NuHAG Numerical Harmonic Analysis Group

Classical Fourier Analysis and THE Banach Gelfand Triple

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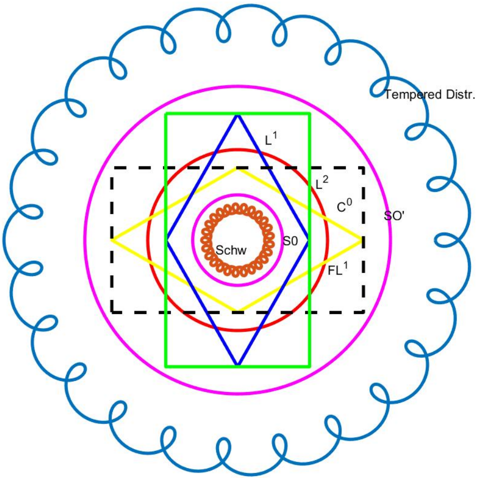
A Zoo of Banach Spaces for Fourier Analysis



Official Abstract (for later reading)

It is the purpose of this presentation to explain certain aspects of Classical Fourier Analysis from the point of view of *distribution theory*. The setting of the so-called *Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ starts from a particular Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ of continuous and Riemann integrable functions. It is Fourier invariant and thus an extended Fourier transform can be defined for $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ the space of so-called **mild distributions**. Any of the $\boldsymbol{L}^p$-spaces contains $\boldsymbol{S}_0(\mathbb{R}^d)$ and is embedded into $\boldsymbol{S}_0'(\mathbb{R}^d)$, for $p \in [1, \infty]$.
We will show how this setting of *Banach Gelfand triples* resp. *rigged Hilbert spaces* allows to provide a conceptual appealing approach to most classical parts of Fourier analysis. In contrast to the Schwartz theory of tempered distributions it is expected that the mathematical tools can be also explained in more detail to engineers and physicists.

Our Aim: Popularizing Banach Gelfand Triples

According to the title I have to first explain what **Banach Gelfand Triples** are, with the specific emphasis on the BGTr $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, arising from the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (as a space of test-functions), alias the *modulation spaces* $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$.

Hence I will describe them, provide a selection of different characterizations (there are *many of them!*) and properties.

Finally I will come to the main part, namely applications or *use of this* (!natural) concept in the framework of **classical analysis**.

The structure of the talk II

Compared to other talks I leave out the main application area of this BGTr, namely so-called *Gabor Analysis* or *Time-Frequency Analysis* (TFA). Specific aspects of this have been described in my talks in the Gabor Analysis section of this conference.

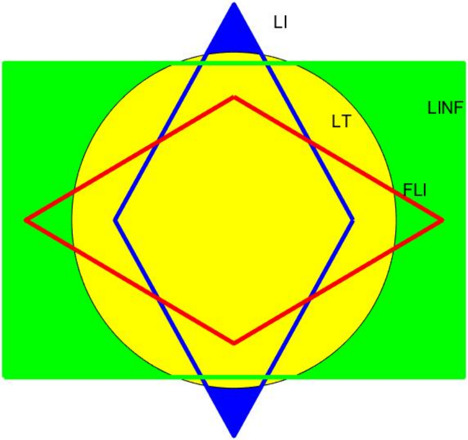
In fact, the *Segal algebra* $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$, which is well defined on LCA (locally compact groups) has been introduced already 1979, at a winter-school in Vienna, organized by H. Reiter (my advisor). This makes the BGTr $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$ useful for applications in *Abstract Harmonic Analysis*.

The Life-Cycle of Tools

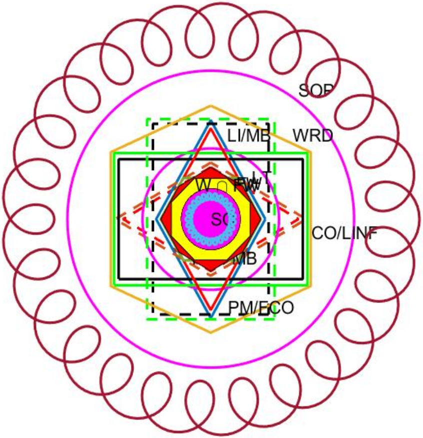
It is psychologically plausible that new tools go through the following development:

1. A *new tool* is developed in order to solve a *concrete problem*;
2. Once it is observed that the new tool can be applied in other (similar or completely different contexts) the "trick" becomes a **method**, and often receives a name;
3. Subsequent analysis of the methods specifies the ingredients and conditions of applications; *optimization*;
4. Exploration of the maximal range of applications;
5. The limitations of the tool are discovered, and so on...
6. *Change of view/problems creates the need for new tools!*
7. Sometimes one then finds many important cases where a simplified version suffices to handle the majority of relevant applications;

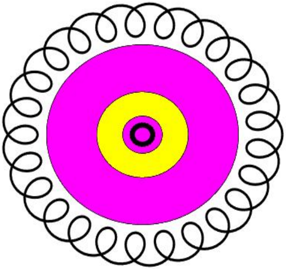
The Standard spaces L^1, L^2, L^∞ and $\mathcal{F}L^1$

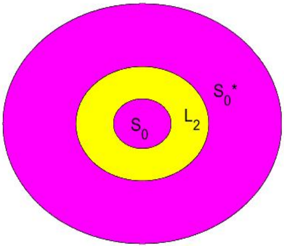


The Universe of Banach Spaces of Tempered Distributions



The Essence: Reducing to the Banach Gelfand Triple





The Minimal Segal Algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

There are many **Segal algebras** in the sense of H. Reiter. By definition they are dense Banach spaces within $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ with isometric and strongly continuous translation, hence (Banach) ideal within $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$, for any LC group (here $G = \mathbb{R}^d$). The intersection of all those Segal algebras is NOT a Segal algebra itself, but if one adds additional pointwise multiplication properties one can find a smallest element in such a family.

For the simple choice $\boldsymbol{A} = \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ the Wiener algebra (which I use to denote by) $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$ is the corresponding Segal algebra, while for $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ it is the mentioned Segal algebra $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
So let us consider a FEW out of the MANY different characterizations of this Banach spaces (of continuous and integrable) functions on $G = \mathbb{R}^d$.

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The key-players for time-frequency analysis

What we need in order to give the first, simple definitions (for the case of $G = \mathbb{R}^d$) are the following ingredients:

Time-shifts and Frequency shifts (II)(modulations)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

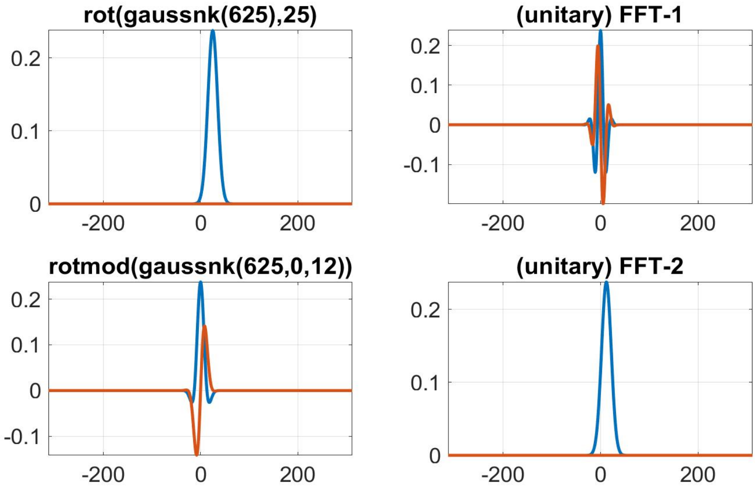
$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

Behavior under Fourier transform (modulation goes to translation)

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t, \omega);$$



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Alternative Function Spaces

There are many situations where the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ is the most appropriate function spaces in the context of Gabor Analysis. We will first describe the space shortly (using a few of its characterizations) and then list a few examples where and why it is a convenient and frequently used tool in the context of Gabor Analysis.

The (meanwhile) classical way to introduce $\boldsymbol{S}_0(\mathbb{R}^d)$[1] is making use of the Short-Time Fourier transform of a signal $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined by

$$V_g(f)(t,\omega) = \langle f, M_\omega T_t g \rangle, \quad t,\omega \in \mathbb{R}^d. \tag{1}$$

Fixing any non-zero Schwartz function, typically $g = g_0$, with $g_0(t) = e^{-\pi t^2}$, one sets:

$$\boldsymbol{S}_0(\mathbb{R}^d) := \{ f \in \boldsymbol{L}^2(\mathbb{R}) \,|\, V_g(f) \in \boldsymbol{L}^1(\mathbb{R}^{2d}) \}. \tag{2}$$

[1] This so-called Segal algebra can be defined over general LCA groups.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Reconstruction Formula

The Moyal formula implies immediately the following inversion formula "in the weak sense"

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda) g d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{3}$$

It follows from **Moyal's formula** (energy preservation):

$$\|V_g(f)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} = \|g\|_2 \|f\|_2, \quad f, g \in \boldsymbol{L}^2. \tag{4}$$

This setting is well known under the name of coherent frames when $g = g_0$, the Gauss function. Its range is the *Fock space*. But for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one has (according F. Weiss) even norm convergence of Riemannian sums, for *any* $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.

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OUTLINE and GOALS

It is the purpose of this talk to highlight a few topics arising from Gabor Analysis, which are also of great use also for other branches of *functional analysis* and application areas such as communication theory or (quantum) physics.

1. Quick overview over the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$; including w^*-convergence;
2. Convolution and pointwise multiplication;
3. System Theory: impulse response and transfer function;
4. Shannon's Sampling Theory ($>>$ CD player);
5. Fourier transform defined on $\boldsymbol{L}^p(\mathbb{R}^d)$;
6. Spectrum and Spectral Analysis Problem;
7. Kernel Theorems; $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \equiv \boldsymbol{S}_0'(\mathbb{R}^{2d})$;
8. Soft transitions and w^*-convergence.

Usual Approach to Fourier Analysis

Obviously Fourier Analysis is a very classical field, about **200 years** old (J.B.Fourier published his seminal paper in 1822).
Since the arrival of **Lebesgue integration** in the early 20th century the setting appears to be quite well understood:
Most books in the field follow a very similar pattern:

Fourier series $>>$ Fourier transforms $>>$ *FFT*

Sometimes it is argued, that the FT can be *even extended to generalized functions, so-called distributions*, e.g. using the **Schwartz theory of tempered distributions**.

Abstract Harmonic Analysis emphasizes the fact that one can do Fourier Analysis over any LCA (locally compact Abelian) group G. which always carries a (unique) Haar measure, and has a dual group $\widehat{G}$. According to Pontryagin $\widehat{\widehat{G}} \equiv G$.

The role of $(L^1(\mathbb{R}^d), \|\cdot\|_1)$

Common to most presentations of the Fourier transform (now we restrict our attention to the Euclidean setting) is the (somehow) natural view-point that the FOURIER TRANSFORM is an INTEGRAL transform, given by the following pair of (mutually inverse) formulas:

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i s \cdot t} \, dt, \quad t, s \in \mathbb{R}^d \tag{5}$$

Here $s \cdot t$ denotes the Euclidean *scalar product* of the two vectors $s, t \in \mathbb{R}^d$. The inverse Fourier transform then takes the form:

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{6}$$

Technical Issues

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, which is *not* satifsied for arbitrary functions $f \in \boldsymbol{L}^1(\mathbb{R}^d)$. In the general case ($f \in \boldsymbol{L}^1(\mathbb{R}^d)$) one can obtain f from $\hat{f}$ using classical summability methods, convergent in the $\boldsymbol{L}^1$-norm. (cf. for example Chap. 1 of [5]).
One often speaks of *Fourier analysis* being the first step, telling us how much energy of f is concentrated at a given frequency ω (namely $|\hat{f}(\omega)|^2$, and the Fourier inversion as a method to build f from the pure frequencies (we talk of *Fourier synthesis*).
Unfortunately (unlike the case of Fourier series and periodic functions) the individual terms, namely the pure frequencies, do *not* to the spaces where f lives ($\boldsymbol{L}^1(\mathbb{R}^d)$ or $\boldsymbol{L}^2(\mathbb{R}^d)$).

Convolution Theorem

One of the (proclaimed) important properties of the Fourier transform is the so-called convolution theorem, i.e. the fact that it turns the *somewhat mysterious operation called CONVOLUTION* into ordinary pointwise multiplication, i.e. we have

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}^d). \tag{7}$$

Again it appears most natural to require that two functions to be convolved should both belong to $\boldsymbol{L}^1(\mathbb{R}^d)$, because this ensures that convolution products can be defined pointwise almost everywhere:

$$f * g(x) := \int_{\mathbb{R}^d} g(x-y) f(y) dy = \int_{\mathbb{R}^d} g(u) f(x-u) du. \tag{8}$$

Convolution and time-invariant linear systems

Aside from probability (cf. above) convolution has its role in *Linear Systems Theory*, in particular in the mathematical description of time-invariant linear systems, meaning linear operators T , mapping signals f to signals $g = T(f)$, with *time-invariance*:

$$T \circ T_x = T_x \circ T, \quad \forall x \in G.$$

A non-trivial, although quite plausible result is the following one

Theorem (hgfei)

Any bounded and translation invariant operator from $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ *into itself (so-called BIBOS system) which commutes with translation is a moving average by some bounded measure, i.e. by some element in the dual space of* $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$*. In fact,* $\mu(f) = Tf(0)$ *describes the system, given by*

Convolution and Fourier Stieltjes transforms

The fact, that the collection of all TILS is not only a Banach space, but also a Banach algebra under composition of operators (with the operator norm) allows to transfer this composition rule to the *generating measures*.

One can shown, that $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is then a Banach algebra with respect to convolution imposed in this way, containing $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ as a closed, translation-invariant ideal.

The collection of *characters*, i.e. the functions χ_s are the *joint eigenvectors* to all these operators, among them the translation operators. The Fourier transform extends to all of these characters, and is then often called Fourier Stieltjes transform, again with

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2},$$

for $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$ (Convolution Theorem for measures).

Consequences an the Transfer Function

This approach to convolution can be carried out *without the use of measures theory* (!), details can be found in my course notes.

In an engineering terminology the measure μ describing the linear system T via $T(f) = \mu * f$ is the *impulse response of the system*. It can be obtained (!proof) as a w^*-limit of input functions tending to the Dirac measure, e.g. compressed, normalized (in the $\boldsymbol{L}^1$-sense) rectungular pulses.

The Fourier (Stieltjes) transform of the system T is know as the *transfer function* of the system T, and it is characterized by the eigen-vector property:

$$\mu(\chi_s) = \hat{\mu}(s)\chi_s, \quad s \in \mathbb{R}.$$

Plancherel's Theorem

Admittedly the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ also plays a very important role for Fourier analysis, thanks to the **Plancherel Theorem**, telling us (in a sloppy way), that the Fourier transform can be viewed as a *unitary automorphism* of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, i.e. a mapping which preserves the $\boldsymbol{L}^2$-norm (and hence scalar products between elements $\boldsymbol{L}^2(\mathbb{R}^d)$).

$$\|\widehat{f}\|_2 = \|f\|_2, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{9}$$

According to *engineering terminology* the Fourier transform is an *energy preserving* linear transformation.

When we have to *prove this theorem* on resorts to a verification of (9) for $f \in \boldsymbol{L}^1 \cap \boldsymbol{L}^2(\mathbb{R}^d)$ and then extends the integral transform in an abstract way to all of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which is then shown to be a unitary automorphism.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ (hence w^*-dense there) is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is a unitary isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

frametitleBanach Gelfand Triples

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.

For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{10}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies $\chi_\omega(t) = exp(2\pi i \omega t)$ onto the corresponding point measures δ_ω.
Vice versa, (even non-harmonic) trigonometric polynomials are those elements of $\boldsymbol{S}_0'(\mathbb{R}^d)$ which have finite support on the Fourier transform side.

The Problem of Fourier Inversion I

To know that an $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ has a continuous Fourier transform $\widehat{f} \in \boldsymbol{C}_0(\mathbb{R}^d)$ is nice, and also to know that the Fourier transform is linear and *injective*, with a dense, but proper range named $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}) \hookrightarrow (\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ (Lemma of Riemann-Lebesgue), but *how can one recover the function* from its Fourier transform for general $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ (knowing that it may happen that $\widehat{f} \notin \boldsymbol{L}^1(\mathbb{R}^d)$!!)
In this situation a simple abstract result helps:

Lemma

Given $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $h(0) = 1$, then one has:
$f_\rho : s \mapsto h(\rho s) \cdot \widehat{f}(s)$ is in $\boldsymbol{S}_0(\mathbb{R}^d)$ for any $\rho > 0$, and therefore the inverse Fourier transform is well defined in the pointwise sense (even using Riemann integration).
Furthermore $\mathcal{F}^{-1} f_\rho \to f$ in $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$, for $\rho \to 0$.

The Problem of Fourier Inversion II

Arguments supporting the above lemma:

- First of all we find that the dilation invariance of $\boldsymbol{S}_0(\mathbb{R}^d)$ implies that $s \mapsto h(\rho s)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ as well;
- Since $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$ (with corresponding norm estimates), one has on the Fourier transform side

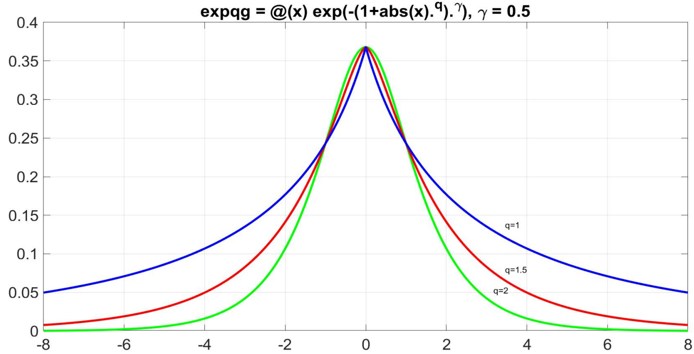
$$\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d).$$

- Finally $g = \mathcal{F}^{-1}(h)$ satisfies $\int_{\mathbb{R}^d} g(t)dt = 1$, and hence one has a bounded approximate unit (within $\boldsymbol{S}_0(\mathbb{R}^d)$) convolved with f (on the time-side), hence norm convergence in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

The same reasoning even implies that one has convergence in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for $1 \leq p \leq 2$, because one finds that even $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{L}^q(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$.

Classical Summability Kernels ARE in $S_0(\mathbb{R}^d)$

The above lemma is of course only useful if one can check that concrete kernels belong to $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$. For the one-dimensional case this has been investigated by F. Weisz.



Why should we care about convolutions?

In engineering, especially in communication theory, the *theory of time-invariant channels*, i.e. operators which are linear and commute with translations (motivated by the time-invariance of physical laws!), also called TILS (time-invariant linear systems) is one of the cornerstones. The collection of all these operators form a commutative algebra, with the pure frequencies as common eigenvalues.

Still in an engineering terminology: Every such system T is a *moving average* or (equivalently) a *convolution operator*, described by the *impulse response* of the system, or alternatively, it can be described as a *Fourier multiplier*, being understood as a pointwise multiplier on the Fourier transform side, by the so-called *transfer function*.

The concrete case of BIBOs systems

From my point of view the most natural setting is that of so-called *BIBOS*, bounded-input-bounded-output systems, or more precisely the bounded linear operators on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ which commute with translations.
It is not hard to show that they are exactly the convolution operators which are arising from a given linear functional $\mu \in \boldsymbol{M}_b(\mathbb{R}^d) := \left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$.
Given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ we can define a convolution operator via

$$\mu * f(x) = \mu(T_x f^\checkmark), \quad x \in \mathbb{R}^d, \tag{11}$$

where $f^\checkmark(x) = f(-x), x \in \mathbb{R}^d$.
Any TILS on $\boldsymbol{C}_0(\mathbb{R}^d)$ is if this form: Given T one finds μ by the rule $\mu(f) := [T(f^\checkmark)](0)$, $f \in \boldsymbol{C}_0(\mathbb{R}^d)$.

More general "multipliers"

According to R. Larsen (his book on the multiplier problem appeared in 1972) it is meaningful to ask for a characterization of the space of "multipliers", i.e. all linear, bounded operators from one translation invariant Banach space into another one, which commute with translations. You may think of such operators from $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ to $(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q)$, for two values $p, q \in [1, \infty]$. Using the concept of *quasi-measures* introduced by G. Gaudry he demonstrates that each such operator is a convolution operator by some quasi-measure, which also has an alternative description as a Fourier multiplier. The drawback of the concept of quasi-measures is the fact, that they do not involve any global condition and thus a general quasi-measure does NOT have a well defined Fourier transform. So the expected claim that the transfer function is the FT of the impulse response (and vice versa) cannot be formulated in that context.

Multipliers between L^p-spaces

The key result for multipliers in the context of $\boldsymbol{S}_0(\mathbb{R}^d)$ can be summarized as follows:

Theorem

Any bounded linear operator from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ which commutes with translations can be described by the convolution with a uniquely $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$, i.e. via

$$Tf(x) = \sigma(T_x[f^{\checkmark}]), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Thus in fact, T maps $\boldsymbol{S}_0(\mathbb{R}^d)$ even into $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$, with norm equivalence between the operator norm of the operator T and the functional norm (in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$) of σ.
If σ is a regular distribution, induced by some test function $h \in \boldsymbol{S}_0(\mathbb{R}^d)$, then we even have that $f \mapsto h * f$ (equivalently given in a pointwise sense) maps $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ into $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$.

Why Linear Functional Analysis

1. When dealing with continuous variables most of the function spaces $\boldsymbol{V}$ that we can think of (even the vector space of all polynomials over $\mathbb{R}$!) are NOT FINITE dimensional;
2. Hence we have to talk about limits of finite linear combinations, i.e. convergence in some *norm* or at least *topology*, i.e. we need (w.l.g.) *Banach spaces* and *topological vector spaces*;
3. Since we cannot describe everything with any fixed basis we have to work with the family of all possible coordinate systems for all possible finite-dimensional subspaces, so in fact with the dual space (constituted by *all bounded linear functionals* on the given space.

Principles of Functional Analysis

1. Theory of Hilbert spaces, orthonormal bases;
2. The Hahn-Banach Theorem, *reflexivity*,
3. w^*-convergence; the Banach-Alaoglou Principles
4. The Closed-Graph Theorem, Open Mapping Theorem;
5. Banach Steinhaus Theorem

Relation to Fourier Analysis

1. The Hilbert space $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$, with Fourier series;
2. The spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, wich are reflexive for $1 < p < \infty$;
3. Existence of solutions in the weak sense (PDE);
4. Automatic continuous embeddings, uniqueness of norms;
5. Convergence of sequences of functionals;
6. Boundedness of maximal operator and pointwise a.e. convergence;

Tools arising from Fourier Analysis

1. **Lebesgue's theory** of the integral is still at the basis of Fourier analysis *when viewed as an integral transform*, with e.g. questions about pointwise convergence and $\boldsymbol{L}^p$-spaces;
2. *Calderon-Zygmund theory* of singular integrals brought up the importance of *BMO* and the *real Hardy spaces*, with their atomic characterization and *maximal functions*;
3. **interpolation theory**, e.g. for *Hausdorff-Young*;
4. Study of differentiability, singular integrals etc. let to the *theory of function spaces* (E. Stein, J. Peetre, H. Triebel).

Distribution Theory

One of the big new tools in the second half of the 20th century was certainly the theory of **tempered distributions** introduced by Laurent Schwartz in the 50th.

1. It gives a (generalized) Fourier transform to many objects which did not "have a FT" so far;
2. It provides a basis for the solution of PDEs (Hörmander);
3. It makes use of **nuclear Frechet spaces**, e.g. in order to prove the *kernel theorem*;

It thus also had a strong influence of the systematic development of the theory of topological vector spaces spaces (J. Horvath (1966), F. Treves (1967), H. Schaefer (1971)).

Distribution Theory and Fourier Analysis

In the context of tempered distributions many things became "kind of simple". Instead of pointwise convergence on can talk about distributional convergence, most operations (including the Fourier transform) are indeed continuous, and since the convergence is relatively weak the space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ of test functions is dense in $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, hence all the operations can be viewed as the natural extension of the classical concepts.
More pratically speaking, using *duality theory* one can define

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d),$$

but alternatively one could take any sequence h_n in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ convergent (distributionally) to σ and define

$$\widehat{\sigma} = \lim_{n\to\infty} \widehat{h_n}.$$

Another Analogy to Schwartz Gelfand Triple

One can regularize distributions from $\boldsymbol{S}_0'(\mathbb{R}^d)$ using *Wiener amalgam* convolution and pointwise multiplier results:

$$\boldsymbol{S}_0 \cdot (\boldsymbol{S}_0' * \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0, \quad \boldsymbol{S}_0 * (\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0 \tag{12}$$

This means, convolution of a *mild distribution* $\sigma \in \boldsymbol{S}_0'$ with some *test function* gives a function which is locally in the Fourier algebra and has enough "smoothness" so that it becomes a pointwise multiplier of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
On the other hand, multiplying $\sigma \in \boldsymbol{S}_0'$ by some $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ provides enough decay (in the sense of global summability in an ℓ^1-sense) so that subsequent convolution by $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ produces again test functions.

Absolutely Convergent Fourier Series

In his studies Norbert Wiener considered the Banach algebra $(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}})$ of absolutely convergent Fourier series. It was one of the early **Banach algebras**, with *Wiener's inversion Theorem* being an important first example.

Later on it was natural to study the dual space, which of course contains the dual space of $(\boldsymbol{C}(\mathbb{T}), \|\cdot\|_\infty)$, which by the Riesz representation theorem can be identified with the bounded (regular Borel) measures on the torus it was natural to call these functions *pseudo-measures*.

Since $\boldsymbol{A}(\mathbb{T})$ can be identified with $\boldsymbol{L}^1(\mathbb{T})$ (viewed as subspaces of $\boldsymbol{L}^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ respectively), it is natural to expect (and prove distributionally) that $\boldsymbol{PM}(\mathbb{T})$ is isomorphic to $\ell^\infty(\mathbb{Z})$ via the (extended) Fourier transform.

Further good properties

The case $a \cdot b < 1$ has further good properties:

1. The minimal norm solution in $\ell^2(\mathbb{Z}^2)$ depends linearly and continuously on f in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.
2. For $f \in \boldsymbol{\mathcal{S}}(\mathbb{R})$ the Gabor series is even convergent in the topology of the Schwartz space (with rapidly decreasing coefficients);
3. It is also possible to replace the Gauss function by another Gauss-like function h (for each fixed a, b with $ab < 1$) such that one has the following[2] *quasi-orthogonal* expansion:

$$f = \sum_{k,n} \langle f, h_{k,n} \rangle h_{k,n}, \quad \in \boldsymbol{L}^2(\mathbb{R}) \tag{13}$$

with unconditional convergence in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.

[2] !ad hoc terminology!

Natural properties

This observation also implies that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow \left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for any $p \geq 1$, or any other function/distribution space on $\mathbb{R}^d$ with isometric TF-shifts.
The *minimality* of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ gives, on the other hand, the continuous embedding of any $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$.

An emerging theory of *Fourier standard spaces* is on the way. They satisfy

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d),$$

and some further conditions, such as

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subset \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subset \boldsymbol{B}.$$

Atomic Decompositions

The minimality implies (inspired by the atomic characterization of *real Hardy spaces* the following result:

Theorem

Given any non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *one has*

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{ \sum_{k \geq 1} c_k \pi(\lambda_k) g, \quad \textit{with} \sum_{k \geq 1} |c_k| < \infty \}.$$

This results also implies that the space is invariant under dilation, rotation and even fractional Fourier transforms.
A long list of sufficient conditions ensures that among others all classical summability kernels belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.

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The Banach Gelfand Triple

Combined with the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ (in the middle) the three spaces $\boldsymbol{S}_0(\mathbb{R}^d)$, $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ form a so-called **Banach Gelfand Triple** or *rigged Hilbert space*.
Such a BGTr is constituted by a Banach space $((\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ in our case) which is continuously and densely embedded into a Hilbert spaces $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$ in our case) and such that this itself is embedded continuously into the dual Banach space (norm continuous, but only w^*-dense).
A **morphism** of Banach Gelfand triples respects each of the three layers, but is also w^*-w^*--continous at the outer level.

Fourier transform

The Fourier transform is a prototype of a *unitary BGTr-automorphosm* for the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
It can be formulated for any LCA group, and reduces in the classical case of $G = \mathbb{T}$ to a well known result, making use of $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$, the algebra of absolutely convergent Fourier series and its dual, the space $\boldsymbol{PM}$ of pseudo-measures.
The classical Fourier transform, establishing a unitary isomorphism (Parseval's identity) from $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ to $\ell^2(\mathbb{Z})$ extends to a BGTr isomorphism between $(\boldsymbol{A}(\mathbb{T}), \boldsymbol{L}^2(\mathbb{T}), \boldsymbol{PM}(\mathbb{T}))$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

Spectrum of a Bounded Function

The fact that $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ for any $1 \leq p \leq \infty$ implies that one go beyond Hausdorff-Young, which shows that for $1 \leq p \leq 2$ on has $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{L}^q(\mathbb{R}^d)$ (with $1/p + 1/q = 1$). Even for $p > 2$ there is a Fourier transform (locally in $\boldsymbol{PM} = \mathcal{F}\boldsymbol{L}^\infty$).
The (known) statement that $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ is of course much weaker than the claim $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d)$.
This fact allows to directly define the *spectrum* of $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$ using the general definiton:

$$\mathrm{spec}(\sigma) := \mathrm{supp}(\widehat{\sigma}), \quad \sigma \in \boldsymbol{S}_0'.$$

If $\mathrm{supp}(\sigma)$ is a finite set it can be shown to be just a finite sum of Dirac measures, i.e. $\sigma = \sum_{k=1}^K c_k \delta_{x_k}$.

Fourier Analysis and Synthesis

Putting ourselves in the setting of $\boldsymbol{S}_0'(\mathbb{R}^d)$ (which contains all of $L^\infty(\mathbb{R}^d)$, hence $\boldsymbol{C}_b(\mathbb{R}^d)$ and even more special the *pure frequencies*, i.e. the exponential functions $\chi_s(t) = exp(2\pi i s \cdot t)$, for $t, s \in \mathbb{R}^d$), we may ask ourselves the following two questions:

1. What are the *pure frequencies* which can be "filtered out of the signal" ($h \in \boldsymbol{C}_b(\mathbb{R}^d)$ or $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$);
2. Can one resynthesize the signal from those pure frequencies (as it is obvious of the case of classical Fourier series expansions of periodic functions);

Distributional convergence

It is clear from the prototypical BGTr $(\ell^1, \ell^2, \ell^\infty)$ that the Hilbert space (here ℓ^2) is not always dense in the dual of the Banach space, here ℓ^∞. However, every element in ℓ^∞ can be approximated in the *coordinate-wise* sense by a bounded sequency in the closed subspace cosp, which is the closure of ℓ^2 within $(\ell^\infty, \|\cdot\|_\infty)$.
Note that the situation is different e.g. in the context of Sobolev space, where for $s > 0$ one has

$$\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{H}_{-s}(\mathbb{R}^d)$$

with dense embeddings.
We will be mostly concerned with sequences which are w^*-convergent. Over $\mathbb{R}^d$ this simplification is fully justified, because $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a separable Banach space.

Distributional, i.e. w^*-convergence in $S_0'(\mathbb{R}^d)$

So let us first recall what on the one hand norm-convergence and w^*-convergence within $S_0'(\mathbb{R}^d)$ in concrete terms, thanks to to the atomic characterization of $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$:

Lemma

Let $0 \neq g \in S_0(\mathbb{R}^d)$ be given. Then one has: Given $\sigma \in \mathcal{S}'(\mathbb{R}^d)$, then belongs to $S_0'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ if and only if the (continuous) function $V_g(\sigma) = \sigma(\overline{g_\lambda})$ is bounded, and:

- $\sup_{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g(\sigma)(\lambda)|$ *defines an equivalent norm on* $S_0'(\mathbb{R}^d)$.
- $\sigma_0 \in S_0'(\mathbb{R}^d)$ *is the w^*-limit of a sequence $(\sigma_n)_{n \geq 1}$ if and only if one has (uniformly over compacts):*

$$\lim_{n \to \infty} V_g(\sigma_n)(\lambda) = V_g(\sigma_0)(\lambda).$$

w^*-density of discrete measures I

In order to demonstrate the possibility of approximating a given distribution $\sigma \in \boldsymbol{S}_0'$ in the w^*-sense by discrete measures we have to look at (quasi)-interpolation operators acting on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
It is clear that by compressing the sampling grid one has uniform convergence for $f \in \boldsymbol{C}_0(\mathbb{R})$, but in fact on can show the non-trivial fact that one has convergence in $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$ for $f \in \boldsymbol{S}_0(\mathbb{R})$ (note that $\Delta \in \boldsymbol{S}_0(\mathbb{R})$ and the sampling sequence $(f(\alpha n))_{n\in\mathbb{Z}}$ is in $\ell^1(\mathbb{Z})$. Hence the PLI, which is an absolutely sum of the form $h = \sum_{n\in\mathbb{Z}} f(\alpha n) T_{\alpha n}\Delta_\alpha$ belongs to $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and the question of convergence is at least meaningful. Similar arguments hold for e.g. cubic B-splines (which also are in $\boldsymbol{S}_0(\mathbb{R})$ and form a BUPU $\Psi = (\psi_n)_{n\in\mathbb{Z}}$.
By taking tensor products one can of course find multi-variate BUPUs, with small support and $\sum_{n\in\mathbb{Z}^d} \psi_n(x) \equiv 1$.

w^*-density of discrete measures II

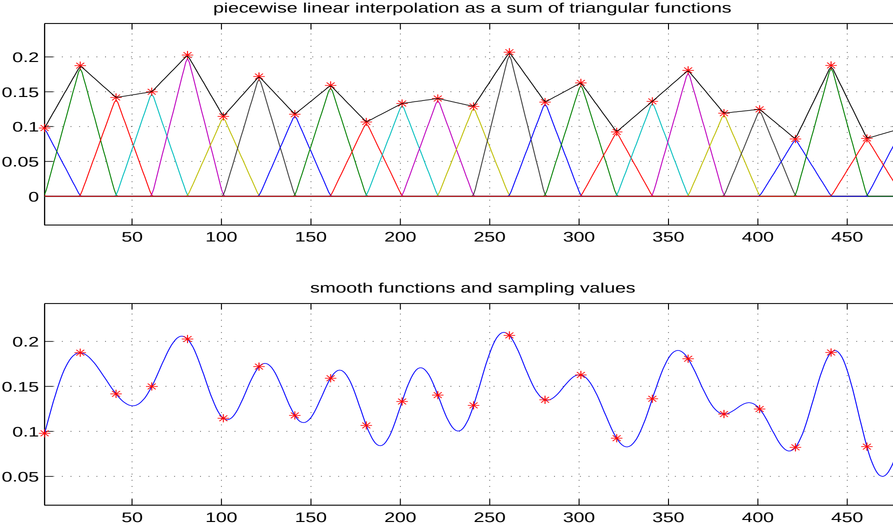
Consequently, given $\sigma \in \boldsymbol{S}_0'$, the action of the adjoint operators converge in the w^*-sense to σ. We just have to verify that the adjoint action can be desribed by the operator

$$D_\Psi^\alpha(\sigma) = \sum_{n \in \mathbb{Z}^d} \sigma(\psi_n^\alpha)\delta_{\alpha n}.$$

This family is also uniformly bounded in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (for $\alpha \to 0$).

w^*-density of discrete measures III

w^*-density of discrete measures IV



A Cauchy approach to distributions

The simple setting of Banach spaces (resp. dual spaces) allows also to work with a very simple, alternative way to introduce the dual space, namely to define

1. weak Cauchy-sequences $(h_n)_{n\geq 1}$ of test functions from $\boldsymbol{S}_0(\mathbb{R}^d)$, in the sense that $(\langle h, f_n\rangle)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{C}$ for any $h \in \boldsymbol{S}_0(\mathbb{R}^d)$;
2. then define equivalence classes of such sequences;
3. then define the norm of such a class (it is finite by Banach-Steinhaus).

It is then not difficult to show that this is just another way (maybe more intuitive to engineers) to describe $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$!

Spectral Analysis

So in the terminology of "mild distributions" one *can prove* and has the follwing statement:

Lemma

An element $s \in \widehat{\mathbb{R}^d}$ in the frequency domain belongs to the spectrum of $h \in L^\infty(\mathbb{R}^d)$ resp. to $\mathrm{supp}(\widehat{h})$ if and only if the corresponding pure frequency χ_s can be obtained as the w^*-limit of a sequence of convolution products $h_n := h * f_n$, for f_n being chosen suitably in $\boldsymbol{S}_0(\mathbb{R}^d)$.

Whenever $h \in \boldsymbol{L}^1 \cap L^\infty(\mathbb{R}^d)$ we know that $\widehat{h}$ is a continuous function. Then it is easy to filter χ_s out of h if $\widehat{h}(s) \neq 0$ (just take modulated Fejer kernels). When s belongs to the closure of this set one has to be a bit more careful!

Spectral Synthesis

Now of course the question of *spectral synthesis*, i.e. the reconstruction of σ (or even $h \in L^\infty(\mathbb{R}^d)$) from the pure frequencies which have been "found within" h appears to be like a natural task.

If we ask the question for the space of bounded measures $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ which we define as $\left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$ then there is also a well-defined support, and one can show (using properties of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$!) that any bounded measure can be described as the w^*-limit of a sequence of finite (discrete) measures of the form $\mu_n = \sum_{k=1}^K c_k^n \delta_{x_k^n}$, with $x_k^n \in \mathrm{supp}(\mu)$.

That a similar property is NOT valid for general $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$, and in fact not even for $h \in L^\infty(\mathbb{R}^d)$, for $d \geq 3$, is one of the remarkable facts of Fourier Analysis.

The Classical Description

I cannot go into the details of the description of the **spectrum** of $h \in L^\infty(\mathbb{R}^d)$ as given in the book of Reiter, but just want to recall some building blocks there.
First, $h \in L^\infty(\mathbb{R}^d)$ is considered as a convolution operator from $\boldsymbol{L}^1(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$ (in fact $\boldsymbol{C}_b(\mathbb{R}^d)$). This convolution operator has a well defined kernel (null-space), which in fact is a (translation-invariant) and closed *ideal* $I = I(h)$ in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.
The set of all points in the Fourier domain where *some* $f \in I$ has non-vanishing FT is called the **cospectrum** of I. The complement of this (open) set is then defined as the *spectrum* of h.
It is one of the interesting

The Euclidean Situation

In the Euclidean Situation interesting objects are for example the Dirac trains supported by lattices $\Lambda \lhd \mathbb{R}^d$:

$$\sqcup\!\sqcup := \sum_{\lambda \in \Lambda} \delta_\lambda .$$

These (unbounded, discrete) measures belong to $\boldsymbol{S}_0'(\mathbb{R}^d)$ and thus have a distributional Fourier transform. In fact, the usual argument in distribution theory making use of Poisson's formula (now not only for $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ but for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$!) gives ($\Lambda = \mathbb{Z}^d$):

$$\widehat{\sqcup\!\sqcup} = \sqcup\!\sqcup . \tag{14}$$

This fact allows to prove *Shannon's sampling Theorem* (based on intuitive arguments).

Sampling and Periodization on the FT side

This result is the key to prove **Shannon's Sampling Theorem** which is usually considered as the fundamental fact of digital signal processing (Claude Shannon: 1916 - 2001).

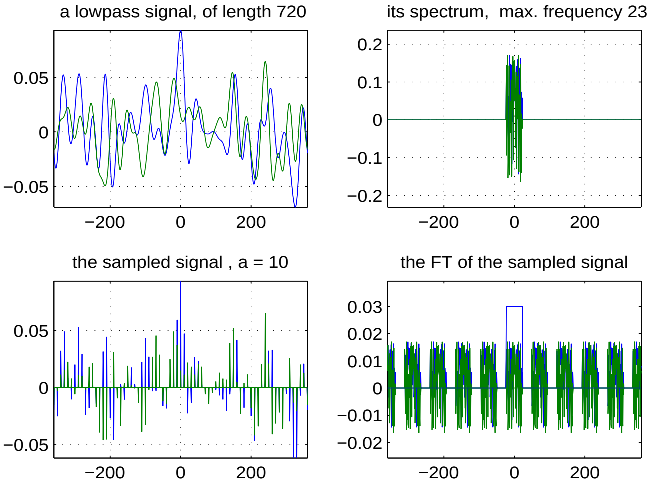
Shannon's Theorem says that one can have **perfect reconstruction** for band-limited functions.

If the so-called *Nyquist criterion* is satisfied (sampling distance small enough), i.e. $\mathrm{supp}(\hat{f}) \subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) g(x - \alpha k), \quad x \in \mathbb{R}^d. \tag{15}$$

For more general sampling lattices the support size conditions have to avoid overlapping of the periodization of the spectrum with respect to the orthogonal lattice $\Lambda^{\perp}$.

A Visual Proof of Shannon's Theorem



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Key aspects of my talk

1. Browse the (long-standing) **history of Fourier Analysis**
2. Show large number of **applications** influencing our life
3. Discussing some of the mathematics behind it
   (take away the touch of mystery?)
4. Describing **time-frequency and Gabor analysis**
5. Suggesting ways to teach Fourier Analysis

Functions, Distributions, Signal Expansions

As a *unifying principle* that allows me to explain the relevant points in the historical development of Fourier Analysis in the last 200 years as well as for a better understanding of how we should teach finally Fourier Analysis in the 21st century I want to focus on the following aspects:

1. What is a function?
2. What does it mean to represent a function on the basis of its Fourier coefficients (e.g. Fourier series expansion, ...)
3. How have these concepts changed over time and what was the effect on the understanding of Fourier Analysis?

NOTE: Recall how the shape of cars is reflecting the available technology and even mathematics of the time (plaster model, CAD, Bezier,...).

Plancherel's Theorem: Unitarity Property of FT

Using the density of $\boldsymbol{L}^1(\mathbb{R}) \cap \boldsymbol{L}^2(\mathbb{R})$ in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ it can be shown that the Fourier transform extends an a natural and unique way to $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$:

Theorem

The Fourier (-Plancherel) transform establishes a unitary automorphism of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, i.e. one has

$$\|f\|_2 = \|\hat{f}\|_2, \quad f \in \boldsymbol{L}^2(\mathbb{R}),$$

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}).$$

In some sense *unitary* transformations of a Hilbert transform is like a change form one ONB to another ONB in $\mathbb{R}^n$.

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the "continuous representation" formula, using $\chi_s(x) = e^{2\pi isx}$, $x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{16}$$

This looks like a perfect orthogonal expansion, but unfortunately the "building blocks" $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

Convolution and the Fourier Transform

Another important fact about the Fourier transform is the so-called **convolution theorem**, i.e. the Fourier transform converts convolution into pointwise multiplication.
Again it is natural to define convolution on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g(x-y)f(y)dy \quad xa.e.; \quad (17)$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}).$$

For positive functions f, g one even has equality. This is relevant for the determination of probability distributions of a sum of *independent* random variables. Assume X has density f and Y has density g then the random variable $X + Y$ has probability density distribution $f * g = g * f$.

Banach algebras

Theorem

Endowed with the bilinear mapping $(f,g) \to f * g$ the Banach space $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ becomes a commutative Banach algebra with respect to convolution.

The **convolution theorem**, usually formulated as the identity

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}), \tag{18}$$

implies

Theorem

The Fourier algebra, defined as $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}) := \{\hat{f} \mid f \in \boldsymbol{L}^1(\mathbb{R})\}$, with the norm $\|\hat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ is a Banach algebra, closed under conjugation, and dense in $\left(\boldsymbol{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$ (continuous functions, vanishing at infinity).

Convolution and time-invariant linear systems

Aside from probability (cf. above) convolution has its role in *Linear Systems Theory*, in particular in the mathematical description of time-invariant linear systems, meaning linear operators T , mapping signals f to signals $g = T(f)$, with *time-invariance*:

$$T \circ T_x = T_x \circ T, \quad \forall x \in G.$$

A non-trivial, although quite plausible result is the following one

Theorem (hgfei)

Any bounded and translation invariant operator from $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ *into itself (so-called BIBOS system) which commutes with translation is a moving average by some bounded measure, i.e. by some element in the dual space of* $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$. *In fact,* $\mu(f) = Tf(0)$ *describes the system, given by*

Convolution and Fourier Stieltjes transforms

The fact, that the collection of all TILS is not only a Banach space, but also a Banach algebra under composition of operators (with the operator norm) allows to transfer this composition rule to the *generating measures*.

One can shown, that $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is then a Banach algebra with respect to convolution imposed in this way, containing $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as a closed, translation-invariant ideal.

The collection of *characters*, i.e. the functions χ_s are the *joint eigenvectors* to all these operators, among them the translation operators. The Fourier transform extends to all of these characters, and is then often called Fourier Stieltjes transform, again with

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2},$$

for $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$ (Convolution Theorem for measures).

Consequences an the Transfer Function

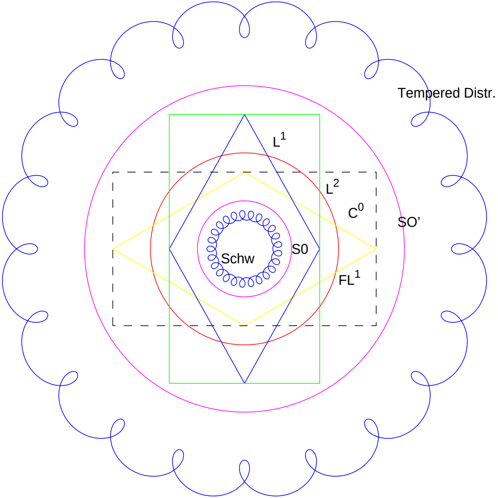
This approach to convolution can be carried out *without the use of measures theory* (!), details can be found in my course notes.

In an engineering terminology the measure μ describing the linear system T via $T(f) = \mu * f$ is the *impulse response of the system*. It can be obtained (!proof) as a w^*-limit of input functions tending to the Dirac measure, e.g. compressed, normalized (in the $\boldsymbol{L}^1$-sense) rectungular pulses.

The Fourier (Stieltjes) transform of the system T is know as the *transfer function* of the system T, and it is characterized by the eigen-vector property:

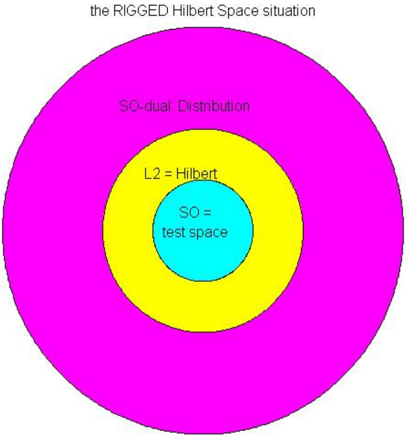
$$\mu(\chi_s) = \hat{\mu}(s)\chi_s, \quad s \in \mathbb{R}.$$

A schematic description: all the spaces



A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as $5/7$ is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

The Banach space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ or $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$, which will serve our purpose. Its dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

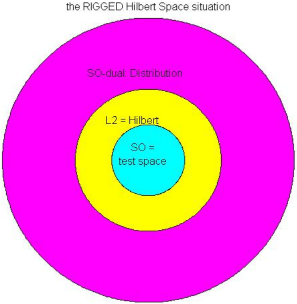
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{19}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The KERNEL THEOREM for $\mathcal{S}(\mathbb{R}^d)$

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{20}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (20) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM: Hilbert Schmidt Operators

The Schwartz Kernel Theorem extends a characterization of $\boldsymbol{L}^2$-kernel-operators of the form to the most general level. It shows that for any "integral kernel" $K(x,y) \in \boldsymbol{L}^2(\mathbb{R}^{2d})$ the following integral operator (continuous analogue of matrix multiplication)

$$T_K f(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x \in \mathbb{R}^d. \tag{21}$$

defines a bounded, in fact compact operator on $\boldsymbol{L}^2(\mathbb{R}^d)$, with singular values in ℓ^2, i.e. a *Hilbert-Schmidt operators*. Altogehter they form a Hilbert space with respect to the *scalar product*

$$\langle T, S \rangle_{\mathcal{HS}} := \mathrm{trace}(TS^*), \tag{22}$$

and corresponding norm $\|T\|_{\mathcal{HS}} := \sqrt{\mathrm{trace}(TT^*)}$.
With this scalar product we have a *unitary isomorphism* between $(\mathcal{HS}, \|\cdot\|_{\mathcal{HS}})$ and $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$.

Extending this situation to $(S_0, L^2, S_0')(\mathbb{R}^{2d})$

If this unitary isomorphism can be extended to a Banach Gelfand Triple isomorphism we have to look what on can say about kernels in $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$.
Of course we look first what one can say for good kernels $K(x,y) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, which (hopefully, and in fact) behave like "continuous matrices". Given such a kernel it is w^*-to-norm continuous from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and it is possible to define $T(\delta_y) \in \boldsymbol{S}_0(\mathbb{R}^d)$. In fact, this one gets (as expected from matrix multiplication)

$$T(\delta_y)(x) = K(x,y)$$

in analogy to the case $T_A(\mathbf{x}) = \boldsymbol{A} * \mathbf{x}$, with

$$\mathbf{a}_{j,k} = \langle T_A(\mathbf{e}_k), \mathbf{e}_j \rangle.$$

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! although $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear space*) One of the corner stones for the kernel theorem is: the tensor-product factorization:

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \tag{23}$$

with equivalence of the corresponding norms.

For $G = \mathbb{R}^d$ it follows readily from the characterization using the atomic decomposition using Gaussians.

The KERNEL THEOREM for S_0 II

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.
This description can be also extended to the Kohn-Nirenberg symbol of an operator (no functions or distributions over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$), or alternatively the Weyl-symbol (in the sense of pseudo-differential operators), and finally the spreading representation.
The symplectic Fourier transform (a further unitary BGTr isomorphism) the transfers the information between the spreading function $\eta(T)$ and the *Kohn-Nirenberg symbol* $\sigma(T)$.

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

In analogy to the matrix case, where the entries of the matrix

$$a_{k,,j} = T(\mathbf{e}_j)_k = \langle T(\mathbf{e}_j), \mathbf{e}_k\rangle$$

we have for $K \in \boldsymbol{S}_0$ the continuous version of this principle:

$$K(x,y) = \delta_x(T(\delta_y), \quad x,y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$*, where the first set is understood as the* w^* *to norm continuous operators from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\mathbb{R}^d)$*, the so-called regularizing operators.*

Motivation for Spreading Representation

Let us know motivate how the BGTr setting can be used to provide a good intutive understanding of the spreading representation of general operators.

As we will see the spreading representation gives us a complete characterization of operators with kernels $K(x,y) \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ with the corresponding spreading symbols in $\eta \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

What are the most simple (non-compact, not HS) operators: the pure TF-shift operators $\pi(\lambda) = M_s T_t$ should correspond to a Dirac at $\lambda = (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

On the other hand one can imagine (and verify) that an operator which can be written as

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} H(\lambda)\pi(\lambda)d\lambda,$$

with $H \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is a operator with kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$.

Spreading function and Kohn-Nirenberg symbol

The following summary has been provided by Götz Pfander (now Univ. Eichstätt):

1. For $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$$

   The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}\,d\omega\Big) f(y)\,dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)\,dy. \end{aligned}$$

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta,u)M_\eta T_{-u} f(x)\,du\,d\eta.$$

   $\widehat{\sigma}$ is called the spreading function of T_σ

Further details concerning Kohn-Nirenberg symbol

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

The kernel $K(x,y)$ can be described as follows:

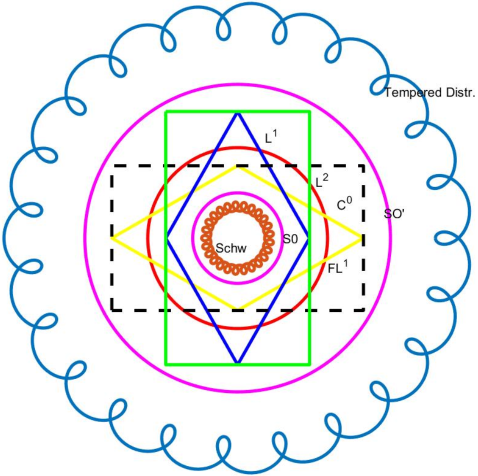
$$K(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x)$$
$$= \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y-x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

Kohn-Nirenberg symbol and spreading function II

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x) f(x-t)\, dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu) f(x-t) e^{2\pi i x\cdot\nu}\, dt\, d\nu$
	$=$
	$\int\int \eta_H(t,\nu) M_\nu T_t f(x), dt\, d\nu\,,$

h

Summarizing the situation: test functions & distributions



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G. Folland: Harmonic Analysis in Phase Space. Princeton University Press, 1989.

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Some further books in the field are in preparation, e.g. on modulation spaces and pseudo-differential operators.
See also www.nuhag.eu/talks.

Thanks!

THANK you for your attention

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