NuHAG Numerical Harmonic Analysis Group

Fourier Standard Spaces and the Multiplier Problem

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at
www.nuhag.eu

Harmonic Analysis and Operator Theory
Istanbul, August 26 - 29th, 2019

Orientation for the reader

This first page is mostly meant for those who read the slides, without having the chance to attend the talk.

To focus of this talk will be on the problem of multipliers, i.e. operators which commute with translations between Banach spaces of functions over LCA groups (locally compact Abelian).

It is organized roughly as follows:

1. First a pseudo-historical review of the subject;
2. Then a modern, distributional view on the subject;
3. Finally some attempts to simplify the traditional approach.

Let us start with a recapitulation of the abstract provided for the conference describing the intentions in more detail. For sake of simplicity we will mostly talk about $G = \mathbb{R}^d$.

Offical Abstract was ... I

The purpose of this talk is to popularize the concepts of Banach Gelfand Triples and Fourier Standard Spaces. For any LCA (locally compact Abelian) group G the Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$ can be introduced, consisting of the Segal algebra SO(G), the Hilbert space L2(G) and the dual space $\boldsymbol{S}_0'(G)$, consisting of so-called "mild" distributions. These space form a chain of natural inclusions via SO in L2 in SO' (density in norm or in the w*-sense), and Wilson basis allow to identify the triple with the prototypical Banach Gelfand Triple $(\ell^1, \ell^2, \ell^\infty)$. Obviously, for $G = T$ (the torus group) this is just $(\boldsymbol{A}T, \boldsymbol{L}^2, \boldsymbol{PM})$ using Wiener's algebra of absolutely convergent Fourier series, whose dual is the space of all pseudo-measures, resp. periodic distributions with bounded Fourier coefficients.

Offical Abstract was ... II

Fourier Standard Spaces are Banach spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ which are sandwiched between $\boldsymbol{S}_0(G)$ and $\boldsymbol{S}_0'(G)$, and which are (roughly speaking) in addition isometrically invariant with respect to time-frequency shifts. In fact, $\boldsymbol{S}_0$ is the smallest such space, and any space of tempered distributions with this property sits inside of $\boldsymbol{S}_0'(G)$. These space form a rich family, including $\boldsymbol{L}^p$-spaces, Wiener amalgam spaces, modulation spaces, and many others.
In this talk we will focus on spaces of **Convolvers** (i.e. kernels of convolution operators) between two such Fourier Standard Spaces or Fourier Multipliers (pointwise multpliers on the Fourier transform side). In doing so we can provide a new approach to the characterization of convolvers for Lp-spaces as members of the dual of the so-called Herz algebra $\boldsymbol{A}'(G)$. This characterization extends to a class of reflexive Banach function spaces.

The Multiplier Problem

According to R. Larsen (see his book on multipliers from 1971 citela71) one can ask for any pair of translation invariant spaces on a LCA group, in which way one could characterize the Banach space of all operators from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ which commute with translations.
If G is discrete the answer to this problem is quite easy, at least if the unit vectors $\{\mathbf{e}_g \mid g \in G\}$ are a basis for the sequence space $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$, because they are just translates of the unit vector $\mathbf{e}_0$, with $0 \in G$ begin the neutral element of the group, hence $\mathbf{b} = \sum_{g \in G} b_g T_g \mathbf{e}_0$ in $\boldsymbol{B}^1$, a sum of translates of $\mathbf{e}_0$, and thus

$$T(\mathbf{b}) = \sum_{g \in G} b_g T_g(T(\mathbf{e}_0)) = \mathbf{b} *_G h$$

for the "impulse response" $h := T(\mathbf{e}_0)$, where $*_T$ denotes convolution in the sense of the group G.

Wendel's Theorem I

It is well known that convolution operators commute with translations, in fact we have

$$T_x(f * g) = (T_x f) * g = f * (T_x g).$$

In a certain sense also the converse is true, as suggested by:

Wendel's Theorem II

Theorem

The space of $\mathcal{H}_{\mathbf{L}^1}(\mathbf{L}^1, \mathbf{L}^1)$ *all bounded linear operators on* $\mathbf{L}^1(G)$ *which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with* $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$. *In terms of our formulas this means*

$$\mathcal{H}_{\mathbf{L}^1}(\mathbf{L}^1, \mathbf{L}^1)(\mathbb{R}^d) \simeq (\mathbf{M}_b(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}_b}),$$

$$\text{via } T \simeq C_\mu : f \mapsto \mu * f, \quad f \in \mathbf{L}^1, \mu \in \mathbf{M}_b(\mathbb{R}^d).$$

Wendel's Theorem III

Lemma

Let us define

$$\mathbf{B}_{\mathbf{L}^1} = \{f \in \mathbf{B} \mid \|T_x f - f\|_{\mathbf{B}} \to 0, \ \ for \ \ x \to 0\}.$$

Consequently we have $(\mathbf{M}_b(\mathbb{R}^d))_{\mathbf{L}^1} = \mathbf{L}^1(\mathbb{R}^d)$*, the closed ideal of absolutely continuous bounded measures on* $\mathbb{R}^d$*.*

This is in contrast to the situation, where the dual spaces $\left(\mathbf{L}^p(\mathbb{R}^d), \, \|\cdot\|_p\right)$ with $1 < p \leq \infty$ are the target.

Wendel's Theorem IV

Theorem

The space of $\mathcal{H}_{\mathcal{G}}(\mathbf{L}^1, \mathbf{L}^p)$ *all bounded linear operators from* $\mathbf{L}^1(G)$ *to* $\mathbf{L}^p(G)$ *which commute is naturally and isometrically identified with* $\mathbf{L}^p(G)$*.*

$$T \simeq C_h : f \mapsto h * f, \quad f \in \mathbf{L}^1, h \in \mathbf{L}^p(G).$$

The relevance of L^p-spaces

If one asks, which function spaces have been used and relevant in those days the list will be quite short: Aside from BV and absolute continuity mostly the family of Lebesgue spaces appeared to be most useful for a study of the Fourier transform.
There are "good reasons". The Fourier transform is given by:

$$\hat{f}(s) := \int_{\mathbb{R}^d} f(t) e^{2\pi i s \cdot t} dt$$

appears to require $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, same with convolution (integrals):

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y) g(y) dy,$$

which turns $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ into a Banach algebra.

and some 50 years later ...

Hans Reiter's book on **Classical Harmonic Analysis and Locally Compact Groups** appeared in 1968, and was describing Harmonic Analysis as the

STUDY OF THE BANACH ALGEBRA $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$,

its behaviour under the Fourier transform, the study of closed ideals (with the hint to the problem of spectral synthesis).

Around that time (1972) Lennart Carleson was able to prove the a.e. convergence of Fourier series in $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$.

Of course we saw the books of Katznelson, Rudin, Loomis and in particular Hewitt and Ross at the same time. Carl Herz called the comprehensive book by C. Graham and C. McGehee a "tombstone to Harmonic Analysis" (1979) (Book Review by C. Herz: Bull. Amer. Math. Soc. 7 (1982), 422425).

Where did Fourier Analysis play a role?

Not to say "everywhere in analysis" let us mention some important developments:

1. L. Schwartz theory of tempered distributions extended the range of the Fourier transform enormously (it was not anymore an integral transform!)
2. L. Hörmander based on this approach (influence of Marcel Riesz!) his treatment of PDEs;
3. J. Peetre an H. Triebel started the theory of function spaces, interpolation theory: Besov-Triebel-Lizorkin spaces;
4. E. Stein and his school developed the theory of maximal functions, Hardy spaces, singular integral operators;

Consequences of Plancherel's Theorem I

One of the cornerstones of Fourier Analysis (especially over $\mathbb{R}^d$) is Plancherel's Theorem, describing the (extended) Fourier transform as a unitary automorphism of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. It is quite clear that one cannot define it as a pointwise (a.e.) integral transform anymore. Instead one has to resort to the use of Cauchy sequences. In other words one makes (mostly) use of the characterization of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ as the (abstract) completion of any dense subspace (e.g. $\boldsymbol{L}^1 \cap \boldsymbol{L}^2(\mathbb{R}^d)$) with respect $\boldsymbol{L}^2$-norm.

$$\mathcal{F}(T_s f) = \chi_{-s}\widehat{f}, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), s \in \mathbb{R}^d,$$

with $\chi_s(t) = exp(2\pi i s \cdot t)$, implies that the study of translation invariant operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ is equivalent to the study (on the FT side) of operators which commute with multiplication by arbitrary trigonometric polynomials.

Consequences of Plancherel's Theorem II

As it is not hard to verify that any such operator has to be a multiplication operator, and subsequently that pointwise multiplier (or transfer function) on the Fourier transform has to belong to $L^\infty(\mathbb{R}^d)$ in order to be bounded:

Theorem

Any operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ commuting with translations (or equivalently, with convolutions by $g \in \boldsymbol{L}^1(\mathbb{R}^d)$-functions) can be described as Fourier multiplier:

$$T(f) = \mathcal{F}^{-1}(h \cdot \widehat{f}), \quad f \in \boldsymbol{L}^2(\mathbb{R}^d),$$

for some $h \in L^\infty(\mathbb{R}^d)$. The identification is isometric:

$$\|h\|_{L^\infty(\mathbb{R}^d)} = |||T|||_{\boldsymbol{L}^2(\mathbb{R}^d)\to\boldsymbol{L}^2(\mathbb{R}^d)} .$$

Consequences of Plancherel's Theorem III

Traditionally one needs then the theory of tempered distributions to describe the operator T as a convolution operator with some distribution. More precisely, it tempting (and OK in the framework of $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$) to denote $\sigma := \mathcal{F}^{-1}(h)$ the inverse Fourier transform of h (it is thus an element of $\mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$, and may be called a peudo-measure).

It is not hard to find out that $\mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$ can be naturally identified with the dual space of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ (a Banach space) and since $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ it is clear that the space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}) = \left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$ (of bounded, regular Borel measures) is continuously (and w^*-densely) embedded into $\boldsymbol{PM}(\mathbb{R}^d)$.

Consequences of Plancherel's Theorem IV

However, one must be careful here with pointwise interpretations. For example, one only can expect that

$$Tf(x) = \sigma * f(x) = \sigma(T_x f^{\checkmark}), \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

The chirp signal $h = exp(2\pi i s^2)$ in $\boldsymbol{L}^\infty(\mathbb{R})$ is a good example. It has the property of being Fourier invariant (in the distributional sense), i.e. we have $\sigma = \mathcal{F}^{-1}(h) = h$. Since $SINC = \mathcal{F}^{-1}(boxcar)$ is clearly in $\boldsymbol{L}^2(\mathbb{R})$ one might hope for a pointwise convolution, which of course does not make sense, since there is not a single $x \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \mathrm{sinc}(y - x)\, exp(ix^2)\, dx$$

exists in the Lebesgue sense, since sinc $\notin \boldsymbol{L}^1(\mathbb{R})$.

Convolvers for $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$

It is natural to analyze not the general situation, i.e. with $p \in (1, \infty)$, since the case $p = \infty$ requires special consideration related to the "scandal in system's theory" observed by I. Sandberg (in a series of papers). Essentially, the Hahn-Banach theorem allows to describe operators on $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ which commute with translations, but act trivially on test functions and thus cannot be "represented as convolution operators".
Since $\mathbb{R}^d$ is an Abelian group it very easy to show that any Fourier convolver for $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ is also a convolver for the conjugate index $(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q)$ (with $1/p + 1/q = 1$, as usual).
An application of the Riesz-Thorin (complex) interpolation theorem then implies that any such operator is also bounded on $\boldsymbol{L}^r(\mathbb{R}^d)$, for any $r \in [p, p']$ (for $1 < p < 2$).
In other words, the space of Fourier multipliers is getting (strictly) bigger as p approaches $p = 2$ (from below).

Quasi-measures and Convolvers I

The inclusion just mentioned imply that for bounded linear operators on any of the space $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, with $1 \leq p < \infty$ one can find a representation as a convolver (convolution kernel) by some $\sigma \in \boldsymbol{PM}(\mathbb{R}^d)$, resp. a (pointwise) Fourier multiplier by some $h \in L^\infty(\mathbb{R}^d)$.
But clearly this is not possible anymore, as soon as one allows the use of a target space *different* from the domain. For example, it is clearly that any $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ defines a bounded linear operator from $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ into $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ via

$$T(f) = \mathcal{F}^{-1}(h \cdot \widehat{f}), \quad f \in \boldsymbol{L}^2(\mathbb{R}^d),$$

even for $f \notin L^\infty(\mathbb{R}^d)$.

Quasi-measures and Convolvers II

This was the motivation of G. Gaudry (in the 60th of the last century) to introduce an even larger space, the space of so-called quasi-measures.
Only few years later it was M. Cowling who was able to show that one can identify the space of quasi-measures (defined in a non-trivial way) coincides with space of *local pseudo-measures*, i.e. with the set of all distributions
In the book of Larsen one finds the following double claim (summarized here in words):

Quasi-measures and Convolvers III

Theorem

Given two parameters $p, q \in [1, \infty)$, and a bounded linear operator from $\left(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ into $\left(\mathbf{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$ which commutes with translation, then the following two claims are valid:

1. *There exists some quasi-measure σ such that*

$$T(f) = \sigma * f,$$

   for certain elements f from a dense subspace of $\mathbf{L}^p(\mathbb{R}^d)$.

2. *There exists a quasi-measure τ such that one has*

$$\mathcal{F}(T(f)) = \tau \cdot \widehat{f},$$

   for all f in a specified, dense subset of $\mathbf{L}^p(\mathbb{R}^d)$.

Puzzling Questions

Despite the fact that one has a representation on the time-side 'as well as on the frequency (or Fourier) side there is a number of issues that remain open:

- general quasi-measures do not have a Fourier transform, since there is no global control on the growth; any continuous function defines a quasi-measure;
- hence it is not possible to show that $\mathcal{F}(\sigma) = \tau$.
- it is even unclear whether the formula

$$T(f) = \mathcal{F}^{-1}(\tau \cdot \widehat{f})$$

makes sense, and for which functions $f \in \boldsymbol{L}^p(\mathbb{R}^d)$.

Marcinkiewicz-type Theorems

There are of course many interesting, sufficient conditions (on the Fourier transform side) which ensure that a given function which is smooth enough (essentially) defines a Fourier multiplier (sufficient conditions).

Such results are often based on the Paley-Littlewood characterizations of $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ (dyadic blocks on the Fourier transform side), and go e.g. by the name of *Marcinkiewicz* results.

From a modern point of view one could argue, that wavelet expansions are better suited to characterize $\boldsymbol{L}^p$-spaces and not the Fourier transform as such, and that the dyadic decompositions help to bridge this gap. But this would be another, lengthly discussion.

Rieffel's use of Banach Modules

In his paper on Induced Representations (JFA, 1967) M. Rieffel has provided a very powerful, but abstract view on the subject (mostly referring to E. Hewitt)
It is based on the theory of Banach modules over *Banach algebras*, in our case Banach modules over the Banach convolution algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, such as $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$.
Aside from other (mostly algebraic) properties (such as associativity of convolution in the given context) we are viewing $\boldsymbol{L}^p$-spaces as Banach modules over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, satisfying

$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1}\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{1}$$

Families of Banach spaces of (locally integrable) functions with this property appear in the book of Katznelson by the name of Homogeneous Function Spaces and in the work of H. Reiter as Segal algebras.

Double modules and FouSS

In *retrospect* it is not surprising – from the point of view of time-frequency analysis (going back to D. Gabor) – that in addition to Banach convolution modules over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ that one should also look into the corresponding property on the Fourier transform side, i.e. pointwise Banach modules over the Fourier algebra $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.
In fact, in an (almost forgotten) paper with W. Braun in JFA from 1983 we have studied double modules, i.e. Banach spaces which have BOTH module structures (which do NOT commute)!
Again, for simplicity we work with the unweighted case here only, and call those spaces Fourier Standard Spaces.
The advantage of the setting $G = \mathbb{R}^d$ is the fact, that in a first explanation one can rely on Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ for a quick explanation.

Fourier Standard Spaces

Definition

A Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ of (tempered) distributions is called a **Fourier standard space** if it satisfies the following conditions:

1. $$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathbf{B} \hookrightarrow \mathcal{S}'(\mathbb{R}^d),$$
2. $\mathbf{B}$ is translation and modulation isometrically invariant, i.e.
$$\|M_\omega T_x f\|_{\mathbf{B}} = \|\pi(\lambda) f\|_{\mathbf{B}} = \|f\|_{\mathbf{B}}, \quad \lambda = (x, \omega).$$
3. The Fourier algebra $\mathbf{A}$ defines pointwise multipliers on $\mathbf{B}$:
$$\|h \cdot f\|_{\mathbf{B}} \leq \|h\|_{\mathbf{A}} \|f\|_{\mathbf{B}}, \quad h \in \mathbf{A} := \mathcal{F}\mathbf{L}^1(\mathbb{R}^d),\ f \in \mathbf{B}.$$
4. $\mathbf{B}$ is a Banach convolution module over $\mathbf{L}^1(\mathbb{R}^d)$, with
$$\|g * f\|_{\mathbf{B}} \leq \|g\|_1 \|f\|_{\mathbf{B}}, \quad g \in \mathbf{L}^1(\mathbb{R}^d),\ f \in \mathbf{B}.$$

Some Remarks

Remark

Assuming (3) and (4) one can start equivalently from the situation

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d).$$

Remark

If $\mathcal{S}(\mathbb{R}^d)$ (hence $\boldsymbol{S}_0(\mathbb{R}^d) \supset \mathcal{S}(\mathbb{R}^d)$) is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ then (2) implies automatically (3). In fact, from the continuity of translation (and modulation) one could obtain the "integrated group action" by standard vector-valued integration methods.

Remark

Combining the two actions into the group action of the *reduced Heisenberg group* (via the so-called Schrödinger representation) one has in fact *Heisenberg modules*.

Operations within the family I

The setting of Fourier standard spaces allows to treat a large number of "derived spaces" in a unified viewpoint. In fact, most of the operations which play a role in Fourier Analysis, but also in Gabor or Time-Frequency Analysis can be applied to Fourier Standard Spaces.
Hence one can treat those questions in a more systematic way, avoiding the purely technical questions of integrability and concentrate on various interesting, and sometimes completely overlooked questions.

Constructions within the FSS Family

1. Taking **Fourier transforms**;
2. Conditional dual spaces, i.e. the **dual space** of the closure of $\mathbf{S}_0(G)$ within $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$;
3. With two spaces $\mathbf{B}^1, \mathbf{B}^2$: take **intersection or sum**
4. forming **amalgam spaces** $\mathbf{W}(\mathbf{B}, \ell^q)$; e.g. $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)$;
5. defining pointwise or convolution **multipliers**;
6. projective tensor (and convolution) products, like $\mathbf{A}_p(G)$;
7. using complex (or real) **interpolation methods**, so that we get the spaces $\mathbf{M}^{p,p} = \mathbf{W}(\mathcal{F}\mathbf{L}^p, \ell^p)$ (all Fourier invariant);
8. any **metaplectic** image of such a space, e.g. the **fractional Fourier transform**.

Fourier Multipliers I

Let $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ be two Fourier standard spaces (think of $\boldsymbol{B}^1 = \boldsymbol{L}^p$ and $\boldsymbol{B}^2 = \boldsymbol{L}^q$). Then we can define the space of **multipliers** from $\boldsymbol{B}^1$ to $\boldsymbol{B}^2$.

Definition

$$\boldsymbol{M}_{\boldsymbol{B}^1,\boldsymbol{B}^2} := \{T : \boldsymbol{B}^1 \to \boldsymbol{B}^2, \quad T \circ T_x = T_x \circ T \quad \text{for all} \quad x \in \mathbb{R}^d\}.$$

Given the properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual $\boldsymbol{S}_0'(\mathbb{R}^d)$ one can verify that this is another Fourier standard space. The generalized Fourier transform in the sense of $\boldsymbol{S}_0'(\mathbb{R}^d)$ maps $\boldsymbol{M}_{\boldsymbol{B}^1,\boldsymbol{B}^2}$ onto the space of pointwise multipliers between the Fourier images $\mathcal{F}\boldsymbol{B}^1$ and $\mathcal{F}\boldsymbol{B}^2$, mapping "convolution kernels" into "transfer functions" (engineering terminology).

Fourier Multipliers II

Since $\mathbf{S}_0(\mathbb{R}^d)$ is a subspace of the so-called *quasi-measures* (according to M. Cowling they can be identified with the dual space for the space of test functions from the Fourier algebra with compact support) this last result implies that the action of a multipliers (at least on test functions) can be described as the convolution with some element from $\mathbf{S}_0'(\mathbb{R}^d)$.
The reader can find a lot of other examples about multiplier spaces between $\mathbf{L}^p$-spaces in the book of R. Larsen (1972)

But why should we be interested in these operators

The answer to this question is given by engineers. In their introductory course on "linear system theory" electrical engineering students are introduced to the concept of TILS (translation invariant linear systems), and how they can be treated, using essentially Fourier transforms.
The "physical justification" for the importance of these systems comes from the time-invariance of physical laws. Repeating an experiment ten minutes later a mechanical or electrical system will show the "same reaction", just ten minutes delayed. And smooth systems are well approximated by linear ones...
Starting from the setting of discrete signals they a learn what the impulse response is, and that one can describe such a system by its impulse response or by the transfer function, i.e. the eigenvalues with respect to the system of eigenvectors, namely the pure frequencies (cf. `DFT`,`FFT`, `FFT2`).

The most natural setting over LCA groups I

In order to settle the problem of "representing a TILS" as a convolution operator we consider the most simple norm, namely the sup-norm, and start from $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (or $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$), the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ with respect to the sup-norm. It is a nice Banach algebra (even commutative C^*-algebra) with respect to pointwise multiplication, with *bounded approximate units* (but without! unit element).

In the terminology of engineers a bounded linear operator can be described as a BIBOS (bounded input - bounded output system), which in fact can be justified using the Closed Graph Theorem under mild conditions.

For the characterization of $(\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d)), |||\cdot|||)$ we will need also the dual space of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, i.e. $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, the space of bounded, regular Borel measures. For us this is just a

The most natural setting over LCA groups II

terminology, we will NOT MAKE USE of any measure theoretic argument in what follows (also not in my courses on the subject).

Theorem (hgfei)

Any TILS on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ *is a moving average by some bounded measure* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$*: In fact,* $\nu(f) = Tf(0)$ *describes the system, given by*

$$Tf(x) = [T_{-x}Tf](0) = T(T_{-x}f)(0) = \nu(T_{-x}f = T_x\mu(f).$$

This pairing establishes an isometric bijection between $\left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$ *(the space of bounded measures* $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$*) and the TILS:*

$$\|\nu\|_{\boldsymbol{M}_b} = |||T|||_{\boldsymbol{C}_0}.$$

The most natural setting over LCA groups III

Unfortunately this pairing *associates* with the Dirac measure

$$\delta_{x_0} : f \mapsto f(x_0), \quad f \in \boldsymbol{C}_0(\mathbb{R}^d),$$

the wrong shift operator (in the opposite direction):

$$T_{-x_0} f(z) = f(z + x_0), \quad \text{with } \operatorname{supp}(T_{-x_0} f) = \operatorname{supp}(f) - x_0.$$

This is way instead of the "moving average description" one flips to the concept of convolution, introducing an extra flip, i.e.
$h \mapsto h^{\checkmark}$, with $h^{\checkmark}(z) = h(-z)$.
Then for $\mu := \nu^{\checkmark}$, given by $\mu(f) = \nu(f^{\checkmark})$ we have

$$Tf(z) = \mu(T_z(f^{\checkmark})), \quad z \in \mathbb{R}^d, f \in \boldsymbol{C}_0(\mathbb{R}^d),$$

$$\|\mu\|_{\boldsymbol{M}_b} = |||T|||_{\boldsymbol{C}_0}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

The most natural setting over LCA groups IV

This is the key argument for the proof. Another non-trivial question is the proof that $\mu * f \in \boldsymbol{C}_0(\mathbb{R}^d)$ for $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.
The estimate

$$\|\mu * f\|_\infty \leq \|\mu\|_{\boldsymbol{M}_b} \|f\|_\infty$$

is almost obvious, and also the claim that $\mu * f \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$, i.e. (uniform) continuity of the convolution product is easy.
What is left is the decay at infinity, which requires an argument that every $\mu \in \boldsymbol{C}_0'(\mathbb{R}^d)$ can be approximated in norm by compactly supported measures μ_n. This can in fact done without measure theory (see my course notes for details).

Convolution of Bounded Measures

Once we have an isometric identification of an ALGEBRA of operators (!exercise) and a Banach space (of bounded measures) it is plausible to *transfer* the composition structure to $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. The basis formula, resulting from the identification of Dirac measures with the corresponding translation operators, is the formula (that we expect anyway)

$$\delta_x * \delta_y = \delta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

The point is to approximate (constructively) a bounded measure $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ by a sequence (better net) of uniformly bounded and uniformly tight (concentrated in the same way as μ) $D_\Psi \mu$, in the w^*-sense. Subsequently one can show that convolution of general bounded measures can be identified with the $(w^*$-) limit of the corresponding discrete measure $\mathrm{D}_\Psi \mu_1 * \mathrm{D}_\Psi \mu_2$.

Integrated Group Action

Having established the internal convolution within $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ by transfer of structure one can now look at the external action, often called the integrated group represenation:
It start from a general uniformly bounded and strongly continuous representation ρ of a LCA group G on a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.
That means, that for some $C > 0$ one has

$$\|\rho(x)f\|_{\boldsymbol{B}} \leq C\|f\|_{\boldsymbol{B}}, \quad \forall f \in \boldsymbol{B},$$

and

$$\lim_{x\to 0} \|\rho(x)f - f\|_{\boldsymbol{B}} = 0 \quad \forall f \in \boldsymbol{B},$$

As a consequence any such representation allows to extend the action to an integrated group action, called ρ again, with

$$\|\mu *_\rho f\|_{\boldsymbol{B}} = \|\rho(\mu)(f)\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}_b}\|f\|_{\boldsymbol{B}}.$$

Homogeneous Banach Spaces

Any homogeneous Banach space (of locally integrable functions on $\mathbb{R}^d$) is by consequence a Banach convolution module over $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, because by assumption the translation (often called the regular representation, with $\rho(x) = T_x$) is strongly continuous and uniformly bounded on a homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and in particular on any Segal algebra.
In particular, any Banach space of tempered distributions $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, containing $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ as a dense subspace and with isometric time-frequency shifts

$$\pi(\lambda)f := M_s T_t f = \chi_s \cdot T_x f, \quad t, s \in \mathbb{R}^d$$

will be a (minimal) Fourier Standard Space.

Convolution operators and (S_0, L^2, S_0') I

Theorem

There is a natural isomorphism between $\mathcal{H}_\mathcal{G}(\mathbf{S}_0, \mathbf{S}_0')(\mathcal{G})$ and $\mathbf{S}_0'(G)$, given by the following linear mappings which are both isometric and inverse to each other:

$$\sigma \mapsto C_\sigma : \quad C_\sigma(f)(x) = \sigma(T_x f^{\checkmark})^a, \quad x \in \mathcal{G}, \tag{2}$$

$$T \mapsto \sigma_T : \quad \sigma_T(f) = T(f^{\checkmark})(0), \quad f \in \mathbf{S}_0(G). \tag{3}$$

Moreover, the ultra-weak convergence of a (bounded) net of operators C_{σ_α} corresponds in a one-to-one way to the w^*-convergence of the corresponding distributional kernels (σ_α) in $\mathbf{S}_0'(G)$ which generate these convolution operators.

[a] To make sure: the symbol $T_x f^{\checkmark}$ stands for $T_x(f^{\checkmark})$, here and for the rest.

Convolution operators and (S_0, L^2, S_0') II

A part of the proof actually consists in verifying what could be seen as a corollary to the statement, since obviously $\boldsymbol{S}_0 * \boldsymbol{S}_0' \subset \boldsymbol{C}_{ub}(\mathbb{R}^d)$.

Corollary

Any operator from $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ *to* $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$ *which commutes with translations maps in fact* $\boldsymbol{S}_0(G)$ *into* $(\boldsymbol{C}_b(G), \|\cdot\|_\infty)$.

It also follows that for any FouSS $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ which contains $\boldsymbol{S}_0$ as a dense subspace, the space of multipliers from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to any other FouSS $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ can be represented as convolution operator with SOME element from $\boldsymbol{S}_0'$.
Hence let us turn to a discussion of some concrete cases, with first the case $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$.

The Figa-Talamanca-Herz spaces I

Definition

Given $1 < p < \infty$, the **Herz/Figa-Talamca algebra** $\left(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)}\right)$ is defined as follows:

$$\boldsymbol{A}_p(G) = \{f \in \boldsymbol{C}_0(G) \,|\, f = \sum_{n\geq 1} f_n * g_n, \quad \text{with} \quad \sum_n \|f_n\|_p \|g_n\|_q < \infty\}.$$

Any such representation, with $(f_n)_{n\geq 1}$ in $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ and $(g_n)_{n\geq 1}$ in $\left(\boldsymbol{L}^q(G), \|\cdot\|_q\right)$ is called an *admissible*. The natural norm for $\left(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)}\right)$ is

$$\|f\|_{\boldsymbol{A}_p(G)} = \inf\{\sum_n \|f_n\|_p \|g_n\|_q\}, \qquad (4)$$

where the infimum is taken over all admissible representations.

Facts about *ApG* I

It is not difficult to verify the following facts (for simplicity under the assumption $1 < p < \infty$):

Lemma

1. $(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)})$ *is continuously embedded into* $(\boldsymbol{C}_0(G), \|\cdot\|_\infty)$;
2. $(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)})$ *is a Banach space;*
3. *the compactly supported elements form a dense subspace of* $(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)})$;
4. *translation and modulation act isometrically on* $(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)})$;
5. $lim_{x\to 0}\|T_x f - f\|_{\boldsymbol{A}_p(G)} = 0$ *for every* $f \in \boldsymbol{A}_p(G)$;
6. $(\boldsymbol{A}_p(G), \|\cdot\|_{\boldsymbol{A}_p(G)})$ *is a Fourier Standard spaces.*

Facts about *ApG* II

Remark

It is much less obvious and a deep fact that $(\mathbf{A}_p(G), \|\cdot\|_{\mathbf{A}_p(G)})$ is in fact a Banach algebra with respect to pointwise multiplication, the so-called *Herz algebra*. But we will not need this fact here. In addition, it is not clear to which extent this property extends to similar constructions, e.g. with $\left(\mathbf{L}^p(G), \|\cdot\|_p\right)$ replaced by corresponding *Lorentz spaces* $\mathbf{L}(p,q)(G)$.

Remark

Note that it one may assume that the convolution factors f_n and g_n are taken from a *dense subspace*, e.g. from $\mathcal{F}\mathbf{L}^1 \cap \mathbf{C}_c(G)$, with corresponding modification of the description of the norm on $\mathbf{A}_p(G)$.

Facts about *ApG* III

Towards Herz algebras $\boldsymbol{A}_p(\mathbb{R}^d)$

Lemma

Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a FouSS, and α is an automorphism of $\mathbb{R}^d$, i.e. $\alpha(x) = \boldsymbol{A} * x$ for some non-singular $n \times n$-matrix $\boldsymbol{A}$. Then $\boldsymbol{B}_\alpha := \alpha^*(\boldsymbol{B})$ is a a FouSS as well (with the natural norm $\|\alpha^*(f)\|_{\boldsymbol{B}^\alpha} := \|f\|_{\boldsymbol{B}}$, $f \in \boldsymbol{B}$.

This lemma helps us to simplify the general situation to the following one: Given $n = d + m$, $n, m \in \mathbb{N}$ we want to split $\mathbb{R}^n$ as a direct sum:

$$\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^m.$$

We consider the subgroups $H = H_1 = \mathbb{R}^d \times \{0\}$ and $H_2 = H^\perp = \{0\} \times \mathbb{R}^m$. We write dH_k for the Haar measure on H_k, $k = 1, 2$, viewed as element of $\boldsymbol{S}_0'(\mathbb{R}^n)$.

Facts about *ApG* IV

Lemma

Given a FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *the periodized space* $\boldsymbol{B}_2 = dH_2 * \boldsymbol{B}$ *is well defined if and only if* $dH_1 \cdot \mathcal{F}(\boldsymbol{B})$ *is well defined (both via regularization). In the positive case* $\boldsymbol{B}_2$ *can be identified naturally with a FouSS on* $\mathbb{R}^d$*, and*

$$\mathcal{F}_d\, \boldsymbol{B}_2 = \mathrm{Restr}_{\mathrm{H}_1}(\mathcal{F}\boldsymbol{B}).$$

This is more or less the SLICE THEOREM used heavily in the standard discussion of the Radon Transform.

The Figa-Talamance-Herz algebras $A_p(G)$ I

A meanwhile classical too for the study of multipliers of L^p-spaces is the so-called Figa-Talamanca-Herz algebra, which arose in the 1960th. Although not cited by M. Rieffel in his famous 1967 paper appears nowadays as a concrete version of the module-tensor product described by Rieffel.
We can take the definition of $A_p(G)$ or $A_p(\mathbb{R}^d)$ as usual (a so-called *convolution tensor product*, and obtain immediately that this is a Banach space with the usual quotient norm (infimum over all possible representations). In fact, one could use her any *homogeneous Banach space* in the sense of Y. Katznelson.

The Figa-Talamance-Herz algebras $A_p(G)$ II

Lemma

For $1 < p < \infty$ $(\mathbf{A}_p(G), \|\cdot\|_{\mathbf{A}_p(G)})$ *is a Banach space with the natural quotient norm, continuously and densely embedded into* $(\mathbf{C}_0(G), \|\cdot\|_\infty)$. *It is also a homogeneous Banach space, containing* $\mathbf{S}_0(G)$ *as a dense subspace.*

Due to the density of $\mathbf{S}_0(G)$ in $(\mathbf{A}_p(G), \|\cdot\|_{\mathbf{A}_p(G)})$ the dual space is again a FouSS, thus continuously embedded into $\mathbf{S}_0'$. We will denote it by $\mathbf{PM}_p$ (note that we get $\mathbf{PM}_p = \mathbf{PM}$ for $p = 2$, the space $\mathbf{PM} = \mathcal{F}\mathbf{L}^\infty$ of pseudo-measures).

Multipliers of $A_p(G)$ and $L^p(G)$ I

In the sequel we will make use of a few properties shared by the spaces $(\boldsymbol{L}^p(G), \|\cdot\|_p)$, for $1 < p < \infty$ (but not more):

- A1 $(\boldsymbol{L}^p(G), \|\cdot\|_p)$ is a *homogeneous Banach space*;
- A2 $(\boldsymbol{L}^p(G), \|\cdot\|_p)$ is a *solid BF-space*;
- A3 $(\boldsymbol{L}^p(G), \|\cdot\|_p)$ is a *reflexive BF-space*[1]

Lemma

For $1 < p < \infty$ *and* $1/p + 1/q = 1$ *one has*

$$\mathcal{H}_{\mathcal{G}}(\boldsymbol{L}^p(G), \boldsymbol{C}_0(G)) \equiv \boldsymbol{L}^q(\mathcal{G}), \tag{5}$$

but also

$$\mathcal{H}_{\mathcal{G}}(\boldsymbol{A}_p(G), \boldsymbol{C}_0(G)) \equiv \boldsymbol{A}_p(G)' =: \boldsymbol{PM}_p(G). \tag{6}$$

Multipliers of $A_p(G)$ and $L^p(G)$ II

Theorem

For any LCA group and $1 \leq p \leq \infty$ one has equality of spaces with a natural isometry of the corresponding (operator) norms:

$$\mathcal{H}_{\mathcal{G}}(\boldsymbol{L}^p(G)) = \mathcal{H}_{\mathcal{G}}(\boldsymbol{A}_p(G)) = \mathcal{H}_{\mathcal{G}}(\boldsymbol{A}_p(G), \boldsymbol{C}_0(G)) \qquad (7)$$

Corollary

$$\mathcal{H}_{\mathcal{G}}(\boldsymbol{A}_p(G)) = \mathcal{H}_{\mathcal{G}}(\boldsymbol{A}_p(G), \boldsymbol{C}_b(G)) \equiv \boldsymbol{PM}_p(G). \qquad (8)$$

Multipliers of $A_p(G)$ and $L^p(G)$ III

Corollary

There is an isometric isomorphism between the space $CV_p(G)$ of convolutors of $\left(\mathbf{L}^p(G), \|\cdot\|_p\right)$ and $\mathbf{PM}_p(G)$, the dual space of $(\mathbf{A}_p(G), \|\cdot\|_{\mathbf{A}_p(G)})$. Under the (extended) Fourier transform this space can be identified with the space of p-Fourier multipliers, i.e. the pointwise multipliers on $\left(\mathcal{F}\mathbf{L}^p(\mathcal{G}), \|\cdot\|_p\right)$. Consequently both of these space (for each such p) are again Fourier Standard Spaces.

[1] With dual space $\left(\mathbf{L}^q(G), \|\cdot\|_q\right)$, with $1/p + 1/q = 1$.

The Schwartz Setting

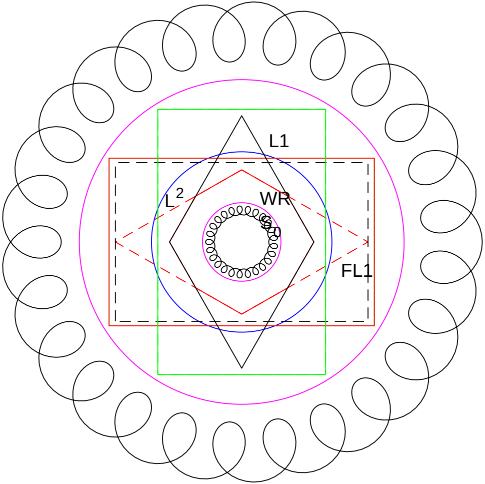


Figure: The classical setting

Lower and Upper Index of a Function Space

We define a *lower* resp. *upper index* for a Fourier Standard space:

Definition

$$\mathrm{low}(\boldsymbol{B}) := \sup\{r \,|\, \boldsymbol{B} \subseteq W(\boldsymbol{B}, \ell^r)\}.$$

Definition

The *upper index* of $\boldsymbol{B}$ is defined as follows:

$$\mathrm{upp}(\boldsymbol{B}) := \inf\{s \,|\, W(\boldsymbol{B}, \ell^s) \subseteq \boldsymbol{B}\}.$$

For $\boldsymbol{B} = \mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$ or $\boldsymbol{B} = \mathcal{F}\boldsymbol{L}^q(\mathbb{R}^d)$ $(1/p + 1/q = 1)$, with $1 \leq p \leq 2$ one has $\mathrm{low}(\boldsymbol{B}) = p$ and $\mathrm{upp}(\boldsymbol{B}) = q$.

Hörmander's argument

If we analyze the argument, why there is no non-trivial convolutor from $\boldsymbol{L}^p(\mathbb{R}^d)$ to $\boldsymbol{L}^r(\mathbb{R}^d)$, if $r < p$, one finds that it is mostly a *global argument* (See Hörmander, 1960).
We all know that a convolution operator can provide increased smoothness, but not better decay (think of a convolution of a positive function by some Gauss function!).
It can be easily adapted to the following claim:

Lemma

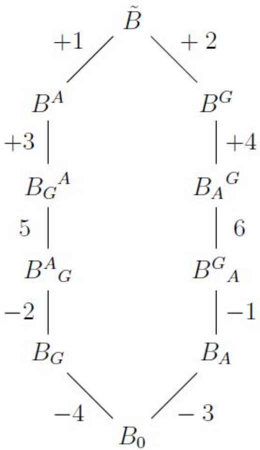
Given two FouSS $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *and* $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$. *Assume that*

$$\text{low}(\boldsymbol{B}^1) > \text{upp}(\boldsymbol{B}^2).$$

Then there is no non-trivial convolution operator from $\boldsymbol{B}^1$ *to* $\boldsymbol{B}^2$.

The Double Module Diagram

The possible 10 derived spaces



Intersections, tempered L^p-spaces

Clearly the intersection (but also the sum, related by duality) of two FouSS is again FouSS, with the norm for $\boldsymbol{B}^1 \cap \boldsymbol{B}^2$:

$$\|f\|_{\boldsymbol{B}^1 \cap \boldsymbol{B}^2} := \|f\|_{\boldsymbol{B}^1} + \|f\|_{\boldsymbol{B}^2}.$$

Since it still has to contain $\boldsymbol{S}_0(\mathbb{R}^d)$ it cannot be trivial. A classical, non-trivial example is found in the work of K. McKennon in the early 70th, who - from the point of view of FouSS - was studying the Banach algebra $\boldsymbol{L}^t_p := \boldsymbol{L}^p \cap \mathbf{Conv}_p$ of all elements of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ which at the same time define bounded "multipliers" on $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$.
The interesting finding of his work was (valid at least for Abelian groups): The space of multipliers of this new space can be identified with the $\boldsymbol{L}^p$-multipliers.

frametitle: bibliography I

W. Braun and H. G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

A. P. Calderon.
Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz.
Studia Math., 26:273–299, 1966.

M. G. Cowling.
Some applications of Grothendieck's theory of topological tensor products in harmonic analysis.
Math. Ann., 232:273–285, 1978.

M. G. Cowling.
The predual of the space of convolutors on a locally compact group.
Bull. Austral. Math. Soc., 57(3):409–414, 1998.

A. Derighetti.
Convolution Operators on Groups.
Springer Berlin / Heidelberg, 2011.

frametitle: bibliography II

H. G. Feichtinger.
Multipliers of Banach spaces of functions on groups.
Math. Z., 152:47–58, 1976.

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

H. G. Feichtinger and M. S. Jakobsen.
The inner kernel theorem for a certain Segal algebra. 2018.

H. G. Feichtinger and M. S. Jakobsen.
Distribution theory by Riemann integrals.
In Pammy Machanda et al., editor, *ISIAM Proceedings*, pages 1–42, 2019.

A. Figa Talamanca.
Multipliers of p-integrable functions.
Bull. Amer. Math. Soc., 70:666–669, 1964.

A. Figa Talamanca and G. I. Gaudry.
Density and representation theorems for multipliers of type (p, q).
J. Aust. Math. Soc., 7:1–6, 1967.

frametitle: bibliography III

G. I. Gaudry.
Quasimeasures and operators commuting with convolution.
Pacific J. Math., 18:461–476, 1966.

C. Herz.
The theory of p-spaces with an application to convolution operators.
Trans. Amer. Math. Soc., 154:69–82, 1971.

L. Hörmander.
Estimates for translation invariant operators in L^p spaces.
Acta Math., 104:93–140, 1960.

Y. Katznelson.
An Introduction to Harmonic Analysis. 2nd corr. ed.
Dover Publications Inc, New York, 1976.

R. Larsen.
An Introduction to the Theory of Multipliers.
Springer-Verlag, New York-Heidelberg, 1971.

K. McKennon.
Multipliers of type (p, p) and multipliers of the group l_p-algebras.
Pacific J. Math., 45(1):297–302, 1973.

frametitle: bibliography IV

J. Griffin and K. McKennon.
Multipliers and the group l_p-algebras.
Pacific J. Math., 49(2):365–370, 1973.

H. Reiter and J. D. Stegeman.
Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.
Clarendon Press, Oxford, 2000.

H. J. Reiter.
L^1-algebras and Segal Algebras.
Springer, Berlin, Heidelberg, New York, 1971.

M. A. Rieffel.
Induced Banach representations of Banach algebras and locally compact groups.
J. Funct. Anal., 1:443–491, 1967.

Thanks and LINKS

More at www.nuhag.eu

All relevant papers are downloadable from

www.nuhag.eu/bibtex

This talk and many other talks on related subjects can be downloaded from

www.nuhag.eu/talks

User: visitor, PWD: nuhagtalks

Thank you for your attention!

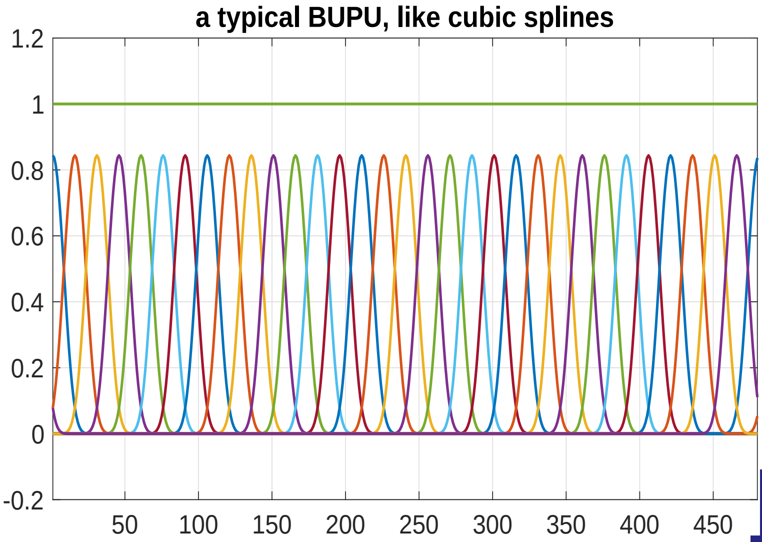
BUPUs: Bounded Uniform Partitions of Unity

Definition

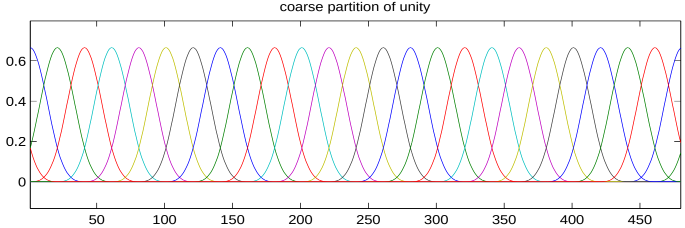
A bounded family $\Psi = (\psi_i)_{i \in I}$ in a Banach algebra $(\mathbf{A}, \|\|_A)$ is called a Bounded Uniform Partition of Unity in $(\mathbf{A}, \|\|_A)$ (a BUPU in $\mathbf{A}$, for short), if there are a relatively separated family $X = (x_i)_{i \in I}$ and some $R > 0$ such that

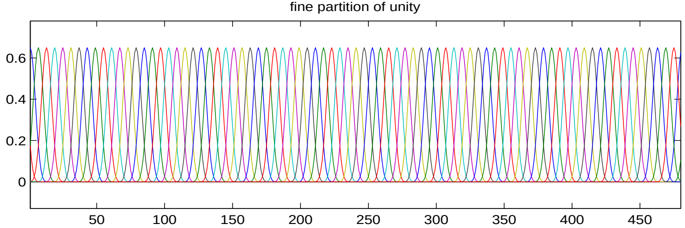
1. $\mathrm{supp}(\psi_i) \subseteq B_R(x_i)$ for each $i \in I$, and
2. $\sum_{i \in I} \psi_i(x) = 1$ for all $x \in \mathbb{R}^d$

The most useful variant are the so-called regular BUPU with $I = \Lambda \lhd \mathbb{R}^d$, a lattice, with $\psi_i = T_\lambda(\varphi)$, for some $\varphi = \varphi_0$, such as a cubic B-spline.

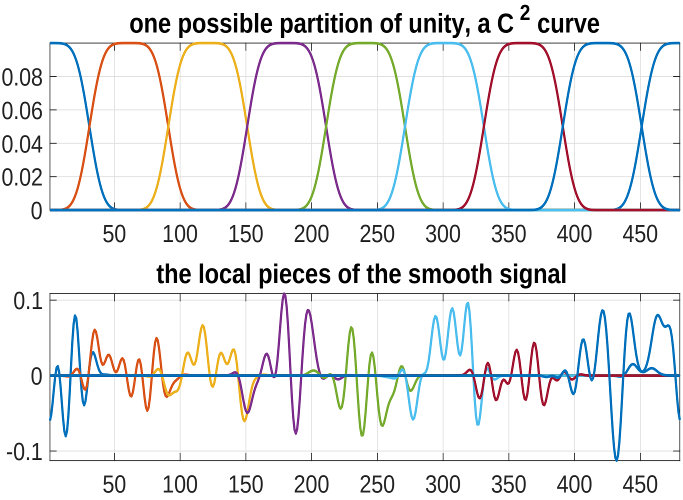

a typical BUPU, like cubic splines
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
50
100
150
200
250
300
350
400
450

BUPUs: Bounded Uniform Partitions of Unity





Wiener Amalgam Norms Vizualized



Recalling the Wiener Amalgam Concept

We recall the concept of BUPUs ideally as translates along a lattice ($T_\lambda \varphi$), with compact support and a certain amount of smoothness, perhaps cubic B-splines.
The Wiener amalgam space $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$ is defined as the set

$$\left\{ f \in \boldsymbol{B}_{loc} \,|\, \|f|\boldsymbol{W}(\boldsymbol{B}, \ell^q)\| := \left(\sum_{\lambda \in \Lambda} \|f \cdot T_\lambda \varphi\|_{\boldsymbol{B}}^q \right)^{1/q} \right\}.$$

There are many "natural results" concerning Wiener amalgam spaces, namely coordinatewise action, e.g.

- duality (if test functions are dense and $q < \infty$);
- convolution and pointwise multiplication;
- interpolation (real or complex).

(S_0, L^2, S_0') as Wiener amalgams

Theorem

1. $S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$
2. $S_0'(\mathbb{R}^d) = W(\mathcal{F}L^\infty, \ell^\infty)$

with equivalence of the corresponding norms.

In analogy with $W(M, \ell^\infty)(\mathbb{R}^d)$, the dual of the Wiener algebra $W(C_0, \ell^1)(\mathbb{R}^d)$, known as the space of *translation bounded measures*, the *systematic* name for $S_0'(\mathbb{R}^d)$ would be space of translation bounded quasi-measures.

$$PM(\mathbb{R}^d) \subset S_0'(\mathbb{R}^d) = W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d) \subset Q(\mathbb{R}^d)$$

in analogy to the inclusion

$$M_b(\mathbb{R}^d) \subset W(M, \ell^\infty)(\mathbb{R}^d) \subset \mathcal{R}(\mathbb{R}^d),$$

the space of Radon measure = locally bounded measures.