NuHAG Numerical Harmonic Analysis Group

Decomposition Spaces and Generalized Coorbit Spaces

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Celebrating the 60th Birthday
of Stephan Dahlke

Motivation and History

I will try to connect various topis that play a role in the work of Stephan Dahlke and myself, all related to the topic of *COORBIT SPACES* from a genetic/historical point of view.
There are ca. 20 papers with the word coorbit in the title and with S. Dahlke as coauthor. You can all find them on his Web-page or at the NuHAG server (most of them, I hope).
The list below also contains papers concerning shearlets (a special case of coorbit theory).

G. Alberti, S. Dahlke, F. De Mari, E. De Vito, and H. Führ.
Recent progress in shearlet theory: systematic construction of shearlet dilation groups, characterization of wavefront sets, and new embeddings.
In *Frames and other bases in abstract and function spaces*, Appl. Numer. Harmon. Anal., pages 127–160. Birkhäuser/Springer, Cham, 2017.

G. S. Alberti, S. Dahlke, V. De, and H. Führ.
Recent Progress in Shearlet Theory: Systematic Construction of Shearlet Dilation Groups, Characterization of Wavefront Sets, and New Embeddings.
ArXiv e-prints, may 2016.

S. Dahlke, M. De, V. De, D. Labate, G. Steidl, G. Teschke, and S. Vigogna.
Coorbit spaces with voice in a Frechet space.
J. Fourier Anal. Appl., 23(1):141–206, 2017.

S. Dahlke, F. de Mari, E. De Vito, S. Häuser, G. Steidl, and G. Teschke.
Different faces of the shearlet group.
J. Geom. Anal., 26(3):1693–1729, 2016.

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Coorbit spaces with voice in a Frechet space.
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S. Dahlke, F. De Mari, E. De Vito, L. Sawatzki, G. Steidl, G. Teschke, and F. Voigtlaender.
On the Atomic Decomposition of Coorbit Spaces with Non-Integrable Kernel.
ArXiv e-prints, jul 2018.

S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl, and G. Teschke.
Generalized coorbit theory, Banach frames, and the relation to α-modulation spaces.
Proc. London Math. Soc., 96(2):464–506, 2008.

S. Dahlke, Gabriele Steidl, and Gerd Teschke.
Multivariate shearlet transform, shearlet coorbit spaces and their structural properties.
In *Shearlets. Multiscale analysis for multivariate data.*, pages 105–144. Boston, MA: Birkhäuser, 2012.

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Shearlet coorbit spaces: traces and embeddings in higher dimensions.
Monatsh. Math., 169(1):15–32, 2013.

S. Dahlke, S. Häuser, G. Steidl, and G. Teschke.
Shearlet coorbit theory.
In *Harmonic and applied analysis*, Appl. Numer. Harmon. Anal., pages 83–147. 2015.

S. Dahlke, S. Häuser, and G. Teschke.
Coorbit space theory for the Toeplitz shearlet transform.
Int. J. Wavelets Multiresolut. Inf. Process., 10(04):1250037, 13 p., 2012.

S. Dahlke, Q. Jahan, C. Schneider, G. Steidl, and G. Teschke.
Traces of shearlet coorbit spaces on domains, 09 2018.

S. Dahlke, G. Kutyniok, P. Maass, C. Sagiv, H.-G. Stark, and G. Teschke.
The uncertainty principle associated with the continuous shearlet transform.

S. Dahlke, G. Kutyniok, P. Maass, C. Sagiv, H.-G. Stark, and G. Teschke.
The uncertainty principle associated with the continuous shearlet transform.
Int. J. Wavelets Multiresolut. Inf. Process., 6(2):157–181, 2008.

S. Dahlke, G. Kutyniok, G. Steidl, and G. Teschke.
Shearlet coorbit spaces and associated Banach frames.
Appl. Comput. Harmon. Anal., 27(2):195–214, 2009.

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Coorbit spaces and Banach frames on homogeneous spaces with applications to the sphere.
Adv. Comput. Math., 21(1-2):147–180, 2004.

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Weighted coorbit spaces and Banach frames on homogeneous spaces.
J. Fourier Anal. Appl., 10(5):507–539, 2004.

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Frames and Coorbit theory on homogeneous spaces with a special guidance on the sphere.
J. Fourier Anal. Appl., 13(4):387–403, 2007.

S. Dahlke, G. Steidl, and G. Teschke.
The continuous shearlet transform in arbitrary space dimensions.
J. Fourier Anal. Appl., 16(3):340–364,, 2010.

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S. Dahlke, G. Teschke, and K. Stingl.
Coorbit theory, multi-α-modulation frames, and the concept of joint sparsity for medical multichannel data analysis.
EURASIP J. Adv. Signal Process., 2008:19, 2008.

S. Dahlke, G. Teschke, and K. Stingl.
Coorbit theory, multi-α-modulation frames and the concept of joint sparsity for medical multi-channel data analysis.
EURASIP J. Adv. Signal Process., pages 5–10 (?19pg), 2008.

H. G. Feichtinger.
Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.

H. G. Feichtinger and K. Gröchenig.
Banach spaces related to integrable group representations and their atomic decompositions, I.
J. Funct. Anal., 86(2):307–340, 1989.

H. G. Feichtinger and K. Gröchenig.
Banach spaces related to integrable group representations and their atomic decompositions, II.
Monatsh. Math., 108(2-3):129–148, 1989.

M. Speckbacher, D. Bayer, S. Dahlke, and P. Balazs.
The α-modulation transform: admissibility, coorbit theory and frames of compactly supported functions.
Monatsh. Math., 184(1):133–169, 2017.

And in the beginning there was Wiener's algebra

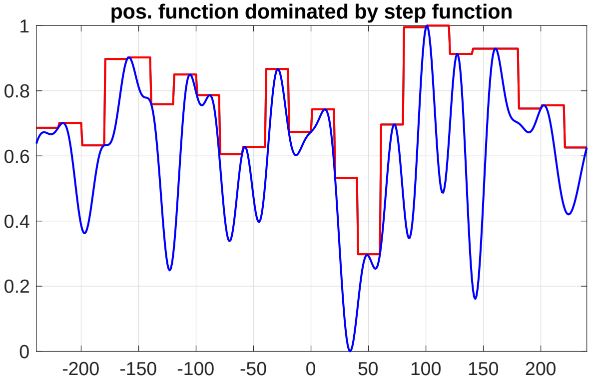
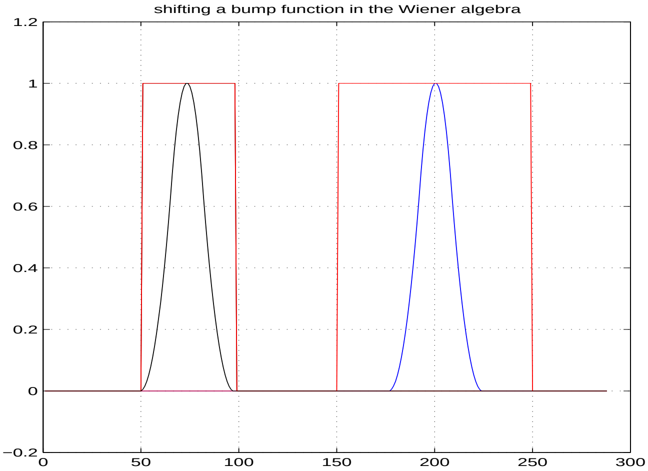
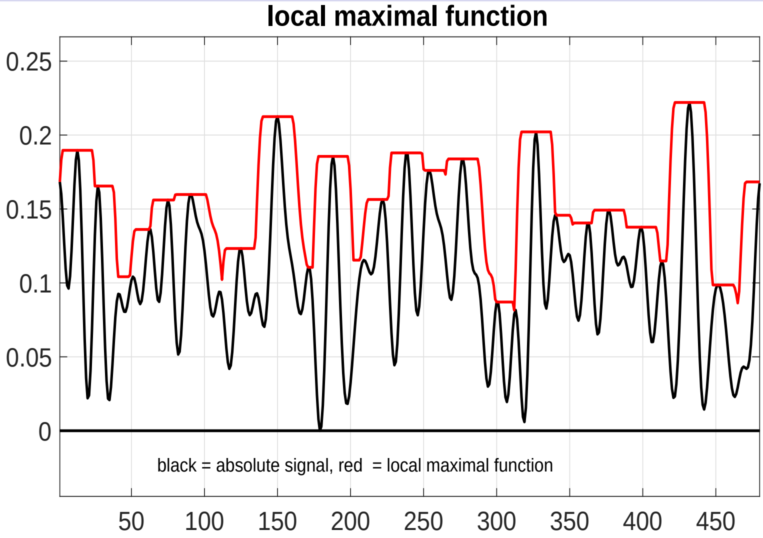


Figure: Wiener's norm on $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)$: upper Riemanian sum
It is a *Segal algebra*, but only with equivalent norm!

Obvious problem with translation invariance



Using a continuous (local) maximums norm



H. Reiter.
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H. G. Feichtinger.
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Archiv d. Math., 29:136–140, 1977.

F. Holland.
Harmonic analysis on amalgams of L^p and ℓ^q.
J. Lond. Math. Soc., 10:295–305, 1975.

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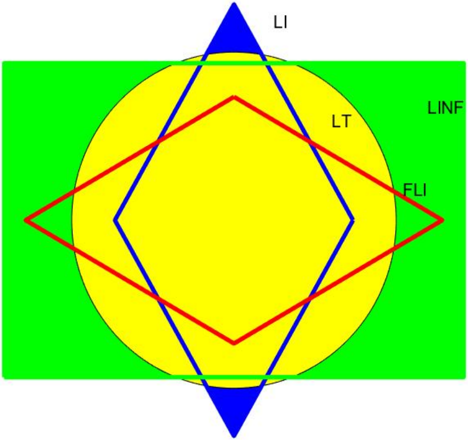
Product-convolution operators and mixed-norm spaces.
Trans. Amer. Math. Soc., 263:309–341, 1981.

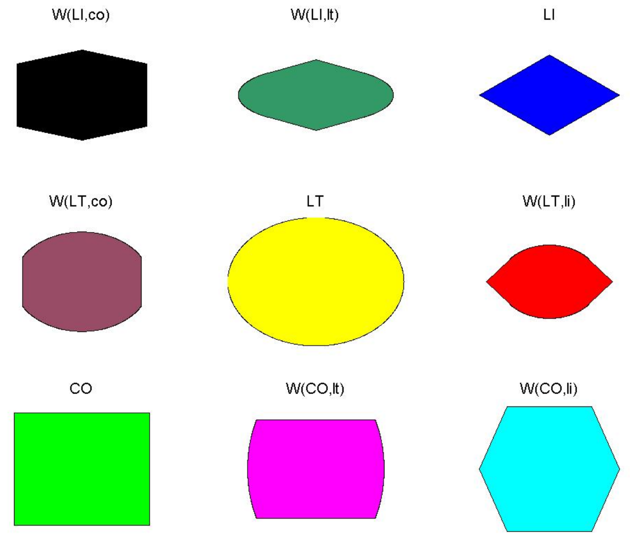
H. G. Feichtinger.
Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.

J. J. F. Fournier and J. Stewart.
Amalgams of L^p and ℓ^q.
Bull. Amer. Math. Soc., New Ser., 13:1–21, 1985.

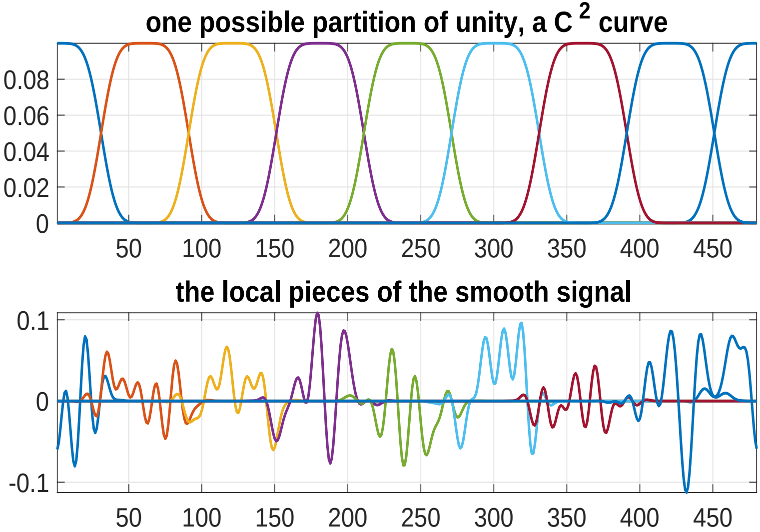
The Standard spaces L^1, L^2, L^∞ and $\mathcal{F}L^1$



Comparing Wiener's algebra to S_0 and $\mathcal{S}$



Using smooth BUPUs



Wiener amalgam spaces $W(B, C)$

Using the idea of BUPUs, i.e. partitions of unity of uniform size of the support, i.e. with $\mathrm{supp}(\psi_i) \subset B_r(x_i) \subset \mathbb{R}^d$, for some $r > 0$ for all $i \in I$ with uniformly bounded action on the *local component* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ it is then possible to define general *Wiener amalgam spaces* $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$.
We will concentrate our comments on the choice $\boldsymbol{B} = \boldsymbol{L}^p(\mathbb{R}^d)$ or $\boldsymbol{B} = \mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$, and global components which are weighted $\boldsymbol{L}^q$-spaces or weighted mixed-norm spaces.
There are natural versions of the Hausdorff-Young inequality for Wiener amalgams.

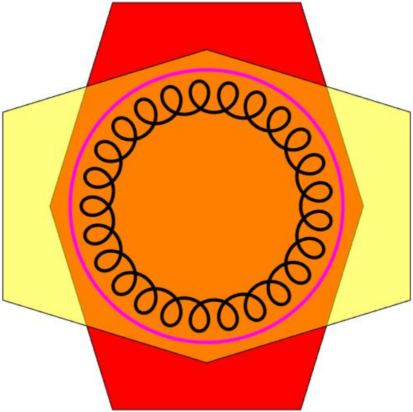
Preparation for Modulation Spaces

In principle the idea of what is nowadays called *modulation spaces* was to describe functions which are characterized by BUPUs, the (infamous) uniform partitions of unity, e.g. from $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)$, by taking the inverse (distributional) Fourier transform.

$$\boldsymbol{M}^{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d).$$

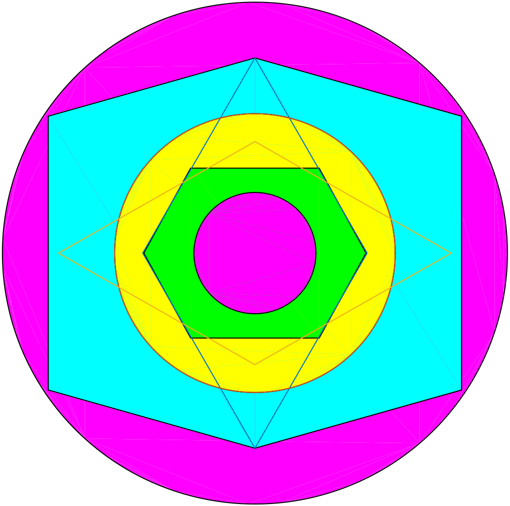
For me the most important case is of course
$\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d)$.

Comparing Wiener's algebra to S_0 and $\mathcal{S}$



Compared to S_0 and S_0'

a more complete picture



Convolution and Interpolation

For the Euclidean case it is natural to use so-called *regular BUPUS* obtained by shifting an appropriate localizing function ψ along some lattice Λ.
Since any lattice is the finite union of much coarser lattices this allows to reduce many questions to the case of quasi-orthogonal decompositions, i.e. one has support sets which do not overlap.
From this it is easy to derive results about (e.g. complex) interpolation but also (one of the most useful properties) *convolution results*, using corresponding properties of the *local* and the *global component* of *Wiener amalgams*, in the general setting.

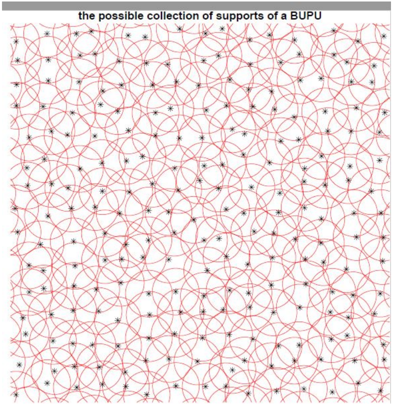


Figure: BUPUsupp1c.jpg

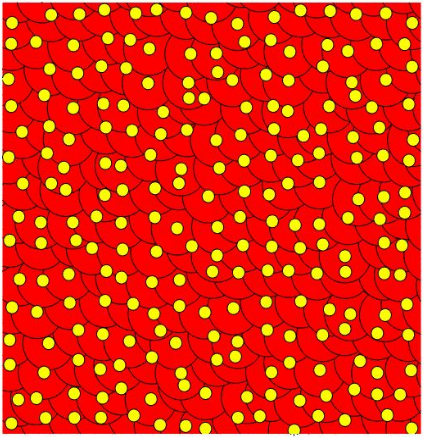


Figure: BUPUsupp2.jpg

Duality, the general case

Once the continuous characterization was available and the equivalence to the discrete characterization using BUPUs was established (in [5]) the question of duality arose for the general (non-Abelian) setting.
Combined with attempts to understand the general setting of Hörmander's work on the Weyl calculus and above all the wish to find a common ground for Besov and modulation spaces this problem was the starting point for *decomposition spaces*.
The PhD thesis of Peter Gröbner was then the break-trough in that direction, featuring the now also well established α-modulation spaces on $\mathbb{R}^d$, with $\alpha \in [0,1)$.

L. Hörmander.
The Weyl calculus of pseudo-differential operators.
Commun. Pure Appl. Math., 32:359–443, 1979.

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Banach spaces of distributions defined by decomposition methods. I.
Math. Nachr., 123:97–120, 1985.

H. G. Feichtinger.
Banach spaces of distributions defined by decomposition methods. II.
Math. Nachr., 132:207–237, 1987.

M. Fornasier.
Banach frames for α-modulation spaces.
Appl. Comput. Harmon. Anal., 22(2):157–175, 2007.

S. Dahlke, M. Fornasier, and T. Raasch.
Adaptive frame methods for elliptic operator equations.
Adv. Comput. Math., 27(1):27–63, 2007.

S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl, and G. Teschke.
Generalized coorbit theory, Banach frames, and the relation to α-modulation spaces.
Proc. London Math. Soc., 96(2):464–506, 2008.

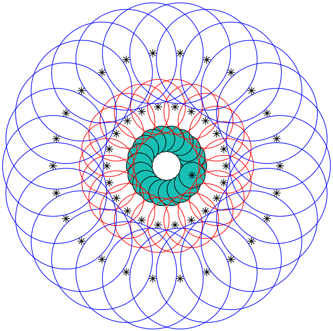

Figure: alphcov02.eps

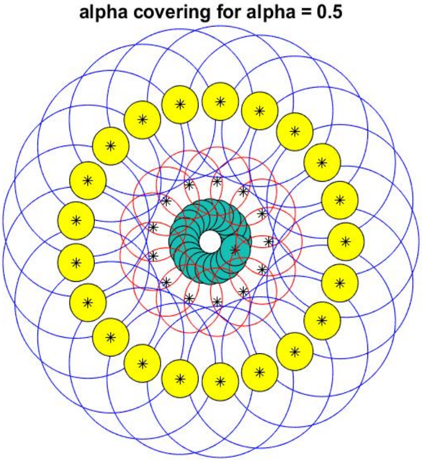

Figure: alphcov03.jpg

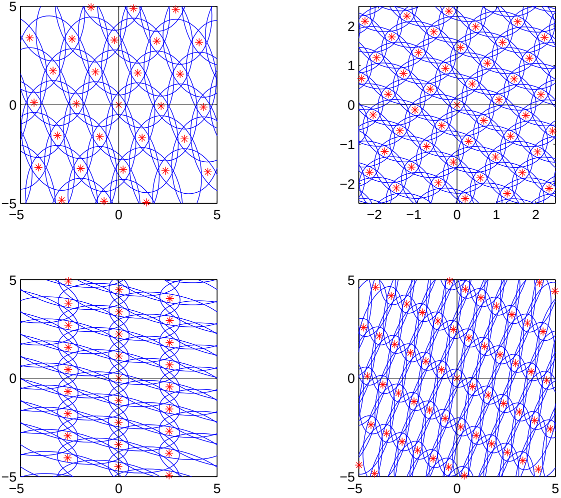

Figure: alphcov02.eps

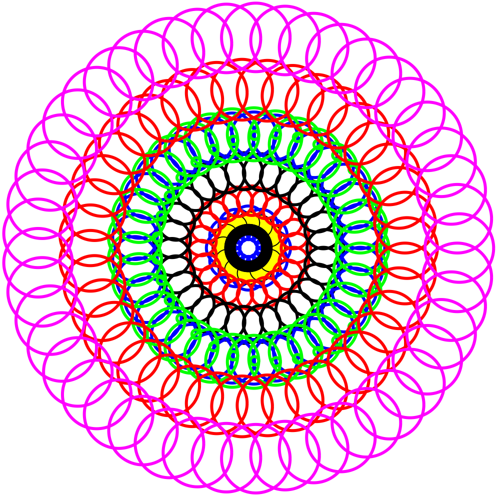

Figure: alphacov003.pdf

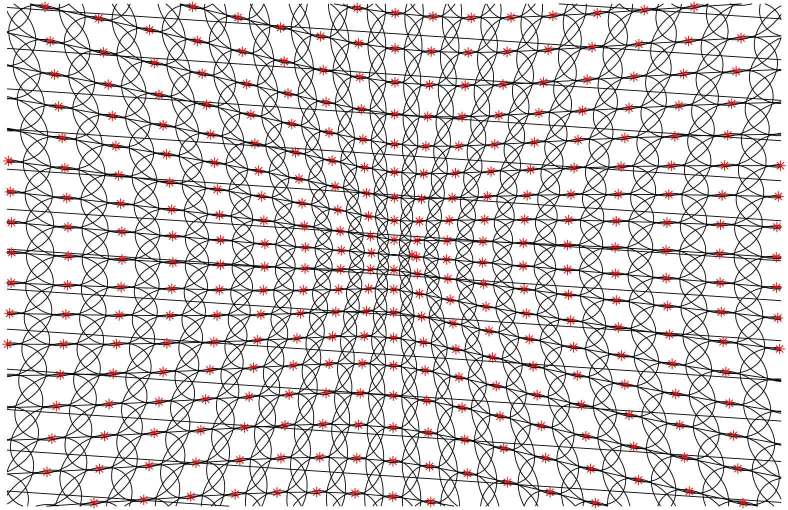

Figure: alphcov04d.pdf

Modulation Spaces inspiring Coorbit Theory

In the development of coorbit theory essentially the *unification aspect* for three situations had been dominant (with further generalizations imminent):

1. the wavelet case;
2. the Gabor (time-frequency) case;
3. Möbius invariant function spaces on the disc;
4. Shearlet groups (more recently, Dahlke et all);

Inspired by the work of J. Peetre (he introduced the term *coorbit spaces* in a more general context, for paracommutators), Charly and myself developed coorbit theory in [24, 25].

The Foundations of Coorbit Theory

Coorbit Theory is based on the following assumptions:

1. There is an *irreducible unitary representation* π of some locally compact group $\mathcal{G}$ on some Hilbert space $\mathcal{H}$;
2. For so-called *admissible elements* φ (in the domain of a densely defined possibly unbounded operator $\boldsymbol{A}$) one can define the continuous *voice transform* on $\mathcal{H}$:

$$V_\varphi(f)(x) = \langle f, \pi(x)\varphi \rangle, \quad f \in \mathcal{H}.$$

3. Given a *solid, translation invariant Banach space* of $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ on $\mathcal{G}$ one defines
4. $\boldsymbol{Co}(\boldsymbol{Y}) : \{f \mid V_\varphi(f) \in \boldsymbol{Y}\}$, with $\|f\|_{\boldsymbol{Co}(\boldsymbol{Y})} := \|V_\varphi(f)\|_{\boldsymbol{Y}}$.

The Foundations of Coorbit Theory II

An important asset for the derivation of the basic properties of coorbit spaces are the following two consequences of the square integrability of the representation.

- For suitably normalized (admissible) atoms/windows one has an isometric embedding of $\mathcal{H}$ into $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$, i.e.

$$\|V_\varphi(f)\|_2 = \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

- The range of V_φ in $\boldsymbol{L}^2(G)$ satisfies a convolution relation:

$$V_\varphi(f) * V_\varphi(\varphi) = V_\varphi(f).$$

- V_φ^* induces a reproducing formula

$$f = \int_G V_\varphi(f)(x)\, \pi(x)\varphi \, dx, \quad f \in \mathcal{H}.$$

Some questions about coorbit spaces

Clearly there are various questions:

1. how can one show completeness of $\boldsymbol{Co}(\boldsymbol{Y})$?
2. What is the reservoir of *generalized functions* for which the voice transform $V_\varphi(f)$ is well defined?
3. Can one prove the continuous "Calderon-type reproduction formula" in the more general context?
4. Is it possible to discretize the continuous transform, in other words, can one have (Banach) frames of the form $(\pi(x_i)\varphi)$?

Banach frames for families

Coorbit theory has been developing great in the last 30 (!) years, but it is still continuing to expand in various directions.
From a modern point of view it provides Banach frames for *families of coorbit spaces*. See Charly's paper on "Desribing functions", establishing the equivalence of the *orbit* and *coorbit* approach.
It is important that all estimates are uniform over "compact families" of parameters (for the corresponding solid BF-spaces).
But these families are also understandable is finite unions of corresponding Riesz projection bases (and this is the basis for what has become known as the *Feichtinger conjecture*).

Generalized Coorbit Theory

The original theory was going for *qualitative results* at the widest possible framework.

Thus there was and still is a lot if room for further development:

- to get more *quantitative results* for concrete examples;
- verify cases where one can get along with weaker assumptions;
- check out whether one work with smaller families of spaces for the case that not all the conditions are satisfied.

A typical obstacle is the lack of *integrability* of the (projective) representation under consideration. This is where the idea of "generalized coorbit spaces" comes in, which has been developed by Stephan with many different coauthors.

The method of sections

Looking at wavelet or Gabor expansions for $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$ and related spaces, i.e. for the Besov-Triebel-Lizorkin spaces linked with the continuous wavelet transform and modulation spaces for the STFT one observes that one has typically two relevant (non-compact) parameters, namely scaling and translation in the $ax + b$-group and time/frequency shifts.
Combining these three commutative families of unitary operators is possible, using the *affine Weyl-Heisenberg group*, but the resulting irreducible representation is *NOT square integrable*.
One way out of this dilemma is to work with sections, at the (high!) price of loosing the simple reproducing formula which was a simple convolution relation for "ordinary coorbit spaces".

α-modulation spaces

As of now the so-called α-modulation spaces on $\mathbb{R}$ are a *prototype* for a family of spaces that have been identified as crossover between generalized coorbit spaces and decomposition spaces. There is obvious work to be done (on the way in our joint project) to extend this result to the multi-dimensional setting. Things look very promising.
Meanwhile it has been shown that they are not just (complex) interpolation spaces between modulation spaces ($\alpha = 0$) and Besov spaces (which can be viewed as limiting case $\alpha = 1$).

Current work related to decomposition spaces

Decomposition spaces have a wide range of applications and I am sure also generalized coorbit spaces. It is natural to adapt the toms used for *atomic decomposition* to the time or location, due to the dependence of the properties of a time/location dependent differential equation or more generally *pseudo-differential equation*. Work in that direction has been done by many people already in the last years.
There are also many papers on the characterization of wavefront sets (J.Toft and others) or mapping properties of frame multipliers (Toft and Gröchenig) in the sense of lifting properties.
In the last years huge progress has been made by Felix Voigtlaender (based on his work with H. Führ) and we are seeing major papers coming up from him.

Applications of decomposition spaces

It is quite easy to invent new function spaces, one has to sometomes hard work to establish their properties and interconnections
BUT it is much harder to prepare the material for REAL APPLICATIONS, in particular those outside the field.
It is one of the very strong sides of Stephan, that he is always looking for such applications, encourages others, in particular young people, to take care of such problems, and to identify possibilities that really demonstrate the usefulness of the concepts discussed so far.
In this way he is a motivator for young people, but also his many collaborators and friends, many of which have come here to celebrate his

HAPPY BIRTHDAY, Stephan!

Happy Birthday to Stephan

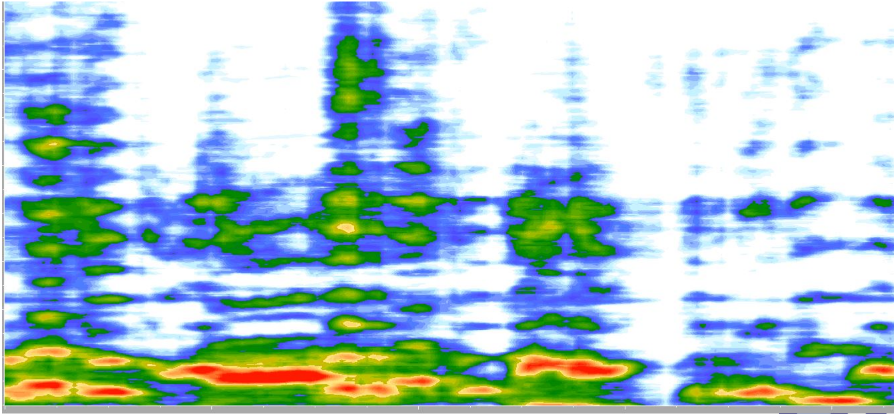

Figure: HappyBSt1.jpg