Current Problems in Gabor Analysis

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Talk at ISAAC 2019
AVEIRO, July 29th, 2019

official abstract I

Gabor Analysis is an important branch of time-frequency analysis. It describes signals or distributions through their series expansions, using time-frequency shifted copies of some *Gabor Atom* or window function. A good choice (for the abstract discussion) is a Gaussian atom because it has optimal decay in both the time and the frequency domain. Very often, Gabor analysis is reduced to the question, whether for a given atom a lattice of the form $\Lambda = a\mathbb{Z}b\mathbb{Z}$ allows to generate a Gabor frame resp. allows to expand any $\boldsymbol{L}^2(\mathbb{R}^d)$-function f in a Gabor series, with square summable coefficients.

However, Gabor Analysis has a much wider scope. It allows to characterize a family of function spaces, the so-called modulation spaces, some of which are useful for certain PDEs, while others, like the Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ are useful for

official abstract II

Abstract Harmonic Analysis and Engineering Applications, from mobile communication to classical system theory.
Although we know - after almost 40 years of research in this area - a lot about Gabor Analysis, and many fundamental facts have been obtained in the last decades, there are still many practical and theoretical questions unsolved. In our presentation we plan to address some of these questions.
One of the main topics will be the application of the general idea of "Conceptual Harmonic Analysis" to the domain of Gabor Analysis. Among other we may ask: What is the best fit of a Gabor atom and a corresponding multi-dimensional lattice, given one of the two ingredients. How well can Anti-Wick operators be approximated by Gabor multipliers, or who can one discretize a given operator, starting from the Kohn-Nirenberg or Weyl calculus or the spreading definition, just to mention a few topics.

The structure of the talk AVEIRO19

I plan to talk about a variety of topics, such as

- The relevance and inspiration of MATLAB experiments
- A number of open problems (subjective!)
- The idea of Conceptual Harmonic Analysis,
  i.e. the emphasis of a closer connection between numerical and abstract Harmonic Analysis
- **Questions to be asked!!**

Citations in the early literature

Typical early references to Gabor expansions (mostly the original case at critical density) sound as follows:

- Useful but numerically instable resp. costly;
- Problem with the *Balian-Low principle*;
- Carrying the hope that eventually problems could be overcome (or that they are unavoidable).

Positive aspects of Gabor analysis (compared to wavelets):

1. Possible for any LCA group;
2. For image processing: it allows smooth building blocks
3. Large number of possible applications:
   Gabor expansions, pseudo-differential operators, modulation spaces, THE Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$

Program of questions to be settled

- first: Establish good and practical suggestions for users demonstrating best practice for the use of Gabor expansions;
- care for reliable and efficient numerical code that can be adapted to quite general situations;
- establish theoretical results that allow in combination with numerical explorations to establish new and valid mathematical results, e.g. estimates with optimal numerical constants;

Lena reconstruction at critical density

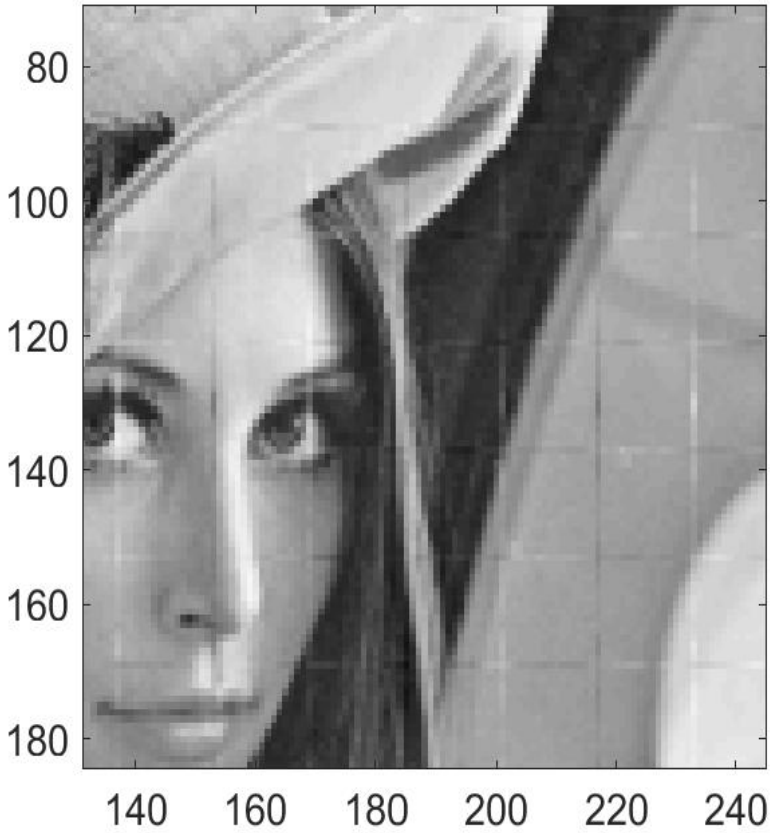

Figure: Reconstruction from critically sampled STFT

JPG Building blocks

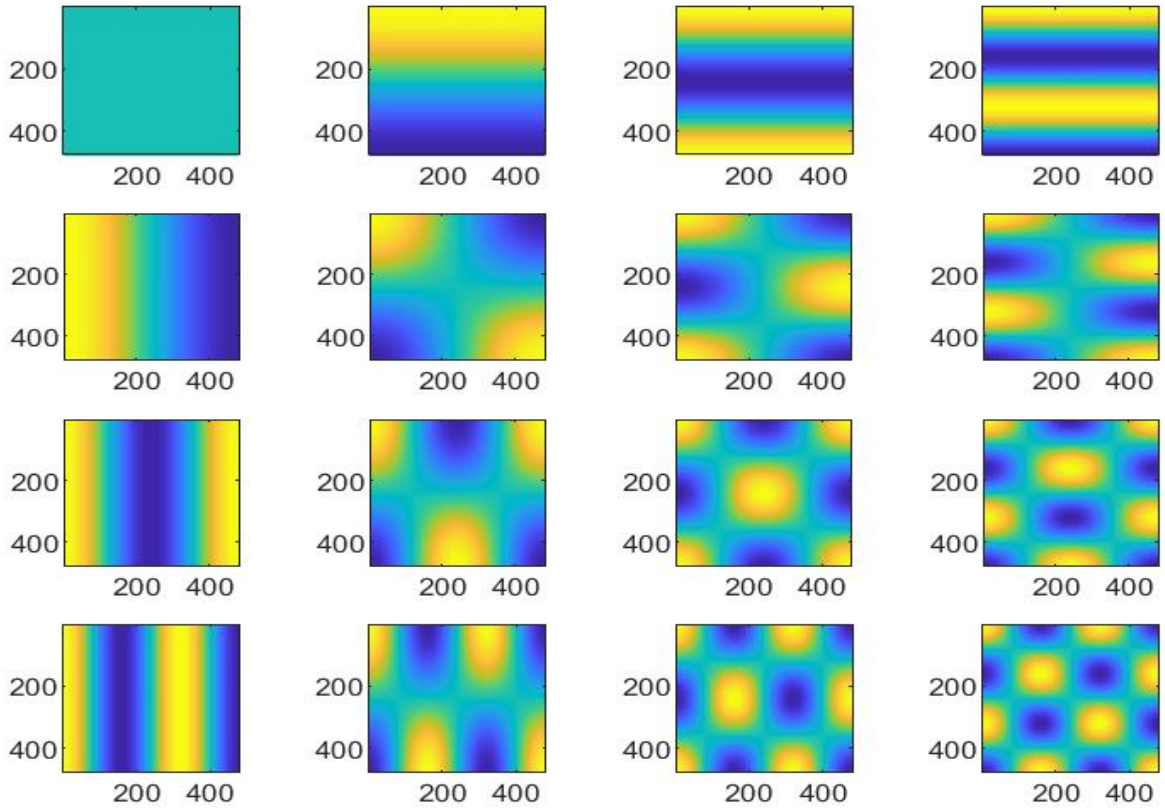

Figure: The first 8 out of 64 building blocks

JPG Building blocks, full catalogue

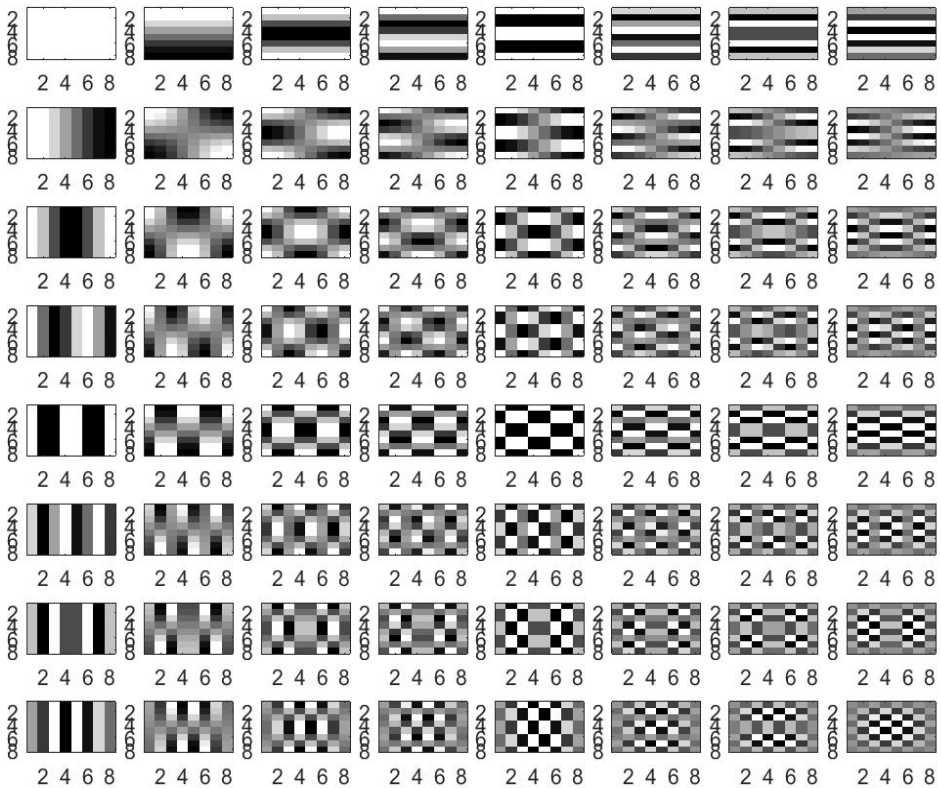

Figure: all 64 building blocks

Current Software

There exists already quite a body of mathematical software, also spread out in the engineering litarature several algorithm, but the most important collection *at least for me* is the LTFAT Toolbox hosted by **ARI (OEAW)** (director: Peter Balazs) and my own collection of MATLAB files developed over 30 years [1].
As a demonstration site for *adaptive Gabor expansions* one can point to the GABORATOR web-page at

`www.gaborator.com`

[1] Partially available at NuHAG, to some extent also a repository of a number of internships financed in the past 20 years.

Collection of Observations

Let me give a number of statements (if you want, a *testimonial of my experience based on uncountable MATLAB experiments*):

1. Much insight can be gained by a simple *inspection* of the situation by *LOOKING at the involved objects*, e.g. Gabor frame matrix, dual Gabor atom, etc.
2. Experiments can *sharpen the intution*, provide stimuli for theoretical statements
3. Avoid that we get lost in purely theoretical and useless speculations
4. Provides a rich reservoir of new and challenging *mathematical questions*

The combination of MATLAB experiments with theoretical developments (e.g. the use of suitable function spaces) has become a work-mode, at last for myself.

Concrete Examples

The insight, that e.g. the problem of dual Gabor atoms can be approximated by a finite setting, using sampling and periodization structures, led naturally to the question:
HOW CAN WE HANDLE the case of lattices with irrational excentricity? It became clear that one has to approximate irrational lattices $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ by rational ones of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$, which led to the study of "varying the lattice constant".
As it is well-known meanwhile (see [4],[6]) one can expect for atoms in the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ that the dual Gabor window depends continuously on the lattice (using the topology of $GL(n, \mathbb{R})$). This is by no means obvious!
Of course such a "continuous dependence" (going by the shape of the dual Gabor atom) was observed experimentally much earlier.

Basic Observations

Two more examples: That there is a lot of structure in the problem can be nowadays observed very easily:
In the early 90s what is now called the *Walnut representation* of the Gabor frame operator (i.e. a description as a sum of multiplication operators combined with shift by multiple of $1/b$ resp. n/b in the discrete setting) become appearent: The Gabor frame matrix has a particular shape: It is zero, except for b side-diagonals, each of which is a-periodic.
Hence it was natural (see various papers and the PhD thesis of Sigang Qiu in Vienna) to store the information about the frame-operator in the format of a small $a \times b$ block matrix.
If one takes the Fourier transform (rowise) of this collection of *basic periods* of the collection of non-trivial side-diagonals (for fixed pair (a, b)) one just comes to the *Janssen represenation* of the Gabor frame operator.

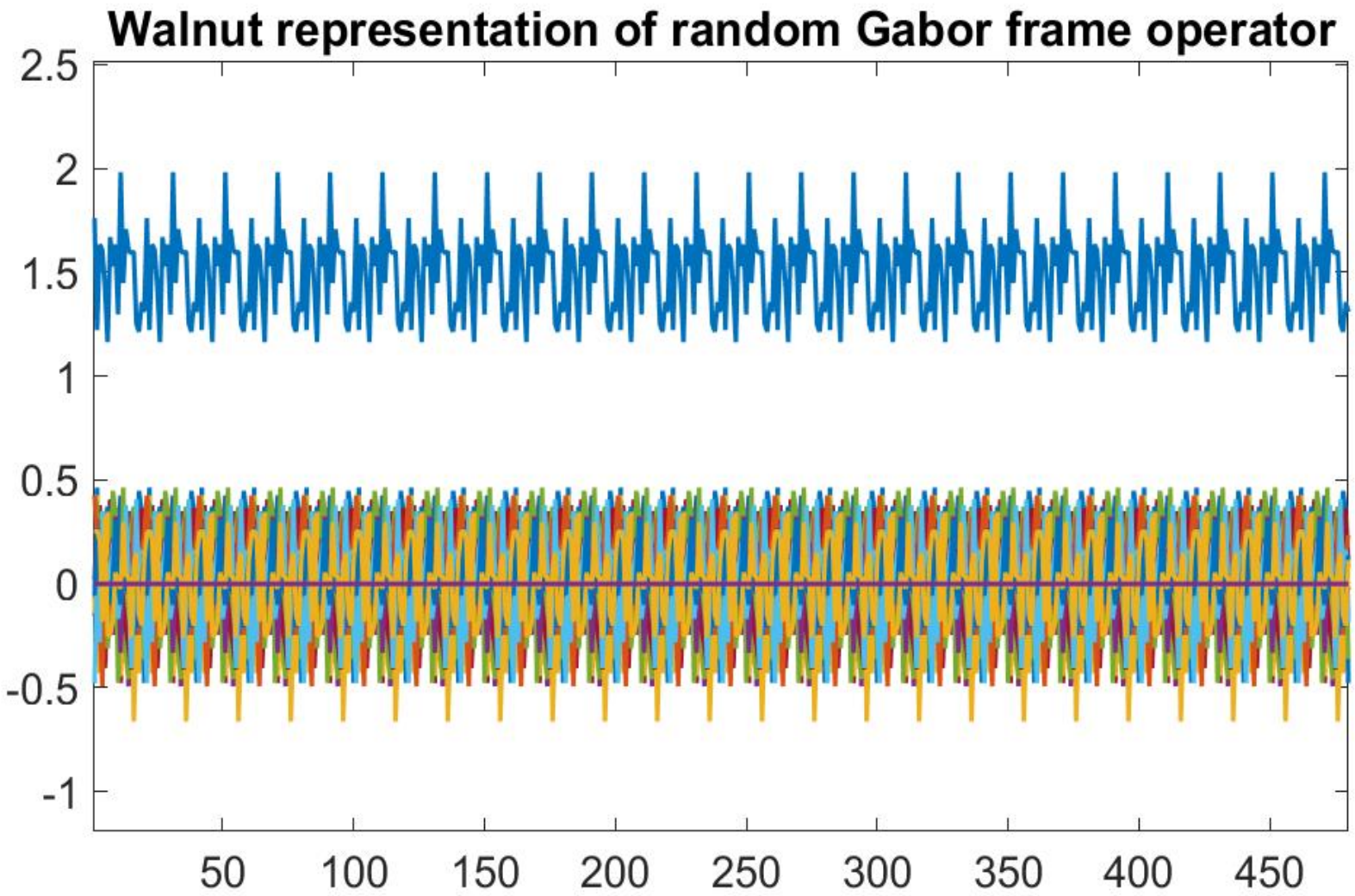

Figure: WalnutRandfr.jpg

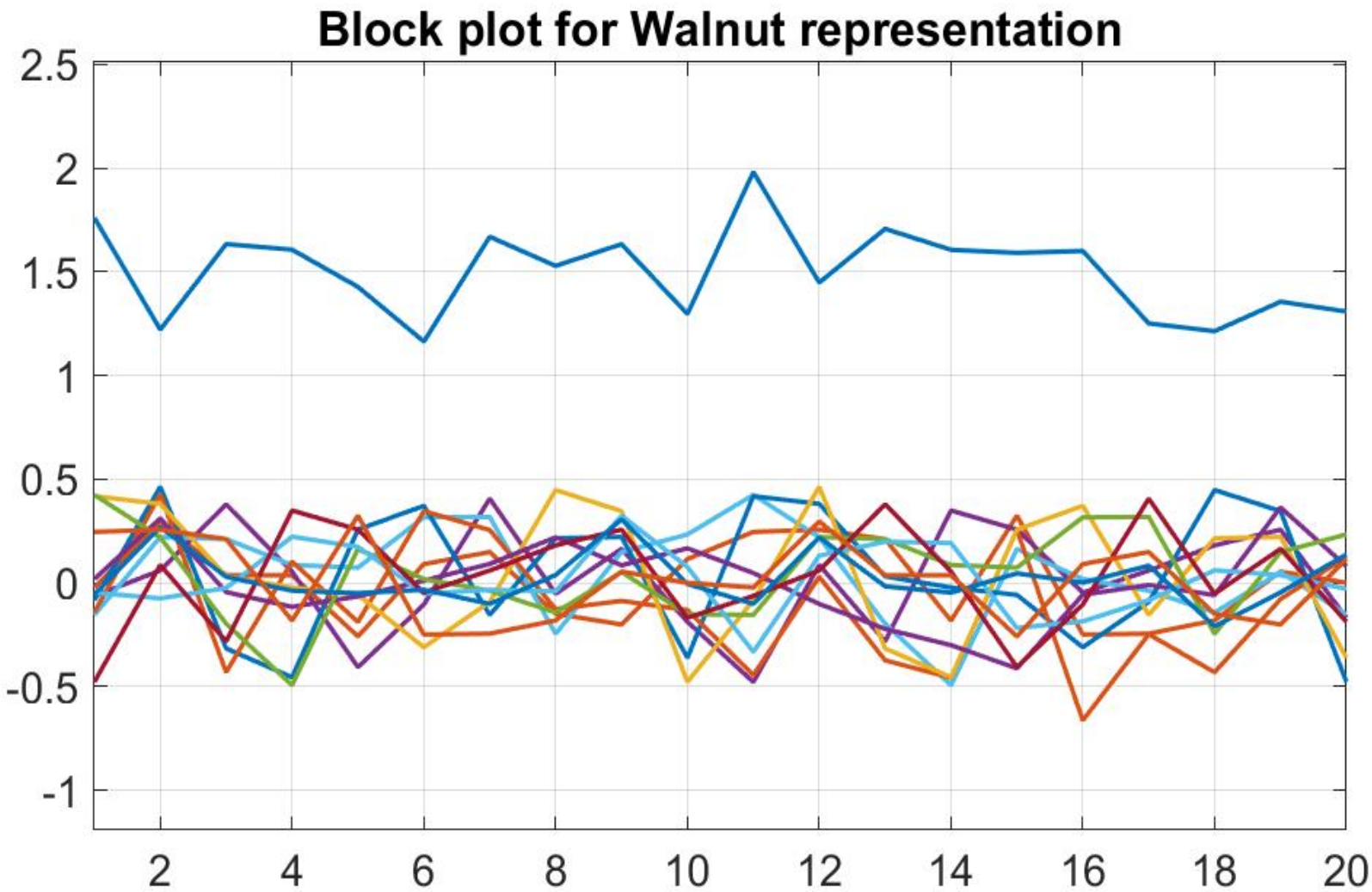

Figure: WalnutRandBL.jpg

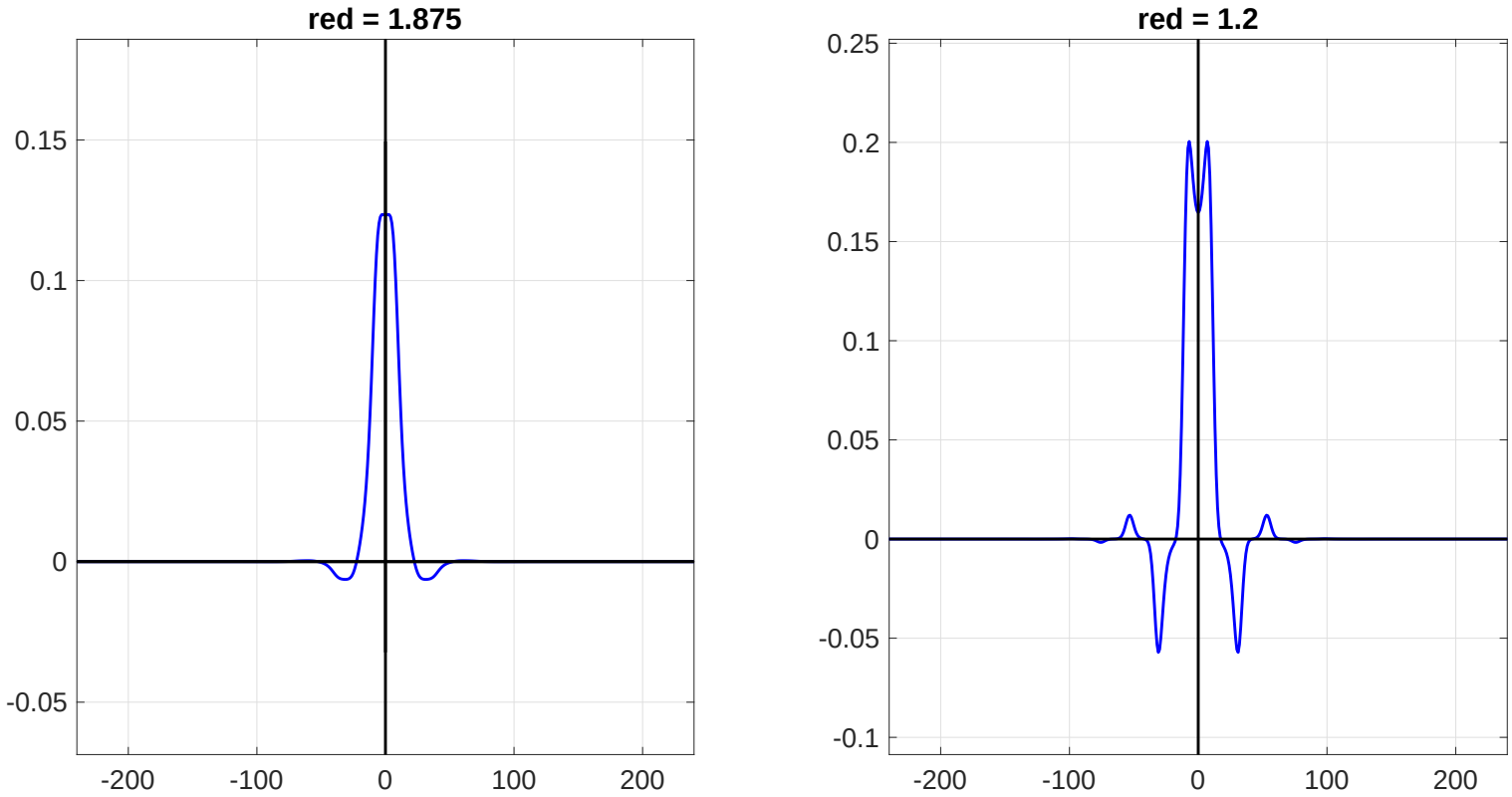

Figure: gaured121875.eps

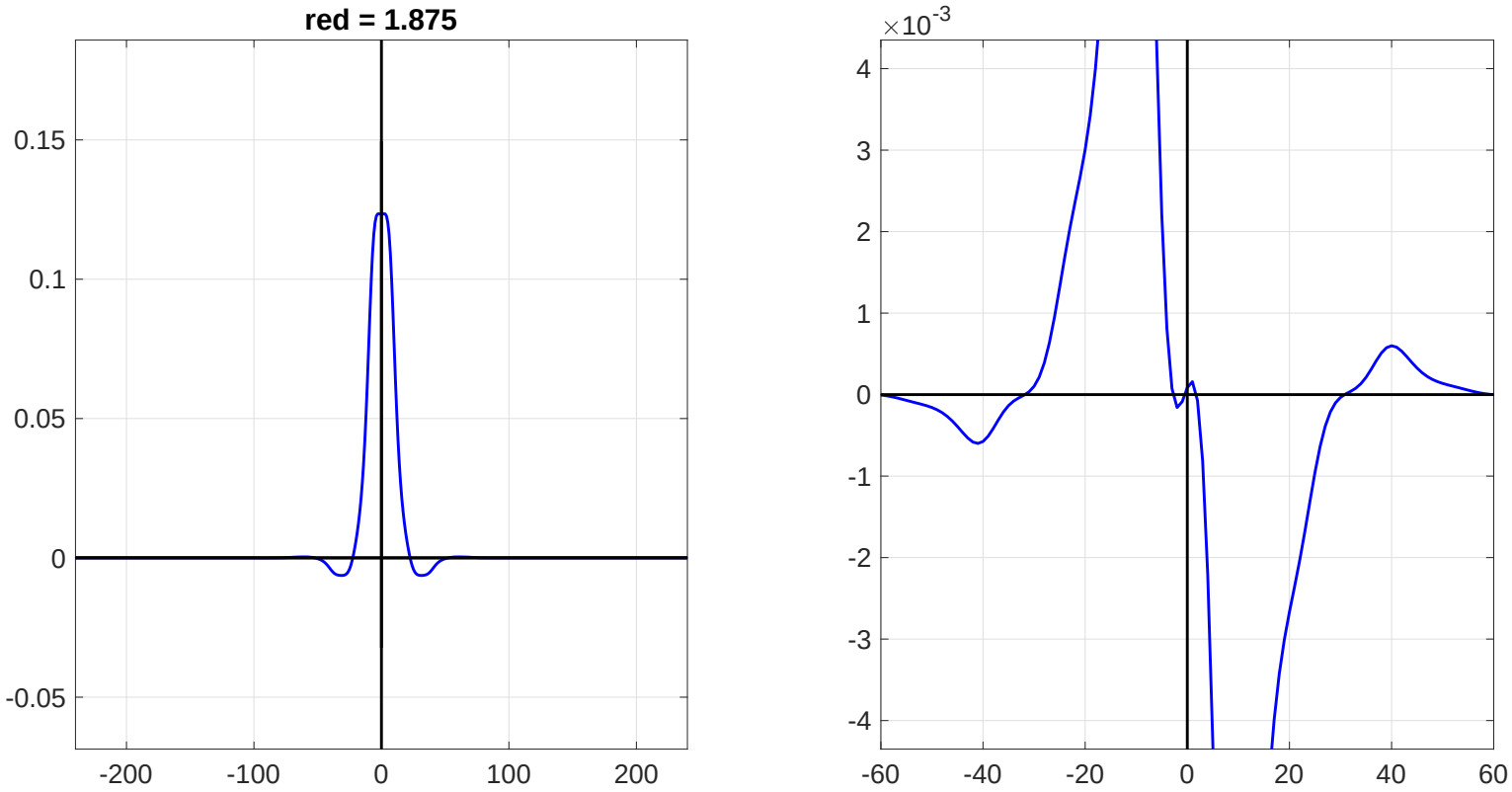

Figure: gaured1875dif.eps

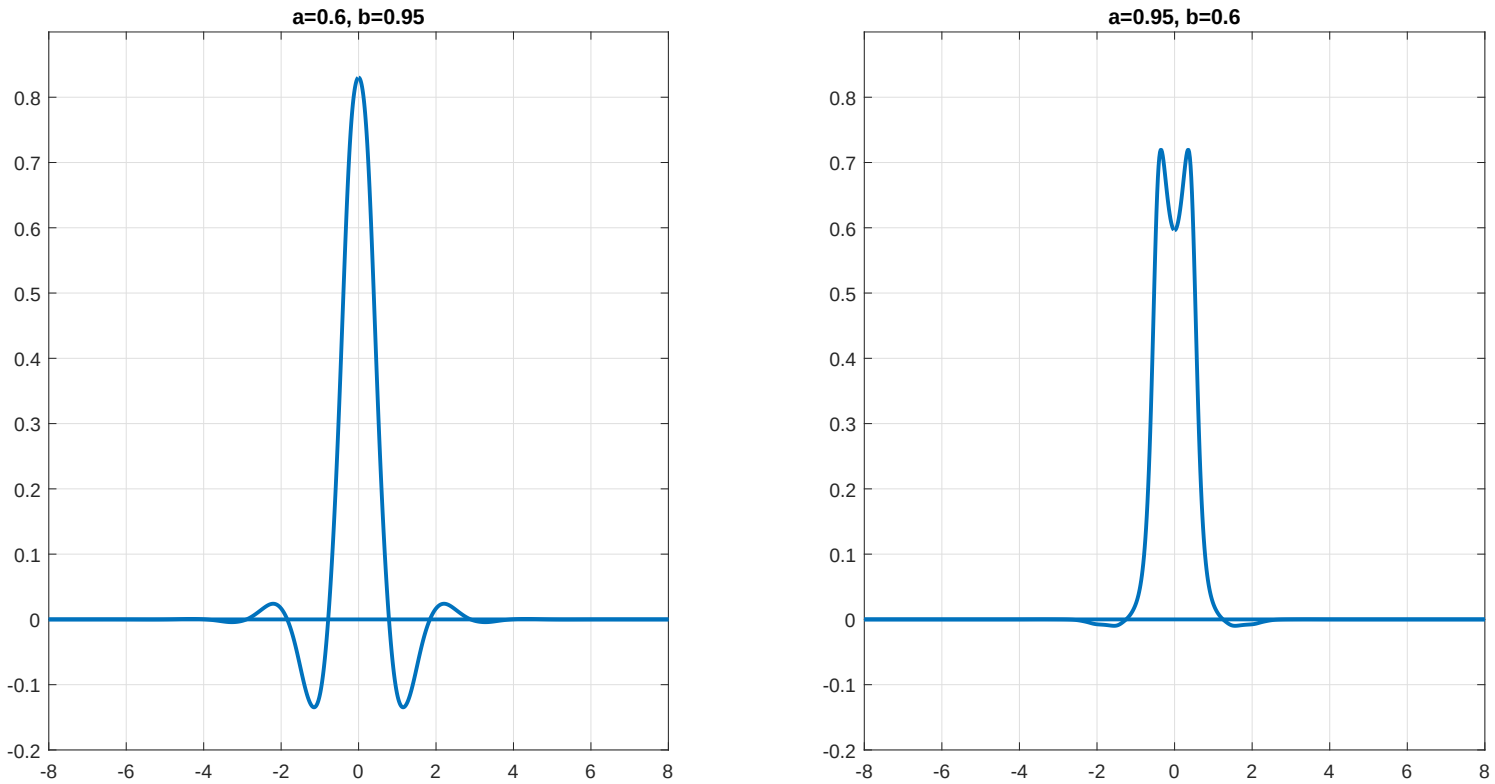

Figure: gau0695.eps

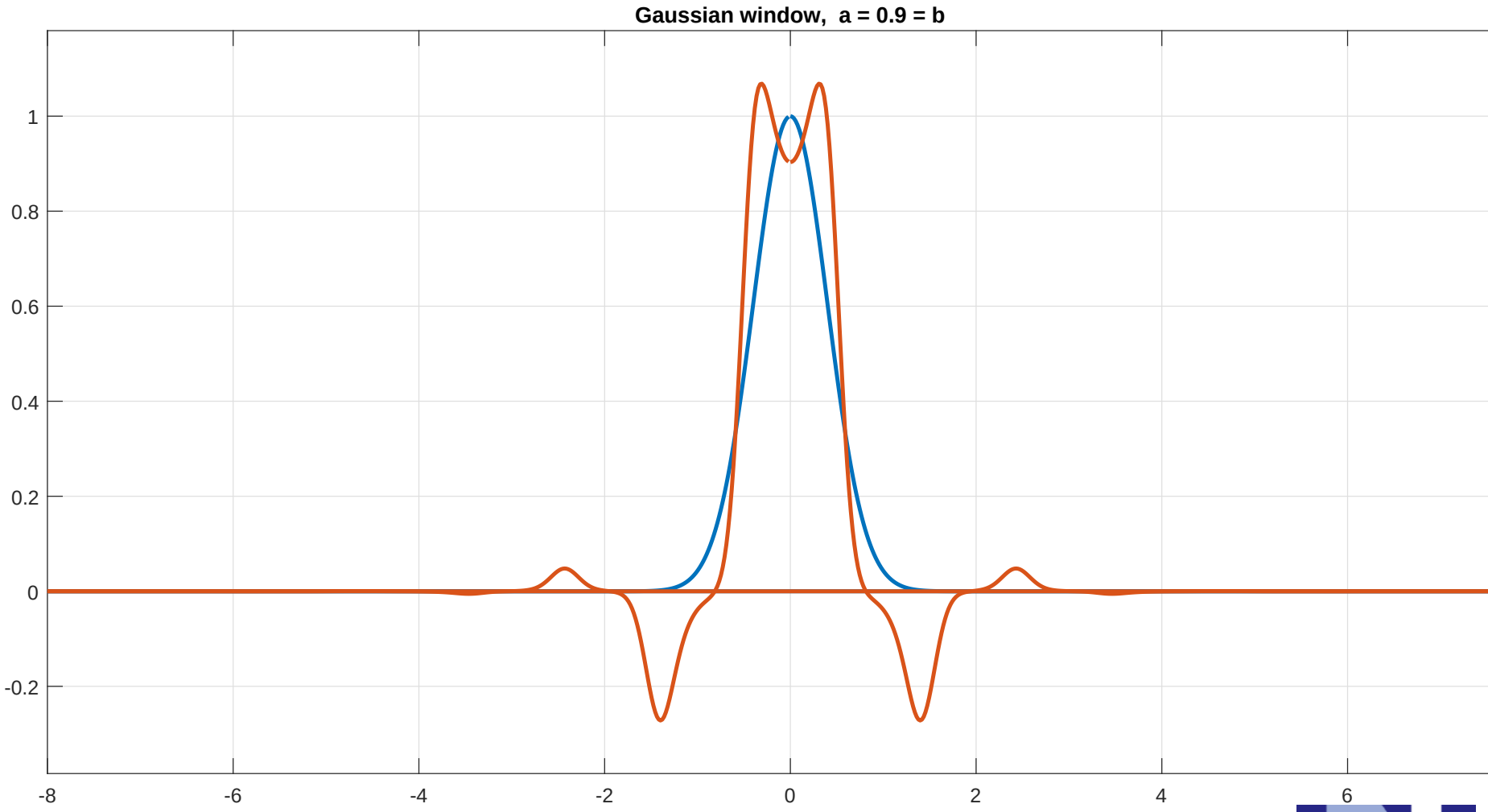

Figure: gau0909.eps

Double preconditioning I

Double preconditioning is based on a reading of those effects.
In the second case the "dipping" near the center comes from the fact that locally near zero the inverse of the Gabor frame diagonal, which is the a-periodic version of g^2 (pointwise absolute square of the Gabor window) is more acute then Gabor (Gaussian) window itself. This occurs when a is relatively large and so the shifted Gaussians are far apart.
In such a case the diagonal part of the Gabor frame operator is dominating and hence its inversion (which is cheap) provides a good and efficient preconditioner!
If the same happens on the Fourier transform side, resp. if b is relatively large, then the dual Gabor atom tends to look like a alternating sum if shifted Gaussians (with exponential decay of the coefficients). Then the corresonding preconditioner is a simple convolution operator.

Double preconditioning II

The experimental findings have been described in two papers with Peter Balazs and others (see [1]), where it was demonstrated to be highly efficient, but we still wait for a corresponding theory to be developed.
Note that both ingredients of this procedure are using commutative subalgebras of the algebra of operators commuting with the TF-shifts from the given lattice (i.e. all of the involved operators have a Janssen representation).
Open questions are:

- For which type of functions does it work (only Gauss-like functions?)?

Double preconditioning III

- For which lattice constants can one *prove* that the doubly pre-conditioned frame operator is close enough to identity, so that one can guarantee its invertibility (on $\boldsymbol{L}^2(\mathbb{R}^d)$ as well as on $\boldsymbol{S}_0(\mathbb{R}^d)$!), without the deep theory of Gröchenig/Leinert ([8]).
- Why is the order of the two preconditioners always of such a low importance that it does not matter?
- What about multiple preconditioning (for more general lattices and in the multi-dimensional setting)?

Historical comments

The problem is that the overlapping elementary functions which form the projection vectors for this transform are NOT orthogonal, and so finding their coefficients is difficult (hence NEURAL NETWORKS are suggested) [3]
This article has more than 2500 citations in Google Scholar.
Porat/Zeevi [11]
Ebrahimi/Kunt [10] Compression : They talk about the computational costs for obtaining the *Bastiaans dual* , to be denoted by γ_B here (see [2]).
[7] explains in which sense the *Bastiaan's dual* window γ_B for the critical case is *weaklky dual* to critical Gabor family $(g_0, \mathbb{Z}^2)$.
Janssen paper on generalized functions [9] discusses among others the fact that γ_B belongs to $\boldsymbol{L}^\infty(\mathbb{R})$, but not to *LpR* for any $p < \infty$.

The multi-dimensional situation

Despite the fact that Gabor Analysis is meanwhile well developed for the setting of general LCA (locally compact Abelian) groups the amount of computational work done for the setting of multi-variate Gabor Analysis is still relatively sparse.

Easy and Challenging

Of course, it is quite simple to explain (and explore computationally) the completely separable case, i.e. the case where lattices $\Lambda = \Lambda_1 \times \Lambda_2$ (sorted by variables) are separable and the windows are equally separable (i.e. $g = g_1 \otimes g_2$).

However there are many more and much more challenging cases!!

Added complications I

Although we have in principle the same questions as in the one-dimensional case, namely to find out what is a *good Gabor frame* we have a significant number of additional complications:

- The number of possible lattices is exploding, in addition there are quite a few numbers of separabilities (by grouping variables);
- There is a clearly growing computational effort required for the higher dimensional case, but also *storage requirements* explode. A dual frame (for an irregular Gabor frame) cannot even be stored, once it would be computed;
- The concept of *good approximate dual frames* gains more importance in such cases (and corresponding modifications of algorithms are required);

Added complications II

This entails the following practical difficulties:

1. Unlike the finite, one-dimensional case one cannot (practically) compute all possible subgroups of a given redundancy: there are simply *far too many such lattices*;
2. Not every lattice is a *symplectic lattice*, hence it appears to be not possible anymore to reduce the problem of a general lattice to the problem of a standard lattice (at the cost of working with a chirped or generalized Gaussian);
3. As already mentioned memory and storage problems may occur also;

What are good Gabor frames? I

There are many ways to describe the quality of Gabor frames. We just give a short list:

- The condition number of the Gabor frame operator (compared to the redundancy of the Gabor lattice Λ);
- The $\boldsymbol{S}_0$-norm of the dual Gabor atom $\widetilde{g} = S^{-1}(g)$ (compared to its $\boldsymbol{L}^2(\mathbb{R}^d)$-norm), describing TF-concentration;
- For a compact description of the family of Gabor multipliers as (infinite) linear combinations of projection operators

$$P_\lambda(f) = \langle f, g_\lambda \rangle g_\lambda.$$

What is the condition number of this family (which is usually a Riesz sequence within the Hilbert space $\mathcal{HS}$ of Hilbert-Schmidt operators)?

What are good Gabor frames? II

Experimentally we have settled a number of practical questions:

1. Given a Gabor atom $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and a redundancy (in the finite/discrete setting), what is the best lattice Λ of the given redundancy?
2. Given a lattice Λ, what is the best generalized Gaussian (i.e. a function which is an exponential function on some quadratic function), still decaying like a Gaussian, for the given lattice;
3. What is - within a given family of Gabor generators (g, Λ) - the optimal family in order to approximate a given operator well by Gabor multipliers based on this pair?

Compound Condition Number I

As a consequence we suggest to define the **compound condition number** as a quality criterion, which has the effect the it leads (at least according to our experiments) to a well-localized Gabor frame with a relatively stable procedure allowing us to approximate underspread operators, typically by Gabor multipliers with the corresponding tight Gabor family.

Definition

$$\kappa\kappa(\mathbf{g}, \mathbf{a}, \mathbf{b}) := \sqrt{\mathrm{cond}(S_{g,a,b}) \cdot \mathrm{cond}(P_g)},$$

where P_g is the condition number of the Riesz basic sequence of the family of projections onto the Gabor atoms, i.e. the quotient of the two Riesz bounds for this family of operators within the Hilbert space of Hilbert-Schmidt operators.

Compound Condition Number II

This figure of merit certainly guarantees that those lattice (for a given Gabor atom g) which show optimal performance one has a well conditioned Gabor frame, but at the same time a stable and numerical realizable procedure to approximate a given operator by Gabor multipliers, preferably with tight Gabor atoms.
Since the interesting parameters describing the lattice of interest (either the separable ones, or the ones given by general matrices) form a compact domain one can expect (due to the continuous dependence of the individual condition numbers on the lattice parameters) that the optimal value is achieved, both within the family of separable lattices as well as in the family of all (reasonable) lattices.

Quantitative Gabor Analysis

I am personally **convinced** that *computational Gabor Analysis* and also the theory of function spaces and operator theory has to move on from purely qualitative estimates to a more quantitative and precise estimates.
Such estimates, combined with a list of possible quality measurements will enable to develop objective criteria concerning the best choice of a Gabor family in a given situation, but also the best performance of a given algorithm providing the necessary information in the fastest way or to carry out the relevant computations for the selected situation in the most efficient way.
Since different situations will require to choose at least different parameters and ingredients we will have to develop a kind of consumer reports along with arguments why the performance testing methods are universally accepted by the community.

Further comments on CHA

Conceptual Harmonic Analysis is the idea to combine the methods from Abstract and Numerical Harmonic Analysis in such a way that new insight can be gained, or that reliable information about a continuous setting can be inferred from the finite-dimensional setting (using *structure preserving* discrete approximations and the suitable function spaces).

This viewpoint entails a whole research program for the community with a large number of concrete questions to be treated.

Unlike NO-GO theorems (such as variants of the Balian-Low principle) I suggest to emphasize the search for best practice results and optimal performance guarantees, given certain situations.

I want to give a few examples for illustration.

Concrete Tasks [Examples only]

- Compute the dual Gabor atom (and condition numbers, tight Gabor atoms etc.) for a given lattice and atom, OR find to optimize for a given atom and level of redundancy;
- Compute the eigenvalues and eigenvectors of a (say compact, self-adjoint) localization operator, up to a given allowed relative error in say a Sobolev norm.
  Such an operator might be given by its Weyl symbol or the Kohn-Nirenberg symbol, or as an Anti-Wick operator (STFT multiplier).
  For the case of radial symmetric decaying weights and Gaussian window this should be discrete Hermite functions and corresponding experiments have been in fact encouraging (see also [5]).

Concrete Tasks [Final Comments]

Possible important further topics of research:

- Why do we need either completely irregular or completely regular Gabor families. Numerical experiments clearly indicate that locally almost regularly (deformed lattices) Gabor families behave quite well and that locally taken approximate duals (!! they have to be TF-local as well) glued together provide good approximate dual frames? Maybe even adapt these *local lattices* to the signal at hand (instrument, tuning, etc.);
- Why using the same window (or finite set of windows) at each point, why not choose those windows, e.g. generalized Gaussians well adapted to the signal; or first: assume the are given in a certain way, smoothly varying or randomly out of a family?


P. Balazs, H. G. Feichtinger, M. Hampejs, and G. Kracher.
Double preconditioning for Gabor frames.
IEEE Trans. Signal Process., 54(12):4597–4610, December 2006.


M. J. Bastiaans.
Gabor's expansion of a signal into Gaussian elementary signals.
Proc. IEEE, 68(4):538– 539, 1980.


J. G. Daugman.
Complete discrete 2-D Gabor transforms by neural networks for image analysis and compression.
IEEE Trans. Acoustics, Speech and Signal Processing, 36(7):1169–1179, 1988.


H. G. Feichtinger.
Gabor multipliers with varying lattices.
In *Proc. SPIE Conf.*, page 14, August 2003.


H. G. Feichtinger.
Gabor expansions of signals: computational aspects and open questions.
In *Landscapes of Time-Frequency Analysis*, volume ATFA17, pages 173–206. Birkhäuser/Springer, 2019.


H. G. Feichtinger and N. Kaiblinger.
Varying the time-frequency lattice of Gabor frames.
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.


H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, 1998.


K. Gröchenig and M. Leinert.
Wiener's lemma for twisted convolution and Gabor frames.
J. Amer. Math. Soc., 17:1–18, 2004.


A. J. E. M. Janssen.
Gabor representation of generalized functions.
J. Math. Anal. Appl., 83:377–394, October 1981.


M. Kunt and T. Ebrahimi.
Image compression by Gabor expansion.
Opt. Eng., 30(7):873–880, July 1991.


M. Porat and Y. Y. Zeevi.
The generalized Gabor scheme of image representation in biological and machine vision.
IEEE Trans. Pattern Anal. Mach. Intell., 10(4):452–468, Jul 1988.