NuHAG Numerical Harmonic Analysis Group

Mathematical Concepts arising from GABOR ANALYSIS

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official abstract I

In his seminal paper from 1946 Denes Gabor was suggesting to expand every function as a double series constituted by Gauss-functions, shifted along the integer lattice and multiplied with harmonic frequencies, as they are used in the classical theory of Fourier series. His suggestions for parameters indicates that he was hoping to have a Riesz basis for the Hilbert space L2(R) in this way, allowing to represent every element f in L2 as a double series, using suitable (and hopefully uniquely determined) coefficients, which can then interpreted as the local energy at a given frequency. The corresponding theory has been developed only after 1980 and is flourishing until now. Concepts like frames and Riesz basic sequences, the Balian-Low Theorem, Gabor multipliers, and modulation spaces, or Banach Gelfand Triples are concepts that have developed during the last decades and have become meanwhile part of the standard repertoire for mathematical

official abstract II

analysis. In this talk, some of these concepts will be illustrated which might be useful for the development of mathematical analysis at large.
Standard reference: K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, 2001.

OUTLINE and GOALS

It is the purpose of this talk to highlight a few topics arising from Gabor Analysis, which are also of great use also for other branches of *functional analysis*.

1. theory of (Banach) frames;
2. Banach Gelfand triples, specifically $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$;
3. Kernel Theorems; $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \equiv \boldsymbol{S}_0'(\mathbb{R}^{2d})$;
4. Conceptual Harmonic Analysis.

Classical Problems

This talk will be structured in the following way: I will try to give my (*certainly subjective*) interpretation of the development of **Fourier Analysis** (first) and subsequently **Gabor Analysis** (viewed as a part of TF-analysis) afterwards.
In each case I will try to point out a few important results and how they have been shaping branches of **functional analysis**, sometimes initiating new tools or even principles which most of the time found afterwards also many applications.
As you will see, in the first part I will describe mostly the analogy between material from standard functional analysis courses and its use in Fourier Analysis.

Why Linear Functional Analysis

1. When dealing with continuous variables most of the function spaces $\boldsymbol{V}$ that we can think of (even the vector space of all polynomials over $\mathbb{R}$!) are NOT FINITE dimensional;
2. Hence we have to talk about limits of finite linear combinations, i.e. convergence in some *norm* or at least *topology*, i.e. we need (w.l.g.) *Banach spaces* and *topological vector spaces*;
3. Since we cannot describe everything with any fixed basis we have to work with the family of all possible coordinate systems for all possible finite-dimensional subspaces, so in fact with the dual space (constituted by *all bounded linear functionals* on the given space.

Principles of Functional Analysis

1. Theory of Hilbert spaces, orthonormal bases;
2. The Hahn-Banach Theorem, *reflexivity*,
3. w^*-convergence; the Banach-Alaoglou Principles
4. The Closed-Graph Theorem, Open Mapping Theorem;
5. Banach Steinhaus Theorem

Relation to Fourier Analysis

1. The Hilbert space $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$, with Fourier series;
2. The spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, wich are reflexive for $1 < p < \infty$;
3. Existence of solutions in the weak sense (PDE);
4. Automatic continuous embeddings, uniqueness of norms;
5. Convergence of sequences of functionals;
6. Boundedness of maximal operator and pointwise a.e. convergence;

Tools arising from Fourier Analysis

1. **Lebesgue's theory** of the integral is still at the basis of Fourier analysis *when viewed as an integral transform*, with e.g. questions about pointwise convergence and $\boldsymbol{L}^p$-spaces;
2. *Calderon-Zygmund theory* of singular integrals brought up the importance of *BMO* and the *real Hardy spaces*, with their atomic characterization and *maximal functions*;
3. **interpolation theory**, e.g. for *Hausdorff-Young*;
4. Study of differentiability, singular integrals etc. let to the *theory of function spaces* (E. Stein, J. Peetre, H. Triebel).

The Life-Cycle of Tools

It is psychologically plausible that new tools go through the following development:

1. A *new tool* is developed in order to solve a *concrete problem*;
2. Once it is observed that the new tool can be applied in other (similar or completely different contexts) the "trick" becomes a **method**, and often receives a name;
3. Subsequent analysis of the methods specifies the ingredients and conditions of applications; *optimization*;
4. Exploration of the maximal range of applications;
5. The limitations of the tool are discovered, and so on...
6. *Change of view/problems creates the need for new tools!*

Distribution Theory

One of the big new tools in the second half of the 20th century was certainly the theory of **tempered distributions** introduced by Laurent Schwartz in the 50th.

1. It gives a (generalized) Fourier transform to many objects which did not "have a FT" so far;
2. It provides a basis for the solution of PDEs (Hörmander);
3. It makes use of **nuclear Frechet spaces**, e.g. in order to prove the *kernel theorem*;

It thus also had a strong influence of the systematic development of the theory of topological vector spaces spaces (J. Horvath (1966), F. Treves (1967), H. Schaefer (1971)).

Distribution Theory and Fourier Analysis

In the context of tempered distributions many things became "kind of simple". Instead of pointwise convergence on can talk about distributional convergence, most operations (including the Fourier transform) are indeed continuous, and since the convergence is relatively weak the space $\mathcal{S}(\mathbb{R}^d)$ of test functions is dense in $\mathcal{S}'(\mathbb{R}^d)$, hence all the operations can be viewed as the natural extension of the classical concepts.
More pratically speaking, using *duality theory* one can define

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), \quad f \in \mathcal{S}(\mathbb{R}^d),$$

but alternatively one could take any sequence h_n in $\mathcal{S}(\mathbb{R}^d)$ convergent (distributionally) to σ and define

$$\widehat{\sigma} = \lim_{n\to\infty} \widehat{h_n}.$$

Absolutely Convergent Fourier Series

In his studies Norbert Wiener considered the Banach algebra $(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}})$ of absolutely convergent Fourier series. It was one of the early **Banach algebras**, with *Wiener's inversion Theorem* being an important first example.

Later on it was natural to study the dual space, which of course contains the dual space of $(\boldsymbol{C}(\mathbb{T}), \|\cdot\|_\infty)$, which by the Riesz representation theorem can be identified with the bounded (regular Borel) measures on the torus it was natural to call these functions *pseudo-measures*.

Since $\boldsymbol{A}(\mathbb{T})$ can be identified with $\boldsymbol{L}^1(\mathbb{T})$ (viewed as subspaces of $\boldsymbol{L}^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ respectively), it is natural to expect (and prove distributionally) that $\boldsymbol{PM}(\mathbb{T})$ is isomorphic to $\ell^\infty(\mathbb{Z})$ via the (extended) Fourier transform.

D. Gabor's Suggestion of 1946

A rough version of Gabor's claim (comparable with Fourier's claim that every periodic function has a representation via the Fourier series expansion) of 1946 says:

Every $f \in \boldsymbol{L}^2(\mathbb{R})$ has an expansion as a Gabor series, with unique coefficients

Here the *atoms* are the standard (Fourier invariant) Gauss-function $g_0(t) = e^{-\pi t^2}$ moved (via time-frequency shifts) of the form

$$\pi(\lambda) = M_n T_k, \quad n, k \in \mathbb{Z},$$

to arbitrary lattice position in *phase space*, thus creating a double indexed family

$$g_{n,k}(t) = e^{2\pi int} g_0(t-k), \quad k, n \in \mathbb{Z}.$$

The choice of the integer (von-Neumann) lattice and the Gauss function had been well motivated (just intuitively).

The Problems with Gabor's Approach

Looking for a Gabor expansion of the form

$$f = \sum_{k,n} c_{n,k} g_{n,k} \quad (1)$$

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raises a few immediate questions, which can be also posed for the families with $g_{k,n} = M_{bn} T_{ak} g_0$, with $a > 0, b > 0$.
Clearly Gabor's choice was $(a, b) = (1, 1)$.

1. How can one determine the cofficients?
2. What kind of convergence should one expect? (pointwise?)
3. Can we represent all the functions $f \in \boldsymbol{L}^2$?
4. Is it true that one has uniqueness?
5. Are there other/better Gabor atoms g?
6. What about different "function spaces"?

Partial Answers

A selection of partial answers is given next:

1. For $a \cdot b < 1$ one can represent every $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, in a norm convergent fashion, but one does not have uniqueness;
2. For $a \cdot b > 1$ the family $(g_{k,n})$ is just a Riesz basis for its closed linear span (suitable for mobile communication);
3. For $a \cdot b = 1$ one has neither nor!! (A.J.E.M. Janssen): Also with ℓ^2-coefficients one cannot reach all of $\boldsymbol{L}^2(\mathbb{R})$, not even if one accepts distributional convergence. Coefficients in $\ell^\infty(\mathbb{Z}^2)$ imply distributional convergence, but already here uniqueness is lost.

We talk about he *oversampled, undersampled* and *critical* case.

Further good properties

The case $a \cdot b < 1$ has further good properties:

1. The minimal norm solution in $\ell^2(\mathbb{Z}^2)$ depends linearly and continuously on f in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.
2. For $f \in \boldsymbol{\mathcal{S}}(\mathbb{R})$ the Gabor series is even convergent in the topology of the Schwartz space (with rapidly decreasing coefficients);
3. It is also possible to replace the Gauss function by another Gauss-like function h (for each fixed a, b with $ab < 1$) such that one has the following[1] *quasi-orthogonal* expansion:

$$f = \sum_{k,n} \langle f, h_{k,n} \rangle h_{k,n}, \quad \in \boldsymbol{L}^2(\mathbb{R}) \tag{2}$$

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with unconditional convergence in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.

[1] !ad hoc terminology!

Frames in Hilbert Spaces

The abstract key concept behind the situation has been captured by the concept of *frames in Hilbert spaces*.
Pratically they are the correct (namely !stable) analogue of "generating systems" in a finite-dimensional vector space.
The usual, formal definition qualifies a countable family $(g_j)_{j\in\mathbb{N}}$ as a *frame* in $\mathcal{H}$ if there are positive constants $A, B > 0$ such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{j\in\mathbb{N}} |\langle f, g_j\rangle|^2 \leq B\|f\|_{\mathcal{H}}^2, \quad f \in \mathcal{H}. \tag{3}$$

fra

This condition is equivalent to the positive definiteness (hence bounded invertibility) the the so-called frame operator

$$Sf = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle f_j, \quad f \in \mathcal{H}. \tag{4}$$

fra

Hence we can always use $f = S^{-1}S(f) = SS^{-1}(f)$.

Coefficients and Atomic Decomposition

Writing as usual $(\widetilde{f_j})$ for the family $(S^{-1}f_j)$ one can use this last formula to find the (minimal norm coefficient) for an *atomic representation* of f:

$$f = \sum_{j \geq 1} \langle f, \widetilde{f_j} \rangle f_j, \quad f \in \mathcal{H},$$

or recover the element $f \in \mathcal{H}$ from the set of coefficients, often a continuous transform:

$$f = \sum_{j \geq 1} \langle f, f_j, \rangle \widetilde{f_j} \quad f \in \mathcal{H}.$$

Of course one would like to have a *minimal* system (f_j), resp. uniqueness of the coefficients, i.e. in fact a Riesz basis for $\mathcal{H}$, but this is in some cases quite impossible.

The Balian-Low Theorem

It is possible to obtain an orthonormal basis for $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ by using a TF-collecton starting from the indicator function of $[0,1]$, but this is not a "good" Gabor frame, because it can have very poorly decaying Gabor coefficients for rather nice functions (for example a simple bump function supported on $[0,2]$, which is non-zero at $t_1 = 1$.

The **Balian-Low** Theorem implies that it is impossible to have a decent function h (e.g. compactly supported with integrable Fourier transform, or integrable with compactly supported Fourier transform, i.e. a *band-limited* function in $\boldsymbol{L}^1(\mathbb{R})$) such that the family $M_n T_k h$ is a Riesz basis for $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.

Alternative Function Spaces

There are many situations where the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ is the most appropriate function spaces in the context of Gabor Analysis. We will first describe the space shortly (using a few of its characterizations) and then list a few examples where and why it is a convenient and frequently used tool in the context of Gabor Analysis.

The (meanwhile) classical way to introduce $\boldsymbol{S}_0(\mathbb{R}^d)^2$ is making use of the Short-Time Fourier transform of a signal $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined by

$$V_g(f)(t,\omega) = \langle f, M_\omega T_t g \rangle, \quad t,\omega \in \mathbb{R}^d. \tag{5}$$ STF

Fixing any non-zero Schwartz function, typically $g = g_0$, with $g_0(t) = e^{-\pi t^2}$, one sets:

$$\boldsymbol{S}_0(\mathbb{R}^d) := \{ f \in \boldsymbol{L}^2(\mathbb{R}) \,|\, V_g(f) \in \boldsymbol{L}^1(\mathbb{R}^{2d}) \}. \tag{6}$$ SOR

[2] This so-called Segal algebra can be defined over general LCA groups.

Robustness Properties

The Moyal formula implies immediately the following inversion formula "in the weak sense"

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda)g d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \qquad (7)$$

STF

It follows from **Moyal's formula** (energy preservation):

$$\|V_g(f)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} = \|g\|_2 \|f\|_2, \quad f, g \in \boldsymbol{L}^2. \qquad (8)$$

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This setting is well known under the name of coherent frames when $g = g_0$, the Gauss function. Its range is the *Fock space*. But for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one has (according F. Weiss) even norm convergence of Riemannian sums, for *any* $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.

Natural properties

This observation also implies that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow \left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for any $p \geq 1$, or any other function/distribution space on $\mathbb{R}^d$ with isometric TF-shifts.
The *minimality* of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ gives, on the other hand, the continuous embedding of any $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$.

An emerging theory of *Fourier standard spaces* is on the way. They satisfy

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d),$$

and some further conditions, such as

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subset \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subset \boldsymbol{B}.$$

Atomic Decompositions

The minimality implies (inspired by the atomic characterization of *real Hardy spaces* the following result:

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Theorem

Given any non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *one has*

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{ \sum_{k \geq 1} c_k \pi(\lambda_k) g, \quad \textit{with} \sum_{k \geq 1} |c_k| < \infty \}.$$

This results also implies that the space is invariant under dilation, rotation and even fractional Fourier transforms.
A long list of sufficient conditions ensures that among others all classical summability kernels belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.

frametitle: bibliography I

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Varying the time-frequency lattice of Gabor frames.
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Usefulness

There is a rich collection of results involving $\boldsymbol{S}_0(\mathbb{R}^d)$ and sometimes the **Banach Gelfand Triple** $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ First of all those relevant for Gabor Analysis:

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Theorem

For every non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ there exists some $\varepsilon > 0$ such that for every ε-dense (and relatively separated) sequence (λ_k) the family $(\pi(\lambda_k)g)_{k\geq 1}$ is a Gabor frame[a]

[a] Every ε-dense sequence contains such a sequence as a subsequence.

In this situation the construction of dual frames also shows some robustness against so-called *jitter error* and geometric deformation (rotation, and even stretching). The key result in this direction is the principle of variation of the lattice constants.

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Approximate dual Gabor windows

The relevance of such a result is among others in the *constructive realization of Gabor systems*, which was for a long time thought as a very difficult and computationally expensive tasks.
Nowadays there is a lot of effective MATLAB code and theory supporting this theory. Among others one has:
Assume that we have some lattice Λ and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Then it is enough to compute the dual atom $\widetilde{g}$ for a close enough lattice (e.g. with rational coordinates) and use it, in order to guarantee that one has

$$\|f - \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda \|_2 < \varepsilon \|f\|_2,$$

for all $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.
Similar statements with the $\boldsymbol{S}_0$-norm hold for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Sampling functions and Gabor Analysis

The effective computation of (approximate) Gabor coefficients is then based on the use of FFT methods combined with (periodization and) sampling of the involved ingredients.
Based on the work of N. Kaiblinger (and HGFei) one has for example (for simplicity we present the $1D$-case:

FT1

Theorem

Given $f \in \boldsymbol{S}_0(\mathbb{R})$ a good approximation of its Fourier transform $\widehat{f}$ can be obtained by piecewise linear interpolation (or cubic spline quasi-interpolation) of the FFT of a sequence of regular samples, passing through zero (with suitable labeling of the samples).

The proof of such a claim relies on the fact that the FFT determines correctly the sampled and periodized version of $\widehat{f}$ from a sampled and periodized version of $f \in \boldsymbol{S}_0(\mathbb{R})$.

The Banach Gelfand Triple

Combined with the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (in the middle) the three spaces $\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ form a so-called **Banach Gelfand Triple** or *rigged Hilbert space*.
Such a BGTr is constituted by a Banach space $\left(\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)\right.$ in our case) which is continuously and densely embedded into a Hilbert spaces $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$ in our case) and such that this itself is embedded continuously into the dual Banach space (norm continuous, but only w^*-dense).
A **morphism** of Banach Gelfand triples respects each of the three layers, but is also w^*-w^*--continous at the outer level.

Fourier transform

The Fourier transform is a prototype of a *unitary BGTr-automorphosm* for the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
It can be formulated for any LCA group, and reduces in the classical case of $\mathcal{G} = \mathbb{T}$ to a well known result, making use of $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$, the algebra of absolutely convergent Fourier series and its dual, the space $\boldsymbol{PM}$ of pseudo-measures.
The classical Fourier transform, establishing a unitary isomorphism (Parseval's identity) from $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ to $\ell^2(\mathbb{Z})$ extends to a BGTr isomorphism between $(\boldsymbol{A}(\mathbb{T}), \boldsymbol{L}^2(\mathbb{T}), \boldsymbol{PM}(\mathbb{T}))$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Euclidean Situation

In the Euclidean Situation interesting objects are for example the Dirac trains supported by lattices $\Lambda \lhd \mathbb{R}^d$:

$$⧢ := \sum_{\lambda \in \Lambda} \delta_\lambda.$$

These (unbounded, discrete) measures belong to $\boldsymbol{S}_0'(\mathbb{R}^d)$ and thus have a distributional Fourier transform. In fact, the usual argument in distribution theory making use of Poisson's formula (now not only for $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ but for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$!) gives

$$\widehat{⧢} = ⧢.$$

This fact allows to prove *Shannon's sampling Theorem* (based on intuitive arguments).

Sampling and Periodization on the FT side

This result is the key to prove **Shannon's Sampling Theorem** which is usually considered as the fundamental fact of digital signal processing (Claude Shannon: 1916 - 2001).
Shannon's Theorem says that one can have **perfect reconstruction** for band-limited functions.
If the so-called *Nyquist criterion* is satisfied (sampling distance small enough), i.e. $\text{supp}(\hat{f}) \subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) g(x - \alpha k), \quad x \in \mathbb{R}^d. \tag{9}$$

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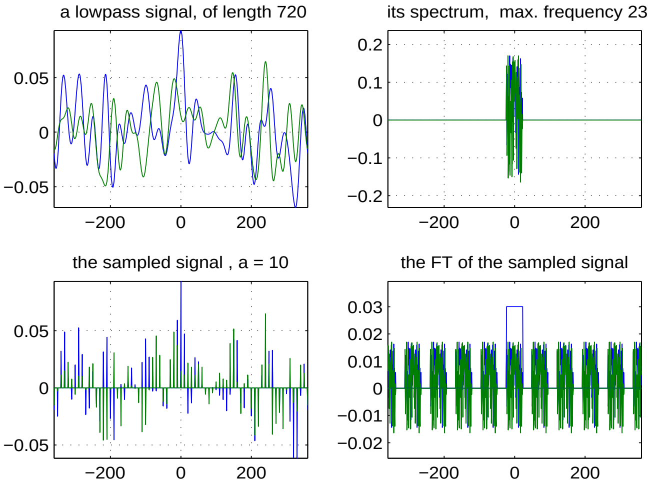
A Visual Proof of Shannon's Theorem



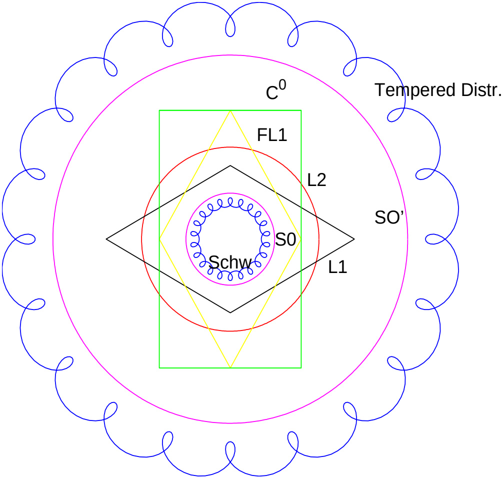
Spectrum of a Bounded Function

The fact that $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ for any $1 \leq p \leq \infty$ implies that one has, beyond $p > 2$ not only Hausdorff-Young: $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{L}^q(\mathbb{R}^d)$ for $1 \leq p \leq \infty$.
The statement that $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ is of course much weaker than the claim $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d)$.
This fact allows to directly define the *spectrum* of $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$ using the general definiton:

$$\mathrm{spec}(\sigma) := \mathrm{supp}(\widehat{\sigma}), \quad \sigma \in \boldsymbol{S}_0'.$$

If $\mathrm{supp}(\sigma)$ is a finite set it can be shown to be just a finite sum of Dirac measures, i.e. $\sigma = \sum_{k=1}^K c_k \delta_{x_k}$.

A schematic description of the situation

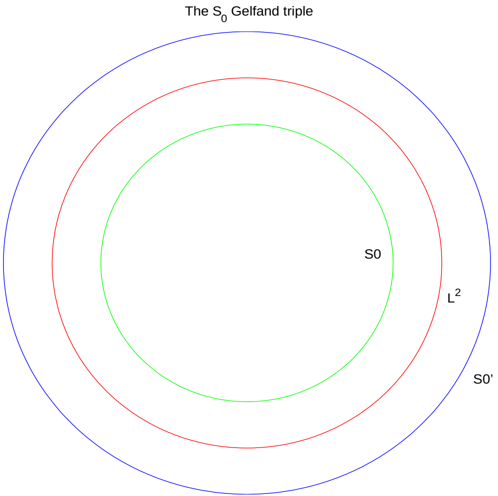


The Banach space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ or $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$), which will serve our purpose. Its dual space $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(S_0, L^2, S_0')(\mathbb{R}^d)$.

The S_0-Banach Gelfand Triple



The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

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Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\widehat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

OVERVIEW old

THE GOAL OF THIS PRESENTATION IS TO CONVEY THE CONCEPTS OF MODULATION SPACES, BANACH FRAMES AND BANACH GELFAND TRIPLES BY DESCRIBING THEM AND SHOW THEIR USEFULNESS IN THE CONTEXT OF MATHEMATICAL ANALYSIS, IN PARTICULAR TIME-FREQUENCY ANALYSIS

- Recall some concepts from linear algebra, especially that of a *generating system*, a *linear independent* set of vectors, and that of the dual vector space;
- already in the context of Hilbert spaces the question arises: *what is a correct generalization of these concepts?*
- Banach Gelfand Triple (comparable to rigged Hilbert spaces) are one way out;

Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \qquad (10)$$

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is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

FACTS

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):

Theorem

Assume that for $g \in \mathbf{S}_0(\mathbb{R}^d)$ the Gabor frame operator

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

is invertible as an operator on $\mathbf{L}^2(\mathbb{R}^d)$, then it is also invertible on $\mathbf{S}_0(\mathbb{R}^d)$ and in fact on $\mathbf{S}_0'(\mathbb{R}^d)$.
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S is (resp. extends to) an isomorphism of the Gelfand triple automorphism for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies $\chi_\omega(t) = exp(2\pi i \omega t)$ onto the corresponding point measures δ_ω.
Vice versa, (even non-harmonic) trigonometric polynomials are those elements of $\mathbf{S}_0'(\mathbb{R}^d)$ which have finite support on the Fourier transform side.

Thanks for your attention

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