Banach Frames for Banach Gelfand Triples

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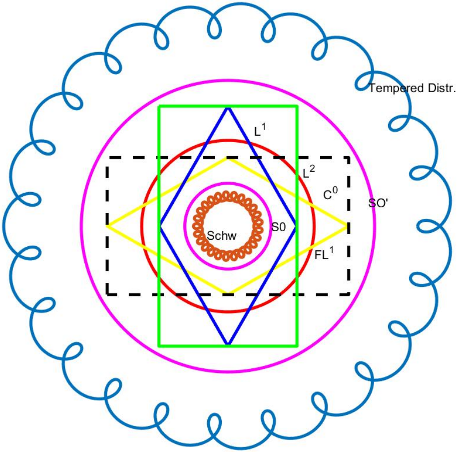
A Zoo of Banach Spaces for Fourier Analysis



In my first talk available at

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I was motivating why the space $(S_0'(\mathbb{R}^d), \| \cdot \|_{S_0'})$ is a natural object, as the largest domain (within $\mathcal{S}'(\mathbb{R}^d)$, if one wants) of all "objects" having a bounded (and always continuous) spectrogram, at least for smooth and compactly supported functions.
We will next describe some basic properties which imply that $(S_0(G), \| \cdot \|_{S_0})$ has a certain minimality property and therefore has many equivalent characterizations. It also is the natural reservoir for the windows for the STFT.

Lemma

For any non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *one has:*

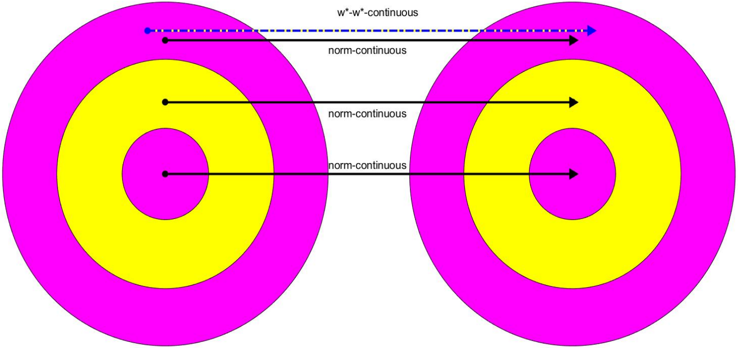
$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) = \boldsymbol{A}(\mathbb{T})_g(\mathbb{R}^d),$$

with equivalence of norms.

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

M. S. Jakobsen.
On a (no longer) New Segal Algebra: A Review of the Feichtinger Algebra.
J. Fourier Anal. Appl., 24(6):1579–1660, 2018.

Banach-Gelfand-Tripel-Homomorphisms



Absolutely Convergent Fourier Series

In his studies Norbert Wiener considered the Banach algebra $(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}})$ of absolutely convergent Fourier series. It was one of the early **Banach algebras**, with *Wiener's inversion Theorem* being an important first example.

Later on it was natural to study the dual space, which of course contains the dual space of $(\boldsymbol{C}(\mathbb{T}), \|\cdot\|_\infty)$, which by the Riesz representation theorem can be identified with the bounded (regular Borel) measures on the torus it was natural to call these functions *pseudo-measures*.

Since $\boldsymbol{A}(\mathbb{T})$ can be identified with $\boldsymbol{L}^1(\mathbb{T})$ (viewed as subspaces of $\boldsymbol{L}^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ respectively), it is natural to expect (and prove distributionally) that $\boldsymbol{PM}(\mathbb{T})$ is isomorphic to $\ell^\infty(\mathbb{Z})$ via the (extended) Fourier transform.

Let me start with a perspective talk ...
The goal of my two presentations is to advertise the concept of Banach frames for Banach Gelfand Triples.
The key aspect of my explanations is the fact, that this is a very natural setting (also from the point of view of applications, e.g. to Gabor Analysis), and secondly that this is not motivated by the idea of generalization, but rather by the wish to find a simple explanation for various cases.

Some historical remarks I

It is well known that the theory of frames was only formulated around 1951 by Duffin and Schaefer and received attention through the work of Daubechies, Grossman and Meyer ("Painless non-orthogonal expansions") published in 1986, in connection with wavelet theory. This is a kind of irony of history, because these concept could or should have appeared as part of basic functional analysis already much earlier in history. I will explain this in two ways.

(1) First of all, generalizing from linear algebra: here the two main concepts are a "generating family of vectors" (in a finite dimensional vector space $\boldsymbol{V}$), and the concept of a "linear independent system", which combined gives us the concept of a basis for a (finite dimensiona) vector space. To have a basis for $\boldsymbol{V}$ is the same as an isomorphism to $\mathbb{C}^n$ (with $\dim(\boldsymbol{V}) = n$);

Some historical remarks II

(2) BOURBAKIsm could have established frame theory from a *structural viewpoint*: Any orthonormal basis defines a *unitary isomorphims* from a separable Hilbert space $\mathcal{H}$ into ℓ^2. Hence one might have formulated special families which establish a weaker property, namely an isomorphism with some closed (and complemented) subspace of the prototypical Hilbert space ℓ^2.

But what was the reason for this shortcoming???

The community had the impression, that the relevant terms have been already found! We all learned in our FA courses:

- A family is *total* (or sometimes called *complete*) if its closed linear space coincides with the given Hilbert/Banach space;
- A set S of vectors (countable or even uncountable) in a Hilbert space $\mathcal{H}$ is called *linear independent* if any FINITE subset of S is linear independent (in the sense of LA);

Linear Algebra Background

Let us shortly recall a few facts from linear algebra, concerning linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ (usually described as $\mathbf{x} \mapsto \boldsymbol{A} * \mathbf{x}$, for some $m \times n$-matrix $\boldsymbol{A}$).
Each such matrix has a **rank** $r := \mathrm{rank}(\boldsymbol{A})$, which is the equal dimension of the column (image space) and the row-space (correctly: column-space of $\boldsymbol{A}^t$).
A matrix is of *maximal rank* (by definition) if $r = min(m, n)$.
For the case $m = n$ this is equivalent to invertibility, for $m < n$ to the fact that the columns of $\boldsymbol{A}$ generate $\mathbb{R}^m$ and for $n < m$ that the columns of $\boldsymbol{A}$ are linear independent.
$\boldsymbol{A}$ is of maximal rank if and only if $\boldsymbol{A}^t$ is of maximal rank!

I WILL DEMONSTRATE the situation quickly (with my bare hands), and indicate the great value of the SVD (Singular Value Decomposition) and the Pseudo-Inverse.

The FOUR SPACES by Gilbert Strang I

Geometrically the situation can be explained as follows:
Both $\boldsymbol{A}$ and $\boldsymbol{A}^t$ have a nullspace, which sits inside of $\mathbb{R}^n$ and $\mathbb{R}^m$.
It is a one-line argument (!Ex!) to show that the nullspace of $\boldsymbol{A}$ is the orthogonal complement of the row-space of $\boldsymbol{A}$ (actually column-space of $\boldsymbol{A}^t$!) and vice versa.
Restricted to that row-space the mapping $\mathbf{y} \mapsto \boldsymbol{A} * \mathbf{y}$ establishes an isomorphism between image of the mapping (= column space of $\boldsymbol{A}$) and the column-space of $\boldsymbol{A}^t$.
The rest are orthogonal projections onto these two r-dimensional subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.
The *pseudo-inverse* is undoing this isomorphism of r-dim. subspaces, thus realizing the MNLSQ solution to $\boldsymbol{A} * \mathbf{x} = \mathbf{b}$.

The FOUR SPACES by Gilbert Strang II

Gilbert Strang promotes the importance of these four space, resp. two orthogonal splittings of domain and target space in his book on Linear Algebra.
The SVD (using eigenvalue decompositions of $\mathbf{A} * \mathbf{A}^t$ or $\mathbf{A}^t * \mathbf{A}$) allows to show that the r-dim. isomorphism can even be put into the form of a diagonal matrix with non-neg. singular values, if appropriate orthonormal bases are chosen.

$$\boldsymbol{S}_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$$

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. In the sequel a Gaussian is used as the window.

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Moreover one can show that $\boldsymbol{S}_0(\mathbb{R}^d)$ is the smallest non-trivial Banach spaces with this property, i.e.it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda) g d\lambda$$

which implies for any space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with $\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$:

$$\|f\|_{\boldsymbol{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda)g\|_{\boldsymbol{B}} \, d\lambda = \|g\|_{\boldsymbol{B}} \|f\|_{\boldsymbol{S}_0}.$$

Basic properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ resp. $\boldsymbol{S}_0(G)$

THEOREM:

- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $\boldsymbol{S}_0(G)$; $[with (\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F}\boldsymbol{S}_0(G) = \boldsymbol{S}_0(\hat{G})$; (Invariance under the Fourier Transform);
- $T_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(G/H)$; (Integration along subgroups);
- $R_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(H)$; (Restriction to subgroups);
- $\boldsymbol{S}_0(G_1)\hat{\otimes}\boldsymbol{S}_0(G_2) = \boldsymbol{S}_0(G_1 \times G_2)$. (tensor product stability);

Basic properties of $S_0'(\mathbb{R}^d)$ resp. $S_0'(G)$

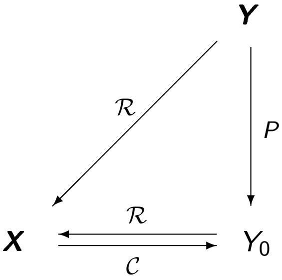
THEOREM: (Consequences for the dual space)

- $\sigma \in \mathcal{S}(\mathbb{R}^d)$ is in $S_0'(\mathbb{R}^d)$ if and only if $V_g\sigma$ is bounded;
- w^*-convergence in $S_0'(\mathbb{R}^d)$ is equivalent to pointwise convergence of $V_g\sigma$ (uniformly over compacts);
- $\left(S_0'(G), \|\cdot\|_{S_0'}\right)$ is a Banach space with a translation invariant norm;
- $S_0'(G) \subseteq \mathcal{S}'(G)$, i.e. $S_0'(G)$ consists of tempered distributions;
- $P(G) \subseteq S_0'(G) \subseteq Q(G)$; (sits between pseudo- and quasimeasures)
- $T(G) = W(G)' \subseteq S_0'(G)$; (contains translation bounded measures);

DIAGRAMS I

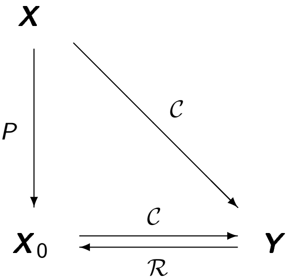
Think of $\boldsymbol{X}$ as something like $\boldsymbol{L}^p(\mathbb{R}^d)$, and $\boldsymbol{Y} = \ell^p$:

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range Y_0 of $\mathcal{C}$ in $\boldsymbol{Y}$:



DIAGRAMS II

Riesz Basis case: E.g. $\mathbf{X}_0 \subset \mathbf{X} = L^p$, and $\mathbf{Y} = \ell^p$:



DIAGRAMS III

A suggestion for making the bring the well established notion of Banach frames closer to the setting we are used from the Hilbert space and ℓ^2-setting and more general solid BK-space (lattices of sequence spaces).

DIAGRAMS IV

Definition

A mapping $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ defines an unconditional (or solid) Banach frame for $\mathbf{B}$ w.r.t. the sequence space $\mathbf{Y}$ if

1. $\exists \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, with $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,
2. $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$ is a solid BK- space over I, i.e. $\mathbf{c} \mapsto c_i$ is continuous from $\mathbf{Y}$ to $\mathbb{C}$ for each $i \in I$, and $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : \ |x_i| \leq |z_i| \ \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$;
3. finite sequences are dense in $\mathbf{Y}$ (at least w^*).

Corollary

By setting $h_i := \mathcal{R}\mathbf{e}_i$ *we have* $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum c_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ *unconditional in* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, *hence* $f = \sum_{i \in I} \mathcal{C}(f)_i h_i$.

DIAGRAMS V

We may talk about Gelfand frames (or Banach frames for Gelfand triples) resp. Gelfand Riesz bases (as opposed to a Riesz projection basis for a given pair of Banach spaces).

We will also talk about BGTr-frames, simply saying that the same form of diagrams is valid as in the linear algebra case, but now with respect to the BGTr $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and $(\ell^1, \ell^2, \ell^\infty)$.

Recall also, that in the context of Riesz bases the term "Riesz projection bases" was introduced by G. Zimmermann in his PhD thesis in the 1990th (under J. Benedetto).

DIAGRAMS VI

A widely known Gelfand-Riesz situation is arising in the context of wavelet constructions, and in particular for so-called *spline-type* spaces (principal shift invariant spaces).
Here the mapping $\mathbf{c} \in (\ell^1, \ell^2, \ell^\infty)(\Lambda)$ to $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ given by

$$\mathbf{c} \mapsto \sum_{\lambda \in Lambda} c_\lambda T_\lambda \varphi$$

is an injective morphism of BGTrs, and the biorthogonal family $(T_\lambda \widetilde{\varphi}$ creates the left inverse BGTr morphism.

Kernel Theorems I

The so-called Kernel Theorem for $\boldsymbol{S}_0$-spaces allows to establish a number of further unitary BGTr-isomorphism. It involves certain types of operators, their (integral) kernels, but also their representation as *pseudo-differential* operators, via the Weyl or Kohn-Nirenberg symbol, or (important for applications in mobile communication) their spreading distribution.
These situations allow to make use of the general principles. In order to understand the transformation one can start from the core spaces, where the analogy with the finite dimensional case is valid in a very natural sense, e.g. if one has $K(x, y) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, then $K(x, y) = T(\delta_y)(x)$, in analogy to the matrices $a_{n,k} = [T(\mathbf{e}_k)]_n$.
The Hilbert space case of the well-known characterization

Kernel Theorems II

Theorem

There is a unitary BGTr isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and* $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))(\mathbb{R}^d)$*, which is a unitary mapping between* $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ *and* $\left(\mathcal{HS}, \|\cdot\|_{\mathcal{HS}}\right)$*, with*

$$\langle T_1, T_2 \rangle_{\mathcal{HS}} = \operatorname{trace}(T_1 \circ T_2^*).$$

Alternative unitary BGTr describe operators via *Kohn-Nirenberg* symbol resp. their spreading representation, such as $T \in \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} H(\lambda)\pi(\lambda), \quad H \in \boldsymbol{S}_0(\mathbb{R}^{2d}).$$

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
Gabor Book 1998.

Wilson Bases

For the case $G = \mathbb{R}^d$ one can derive the kernel theorem also from the description of operators mapping ℓ^1 to ℓ^∞ or vice versa (in a w^*-to-norm continuous way).
The key is the fact, that local Fourier basis, but in particular the so-called Wilson bases are suitable for modulation spaces. In our situation we can formulat the following

Theorem

Any ON Wilson basis (obtained by a smart pairwise recombination of the elements of a tight Gabor frame of redundancy 2) establishes a unitary BGTr between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ *and* $(\ell^1, \ell^2, \ell^\infty)$.

H. G. Feichtinger, K. Gröchenig, and D. F. Walnut.
Wilson bases and modulation spaces.
Math. Nachr., 155:7–17, 1992.

Frame Operator

One of the key results (based on a modern variant of Wiener's inversion theorem) by Gröchenig and Leinert implies that for the case of a regular Gabor system (g, Λ), with $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one also has: $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem

Assume that $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ has the property, that $(\pi(\lambda)g)_{\lambda\in\Lambda}$ defines a Gabor frame, i.e. such that $S_{g,\Lambda}$ is invertible as operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Then this operator is also a Banach Gelfand Triple automorphims for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Note: First $g \in \boldsymbol{S}_0$ implies boundedness as BGTr morphism. The strong point is the conclusion from invertibility at the Hilbert space level to invertibility as BGTr morphism, or equivalently $\widetilde{g} = S_{g,\Lambda}^{-1}(g) \in \boldsymbol{S}_0(\mathbb{R}^d)$.

BGTr-Frames I

Already in the book chapter with G. Zimmermann (1998) we have shown (without using the BGTr terminology) that for a dual pair of Gabor atoms $(g, \widetilde{g})$ (both in $\boldsymbol{S}_0(\mathbb{R}^d)$!) we can say, that the coefficient mapping $f \mapsto V_g(f)|_\Lambda$ defines a BGTr-frame for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ (with respect to the prototypical BGTr $(\ell^1, \ell^2, \ell^\infty)$.
In other words: The synthesis mapping

$$(c_\lambda) \mapsto \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) \tilde{g}$$

is a BGTr-morphism from $(\ell^1, \ell^2, \ell^\infty)$ into $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which is the left inverse to the sampling operator

$$f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}.$$

BGTr-Frames II

In particular, $g \in \mathbf{S}_0'(\mathbb{R}^d)$ belongs to $\mathbf{S}_0(\mathbb{R}^d)$ resp. $\mathbf{L}^2(\mathbb{R}^d)$ if and only if the sampling sequence belongs to ℓ^1 or ℓ^2 respectively.

H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, 1998.

H. G. Feichtinger and F. Luef.
Wiener amalgam spaces for the Fundamental Identity of Gabor Analysis.
Collect. Math., 57(Extra Volume (2006)):233–253, 2006.

Stability I

For many reasons it is important to know about the stability of Gabor frames, either regular (i.e. for a lattice of the form $\mathbf{A} * \mathbb{Z}^{2d} \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, for a non-singular matrix $\mathbf{A}$, or for general, so-called irregular families), in the following sense:
Early results in Gabor analysis (joint work with Janssen, Kaiblinger and Gröchenig) establish the following facts:

1. There are windows $g \in \mathbf{L}^2(\mathbb{R})$ such that the family $((\pi(\lambda)g)_{\lambda\in\Lambda})$ is a Bessel family for all rational lattices of the for $a\mathbb{Z} \times b\mathbb{Z}$, for $a, b \in Qst$, but which is not a Bessel family for all rational multiples of some specified irrational lattice. Hence Bessel bounds are locally unbounded!

H. G. Feichtinger and A. J. E. M. Janssen.
Validity of WH-frame bound conditions depends on lattice parameters.
Appl. Comput. Harmon. Anal., 8(1):104–112, 2000.

Stability II

2. In contrast, one can say that for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and any compact set of matrices $2d \times 2d$ matrices $\boldsymbol{A}$ there is a common Bessel bound for all the corresponding lattices.
3. This is used in order to derive that the mapping $(g, \Lambda) \mapsto \widetilde{g} = S_{g,\Lambda}^{-1}(g)$ is continuous from $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \times M_{2d} \to (\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ on the open set of all pairs (g, Λ) generating Gabor frames.
   In other words: Assume that (g, Λ) establishes a BGTr-frame for $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ with respect to $(\ell^1, \ell^2, \ell^\infty)$ for some g, Λ, then nearby couples have the same property, with similar dual atoms (in fact: good approximate dual frames are just those using nearby lattices, e.g. rational ones).

H. G. Feichtinger and N. Kaiblinger.
Varying the time-frequency lattice of Gabor frames.
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.

We next come to the consequences of abstract coorbit theory to the concrete case of Gabor Analysis.
Generally speaking one can say that coorbit theory shows, how discretization of convolution operators on suitable locally compact groups can be used to derive the existence of (many) frames, in fact, Banach frames for BGTrs (!) whenever one can start from an irreducible, integrable (hence square integrable) group representation on some Hilbert space.
Wavelets (with the $ax + b$-group), and the STFT (with the reduced Heisenberg group $\mathbb{R}^d \times \widehat{\mathbb{R}}^d \times \mathbb{T}$) are the key examples, shearlets, Blaschke groups, and many other (Lie) groups provide further examples.

H. G. Feichtinger and K. Gröchenig.
Banach spaces related to integrable group representations and their atomic decompositions, I.
J. Funct. Anal., 86(2):307–340, 1989.

Irregular Gabor Frames I

While for the bad $\boldsymbol{L}^2(\mathbb{R})$-atoms described above there is no chance for having a Gabor frame, even for arbitrary fine lattices, coorbit theory allows us to make the following statement:
Recall, that a (discrete, countable) set $X = (x_i)_{i \in I}$ is called *relatively separated* if it is the finite union of separated sets (with some positive minimal distance between points of the subset).

Theorem

Given $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ there exists some $\delta = \delta(g) > 0$ such that any δ-fine discrete set X which is relatively separated generates a BGTr Banach frame for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ with respect to $(\ell^1, \ell^2, \ell^\infty)$. Moreover the bounds are uniform for compact subsets of atoms $g \in \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and all equally dense and well-spread sets X.

Irregular Gabor Frames II

Even in this more general situation one can show that one has stability with respect to rigid (and even non-rigid) deformation, UNLIKE the ordiary $L^2(\mathbb{R})$-case (take the box-funtion $\chi_{[0,1]}$ and $a = b = 1$, which is an ONB for $(L^2(\mathbb{R}), \|\cdot\|_2)$).

G. Ascensi, H. G. Feichtinger, and N. Kaiblinger.
Dilation of the Weyl symbol and Balian-Low theorem.
Trans. Amer. Math. Soc., 366(7):3865–3880, 2014.

This situation is of course related to what has turned (thanks to Pete Casazza) into the Feichtinger conjecture.
Note also, that in the absence of relative separation one can still obtain weighted frames of uniform quality for a given $\delta > 0$, small enough, depending on the quality of $V_g(g)$.

frametitle: bibliography I

H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, 1998.

H. G. Feichtinger and N. Kaiblinger.
Quasi-interpolation in the Fourier algebra.
J. Approx. Theory, 144(1):103–118, 2007.

H. G. Feichtinger and N. Kaiblinger.
Varying the time-frequency lattice of Gabor frames.
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.

H. G. Feichtinger and F. Weisz.
Inversion formulas for the short-time Fourier transform.
J. Geom. Anal., 16(3):507–521, 2006.

H. G. Feichtinger.
Choosing Function Spaces in Harmonic Analysis,
Vol. 4 of The February Fourier Talks at the Norbert Wiener Center, 2015.

frametitle: bibliography II

H. G. Feichtinger.
Elements of Postmodern Harmonic Analysis.
In *Operator-related Function Theory and Time-Frequency Analysis. The Abel Symposium 2012, Oslo, Norway, August 20–24, 2012*, pages 77–105. Cham: Springer, 2015.

H. G. Feichtinger.
Modulation Spaces: Looking Back and Ahead.
Sampl. Theory Signal Image Process., 5(2):109–140, 2006.

H. G. Feichtinger and M. S. Jakobsen.
Distribution theory by Riemann integrals. Arxiv, 2018.

M. S. Jakobsen and H. G. Feichtinger.
The inner kernel theorem for a certain Segal algebra. 2018.

M. S. Jakobsen.
On a (no longer) New Segal Algebra: A Review of the Feichtinger Algebra.
J. Fourier Anal. Appl., 24(6):1579–1660, 2018.

Book References

K. Gröchenig: Foundations of Time-Frequency Analysis, 2001.
H.G. Feichtinger and T. Strohmer: Gabor Analysis, 1998.
H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, 2003. both with Birkhäuser.
G. Folland: Harmonic Analysis in Phase Space, 1989.
I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.
G. Plonka, D. Potts, G. Steidl, and M. Tasche. Numerical Fourier Analysis. Springer, 2018.

Some further books in the field are *in preparation*, e.g. on modulation spaces and pseudo-differential operators (Benyi/Okoudjou, Cordero/Rodino).

See also www.nuhag.eu/talks.

Conclusion and OUTLOOK I

1. Many problems in Time-Frequency and Gabor Analysis require the use of elements of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. But also classical subjects (questions of multipliers, spectral synthesis, etc.) can be treated more conveniently if formulated in the context of the BGTr.
2. When it comes to frame theory one should not treat them in the setting of pure Hilbert spaces alone, but in the more (also practical useful) context of the BGTr, which offers among others a lot of flexibility (e.g. varying the lattice constant).
3. The best way to understand the situation is in terms of diagrams (as they can be described in any *category*)!

Conclusion and OUTLOOK II

4. The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and more generally modulation spaces appear to be an appropriate framework, due to its Fourier invariance and the existence of a *kernel theorem*.
5. Much work is needed in order to fill the idea of Conceptual Harmonic Analysis (marriage of Abstract and Harmonic Analysis with Computational Harmonic Analysis) with life and to turn it into a foundation of modern Fourier and Time-Frequency Analysis.

Conclusion and OUTLOOK III

Let us finally collect a few more recommendations:

- Good discretizations are structure preserving! For example: a good version of a discrete Gauss function should be (up to the normalization) be FFT invariant!
- Good discretization should show as many properties already for relatively small dimensions as close as possible to the continuous situation;
- Only then one can hope that numerical experiments and computations are supporting the theory and vice versa!
- Maybe the best estimates are not obtained for the classical function spaces such as $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, but maybe via modulation spaces $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$.

Thanks for your attention

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