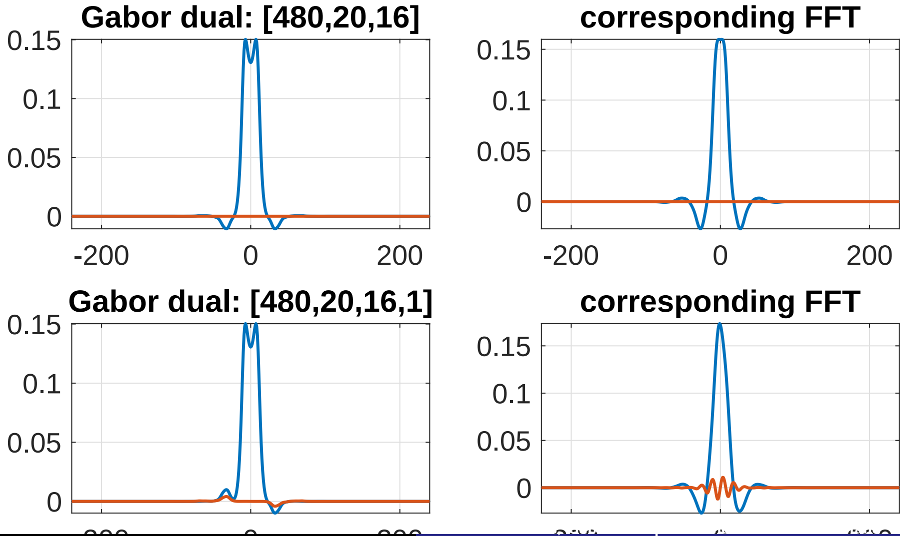


Figure: compadilat2z.pdf

Different Gabor lattices and adjoint lattices



Dual atoms for different Gabor lattices



Gabor Analysis and Functional Analysis

Finally we can discuss Gabor Analysis in the realm of continuous functions (say over $\mathbb{R}^d$, $d \geq 1$), for signals in the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, or more general in suitably chosen Banach spaces. Here we have a number of different new effects:

1. First of all, we deal with *infinite dimensional* signal spaces;
2. Consequently two norms typically are *not* equivalent;
3. The natural objects, namely *pure frequencies* $\chi_s(t) := exp(2\pi i s \cdot t)$ do not belong to $\boldsymbol{L}^2(\mathbb{R}^d)$, nor do the Dirac measures (unlike unit vectors in $\mathcal{H} = \mathbb{C}^N$).

Frames as stable sets of generators

Usually we think that the correct analogue of a set of generators in an (infinite dimensional) Hilbert space $\mathcal{H}$ is the assumption that a set $(g_i)_{i\in I}$ is **total**, i.e. that the closed linear span of the set coincides with the whole Hilbert space.
This means of course that, given a vector $h \in \mathcal{H}$ for $\varepsilon > 0$ there exists some finite linear g such that $\|h - g\|_{\mathcal{H}} < \varepsilon$.
But of course it may be much harder to have a representation

$$h = \sum_{i\in I} c_i g_i$$

with the additional condition

$$(\sum_{i\in I} |c_i|^2)^{1/2} \leq C\|h\|_{\mathcal{H}}.$$

Frames as stable sets of generators

The better approach to what is now called **frame theory** is via the so-called *frame operator*

$$S(f) = \sum_{i \in I} \langle f, g_i \rangle g_i,$$

which is supposed to be invertible (in the case of a frame) and then entails the formulas

$$S(S^{-1} f) = f = S^{-1}(Sf),$$

respectively

$$f = \sum_{i \in I} \langle f, g_i \rangle \widetilde{g_i} = \sum_{i \in I} \langle f, \widetilde{g_i} \rangle g_i,$$

with $\widetilde{g_i} = S^{-1}(g_i)$.

Sampling the Spectrogram, Gabor expansions

Translated back to the Gabor setting we have this:
a) reconstruction of a signal from a sampled spectrogram;
b) or care for the representation of a signal f as a superposition of Gaborian building blocks (i.e. Gabor expansion);
c) as a compromise one can ask for a tight (Gabor) frame representation, using $h_i = S^{-1/2} g_i$, the *canonical tight frame*:

$$f = \sum_{i \in I} \langle f, h_i \rangle h_i, \quad f \in \mathcal{H}.$$

This is most useful for the discussion of *frame multipliers*

$$T_m(f) = \sum_{i \in I} m_i \langle f, h_i \rangle h_i, \quad f \in \mathcal{H}, \tag{3}$$

which certainly defines a bounded operator for $m \in \ell^\infty(I)$.

Riesz Basic Sequences

A similar situation occurs with the concept of *linear independence*. There is a natural, perhaps *naive* version, which restricts the attention to arbitrary finite subsets of an infinite set.
However, we are convinced that the correct version if this concept to Hilbert spaces is that of a Riesz basis for a closed subspace (in the same spirit as B-splines are a Riesz basis for the cubic splines in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$), resp. a *Riesz basic sequence*.
The biorthogonal system can be used to describe the orthogonal projection from the Hilbert space onto the closed span of this family. It is obtained from the original system using as coefficients the rows/columns of the Gramian matrix of the family.
Again, the **square root of the inverse Gramian** provides a method that helps to do a *symmetric orthogonalization* (cf. wavelet theory).

Tight Gabor frames and Symmetric ONB

In the same way as the frame operator allows us to generate a tight frame by applying $S^{-1/2}$ to the given Gabor atom we can obtain an orthonormal Gaborian basis for the Gabor family over the adjoint lattice, and again these two natural objects coincide (up to suitable normalization).
There is nothing like this in wavelet theory, this is a very specific property of Gabor systems.
Moreover, based on deep results from harmonic analysis (and operator theory) the assumption $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ implies that both the dual atom and $h = S^{-1/2}g$ belong to $\boldsymbol{S}_0(\mathbb{R}^d)$ as well!
Thus for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the frame operator $S = S_{g,\Lambda}$, its inverse and inverse square root are all BGT-isomorphism.

A General Principle

When it comes to the discussion of BGT morphisms it is a good idea to think of the following three steps:

1. First establish the result at the level of test functions $((\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}))$; Here one can consider most statements just as the continuous analogue of the corresponding statements from linear algebra!
2. Then extend it to the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ by continuity, ideally using an isometry (cf. proof of Plancherel's Theorem).
3. Then extend the mapping further either using w^*-w^*--continuity, or (equivalently and much better!) using duality arguments!

In fact, the outer layer typically reveals the true value of facts difficult to grasp at a technical level ($!e^{2\pi i} = 1$).

The Kernel Theorem I

Clearly a linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^m$ have a matrix representation: $T(\mathbf{x}) = \boldsymbol{A} * \mathbf{x}$, where the entries are of the form

$$a_{j,k} = \langle T(\mathbf{e}_k), \mathbf{e}_j \rangle, 1 \leq k \leq m, 1 \leq j \leq n.$$

Hence one can expect that the continuous version allows to write at least (certain integral) operators as

$$T(f) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

It turns out, that for $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ these operators map $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$ in a w^*-to-norm continuous fashion and *vice versa*. Moreover in analogy to the discrete case one has

$$K(x,y) = \delta_x(T(\delta_y)), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem II

Extending to the Hilbert space setting one finds that kernels in $\mathbf{L}^2(\mathbb{R}^{2d})$ give rise to the well-known Hilbert Schmidt operators. In fact this is a unitary mapping, using the fact

$$\|K\|_{\mathbf{L}^2} = \|T\|_{\mathcal{HS}} := \mathrm{trace}(T \circ T*).$$

The outer layer describes the most general operator. The correspondence identifies $\mathbf{S}_0'(\mathbb{R}^{2d})$ with the space of all bounded linear operators from $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ to $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$. In this setting one can even describe multiplication or convolution operators, in particular the identity operator, which corresponds to the distribution $F \mapsto \int_{\mathbb{R}^d} F(x,x)dx, \quad F \in \mathbf{S}_0(\mathbb{R}^{2d})$, which is well defined since the restriction of $F \in \mathbf{S}_0(\mathbb{R}^{2d})$ to the diagonal is in $\mathbf{S}_0(\mathbb{R}^d)$ and hence integrable.

The Kernel Theorem III

If one tries to rewrite the functional (representing the identity operator) in the usual way (or observing that of course the identity operator is an operator which commutes with translation, and thus has to be a convolution operator, with the usual Dirac measure $\delta_0 : f \mapsto f(0)$) we have

$$f(t) = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

which is only possible if one has in each row

$$K(x,.) = \delta_x, \quad x \in \mathbb{R}^d.$$

(this is more or less the transition from the *Kronecker delta* describing the unit-matrix to the Dirac delta, and is another way of expressing the "sifting property" of δ_0.)

The Kernel Theorem IV

The composition law for matrices is the unique way of combining information about two linear mappings which can be composed (Domino rule) into a new matrix scheme, via standard matrix multiplication rules: $\boldsymbol{C} = \boldsymbol{A} * \boldsymbol{B}$. Thus one expects for the composition of operators a similar composition law for their kernels, something like

$$K(x,y) = \int_{\mathbb{R}^d} K_1(x,z) K_2(z,y) dz, \quad x, y \in \mathbb{R}^d.$$

If one make use of the kernel for the Fourier transform, i.e. $K_2(z,y) = exp(-2\pi i \langle y, z \rangle)$ and $K_1(x,z) = exp(2\pi i \langle x, z \rangle)$, then, even if the integrals do not make sense anymore in the Lebesgue sense, it still suggest to claim that the resulting product operator is the identity operator, which gives a meaning to formulas appearing in engineering books on the Fourier transform.

Formulas as found in Engineering Books

The so-called *sifting property* of the Dirac delta, namely the formula

$$\int_{-\infty}^{\infty} f(s)\delta(s-t)ds = \int_{-\infty}^{\infty} f(s)\delta(t-s)ds = f(t) \qquad (4)$$

describes the Fmo "distributional kernel" of the identity mapping. Applying the above composition rule to the Fourier and inverse Fourier transform kernels given above the exponential law one easily finds that one should have this (!!symbolically!!)

$$\int_{-\infty}^{\infty} e^{i2\pi\nu(t-\tau)}d\nu = \delta(\tau - t). \qquad (5)$$

While mathematicians shake their heads this symbolic formula makes a lot of sense, even if the integrals do not converge. Taking them in a pointwise sense is of course also a very risky thing. But it is also clear that (5) cannot be used to "prove" that the inverse Fourier kernel induces the inverse mapping.

The Kernel Theorem V

Summarizing one can say that the correspondence between the operator kernel and the corresponding operator extends from the well-known characterization of Hilbert-Schmidt operators via $\boldsymbol{L}^2$-kernels to a *unitary Banach Gelfand Triple isomorphism* between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Next we will show that there are various other representations of such operators, e.g. in the spirit of *pseudo-differntial* operators in the frame-work of the *Weyl calculus* or for us more important the Kohn-Nirenberg setting which is closely related to the *spreading representation* of an operator.

The Spreading Function I

Again we start from the case of $\mathcal{G} = \mathbb{Z}_N$, with the representation of linear operators from $\mathbb{C}^N = \ell^2(\mathbb{Z}_N)$ into itself via $N \times N$-matrices. In this setting we have N different *cyclic shift operators*, and also the unitary Fourier transform, which produces another set of frequency shift operators or *modulation operators*. Combining them we obtain a collection of N^2 TF-shifts living in the N^2-dimensional linear space of all complex $N \times N$-matrices. Viewing $\mathbb{C}^{N^2}$ as Euclidean space, resp. endowing these matrices with the Hilbert-Schmidt (also called the Frobenius norm

$$\|\boldsymbol{A}\|_{\mathcal{HS}} := \sqrt{\sum_{j,k} |a_{j,k}|^2} = \mathrm{trace}\boldsymbol{A} * \boldsymbol{A}',$$

it is easy to see that up to the scaling factor $\sqrt{N}$ these operators form indeed an ONB for this space.

The Spreading Function II

The coefficients in this representation form a function over the finite phase space $\mathbb{Z}_N \times \mathbb{Z}_N$, which is called the *spreading function* of the given operator, denoted by $\eta(T)$.
Since shift operators are sitting on (cyclic) side-diagonals, and modulation operators are just pointwise multiplication operators (by the rows of the DFT matrix, i.e. the pure frequencies) it is clear that the spreading coefficients can be easily computed by first viewing the matrix as a collection of N (cyclic) side-diagonals and then taking a one-dimensional Fourier transform.
In the continuous domain the first step is some *automorphism of* $\mathbb{R}^{2d}$, followed then by a *partial Fourier transform*.
The spreading function of a rank-one operator $f \mapsto \langle f, h \rangle g$ with $g, h \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be shown to be equal to the STFT $V_h(g)$, and belongs to $\boldsymbol{S}_0(\mathbb{R}^{2d})$ (kernel is: $K(x,y) = \overline{h(y)} g(x)$).

The Spreading Function III

It is now easy to show that this transformation from kernels to spreading functions not only preserves the $\boldsymbol{S}_0$-property, but also extends (in the expected way) to a BGTr between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
It can be characterized by the fact that the time-frequency shifts $\pi(\lambda) = M_s T_t$, for $\lambda = (t, s)$ is mapped into $\delta_\lambda \in \boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Looking at the Gabor multipliers we see that they constitute a weighted (infinite) sum of projection operators on certain Gabor atoms of the form $\pi(\lambda)h$. Writing P_h for the orthogonal projection of f onto h we have

$$P_{h_\lambda} = (\pi \otimes \pi^*)(\lambda) P_h := \pi(\lambda) \circ P_h \circ \pi(\lambda)^*.$$

The Spreading Function IV

Using the composition rules for TF-shifts (i.e. some algebra, involving phase factors, because they form only a *projective representation*) one finds for the spreading functions

$$\eta((\pi \otimes \pi^*)(\lambda) T) = M_\chi \cdot \eta(T)$$

i.e. multiplication by the pure frequency χ depending on λ. This suggests to introduce the so-called KNS (Kohn-Nirenberg symbol) of T by taking a symplectic Fourier transform of $\eta(T)$, namely $\kappa(T) := \mathcal{F}_s(\eta(T))$. We then have the rule

$$\kappa([\pi \otimes \pi^*](\lambda) T) = T_\lambda \kappa(T).$$

The Spreading Function V

Again, we have a natural unitary BGTr isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and the corresponding KNS in $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, now characterized by the fact that TF-shifts $\pi(\lambda)$ correspond to certain pure frequencies.

The best approximation of a given $\mathcal{HS}$-operator is then equivalent to the approximation of $\kappa(T)$ by linear combinations of shifted copies of $\kappa(P_h) = \mathcal{F}_s(V_h(h)) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

For such approximations one has well-known and efficient formulas for the case that $(P_{h_\lambda})_{\lambda \in \Lambda}$ forms a Riesz basic sequence in the $\mathcal{HS}$-operators.

One can also show that Gabor multipliers are "slowly varying systems" or underspread operators, and that on the other hand such operators are well approximated by Gabor multipliers.

Anti-Wick Operators

There is also the continuous analogue of a Gabor multiplier, which one might call an *STFT-multiplier*. In the literature such operators are known as *Anti-Wick* operators.
For the case that the pointwise multiplier is a bounded function, well concentrated around the origin in phase space, or the indicator function of a some fixed bounded set one talks about a localization operator.
Function in $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ or even in $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ give rise to operators in $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$, thus establishing another BGTr-homomorphism.
The most important consequence is the approximation of an Anti-Wick operator by Gabor multipliers in the corresponding operator norm (at all three levels!).

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