NuHAG Numerical Harmonic Analysis Group

Convolutions, Fourier Transforms and Rigged Hilbert Spaces

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Personal Remarks I

As some of you may know, I am an Abstract Harmonic Analysist who has turned into an Application Oriented Mathematician, whose work is based on AHA, Function Spaces, but also Numerical Fourier Analysis. The visible expression for this "conversion" is the naming of my work-group as **NuHAG**, the Numerical Harmonic Analysis Group at the Faculty of Mathematics, my university, where I spend 45 years (from 1969 to 2015). Starting as a teacher student in mathematics and physics I was attracted by the clearness of mathematical concepts and confused by the apparent imprecision of some of the physical courses.
Under different circumstances I might have been attracted by the strong school of Mathematical Physics by Walter Thirring

https://de.wikipedia.org/wiki/Walter_Thirring

Personal Remarks II

He was already introducing a lot of mathematical concepts, such as C^*-algebras into theoretical physics, but somehow I never reached this level of physics, because I thought I have first to acquire the necessary mathematical tools, planning to return to physics later on, better equipped with mathematical tools. But somehow I got stuck on the way.

Well, not really. I had my academic career in Harmonic Analysis, I was exposing myself to a variety of applied projects and the contact with the group around Franz Hlawatsch, headed by Wolfgang Mecklenbräuker, and the cooperation with Helmut Bölcskei, based on the work done with Werner Kozek, had a great influence in me, and consequently on my group, and in a way worldwide, within the field of application oriented Harmonic Analysis.

My first courses in analysis have been given in what is known as BOURBAKI style (as e.g. represented in the

Personal Remarks III

"Cours d'Analyse" by Jean Dieudonne, presented to us by Leopold Schmetterer). He was also teaching Classical Fourier and Functional Analysis. So I was well prepared for the courses on Abstract Harmonic Analysis when my then advisor, Hans Reiter, was starting his position in Vienna.

I am talking about this, because only few years ago I realized, that it was no problem for me to accept that one has to learn and to understand the properties of the Banach convolution algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ in order to properly understand the complicated concept of convolution and its properties. In order to do so one has to learn rather abstract concepts: groups, homomorphisms, measure and integration theory and of course various concepts from functional analysis.

But we never heard anything about the background or motivation for all these considerations!

Personal Remarks IV

Well, we heard (but did not understand in detail) that the study of closed ideals of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ has something to do with problem of *spectral analysis*, but for this some strange definitions of the spectrum of a bounded function had to be studied. I myself realized, that there have been various publications at that time which described so called multipliers, i.e. linear operators between certain function spaces which commute with translations, but H. Reiter did not pay much attention to this branch of Fourier analysis.

Already in around 1980 I realized, quite influenced by the work of Paul Butzer and the classical book by Lüke (Signalverarbeitung)

H. D. Lüke. Signalübertragung.
Einführung in die Theorie der Nachrichtenübertragungstechnik.
Berlin-Heidelberg-New York: Springer-Verlag., 1975.

Personal Remarks V

Three periods of HGFei

- 1969 - 1989: Abstract Harmonic Analysis;
- 1989 - 2015: Application Oriented Mathematics, using MATLAB, creating NuHAG;
- 2000 - now: Editor to JFAA: Journal of Fourier Analysis and Applications
- 2006 - now: Conceptual Harmonic Analysis: An attempt to recombine abstract and computational harmonic analysis, to reconciliate the mathematical and the engineering resp. physicists approach to Fourier Analysis.

Overview

- Diagnosis/Problems
- Mathematicians vs. Engineers/Physicists
- Arguments/Techniques
- Therapy (not pathology)!
- Rigged Hilbert Spaces

Linear (Translation) Invariant Systems I

Looking back it would have been really helpful to learn earlier and more about the material taught in engineering courses, concerning linear, time-invariant systems (TILS).
For a mathematician this are linear operators which commute with translations, so they have to be defined on a translation invariant vector space of functions (or distributions, or "signals")!
Of course, also in practice, such a linear operator (or channel) can have different domains (chosen by the user) and correspondingly different target spaces. In certain cases they can be injective, or surjective and even bijective, in other cases not.
Think of Fourier transform defined on $\boldsymbol{L}^1(\mathbb{R})$ or $\boldsymbol{L}^2(\mathbb{R})$. It is injective from $\boldsymbol{L}^1(\mathbb{R})$ to $\boldsymbol{C}_0(\mathbb{R})$, but not surjective, but it is *bijective* on $\boldsymbol{L}^2(\mathbb{R})$, in fact even "energy preserving" (isometric).

Vague and even mysterious explanations for TILS

There are different ways to explain the behaviour of TILS. From a mathematical point of view the goal must always be to "represent" the system as a convolution operator, where the "convolution kernel" is some function, or at least a kind of distribution. In the theory of multipliers in the 1960s alongside with bounded measures the terms *pseudo-measures* and *quasi-measures* appeared.

TILS: Lüke/Ohm

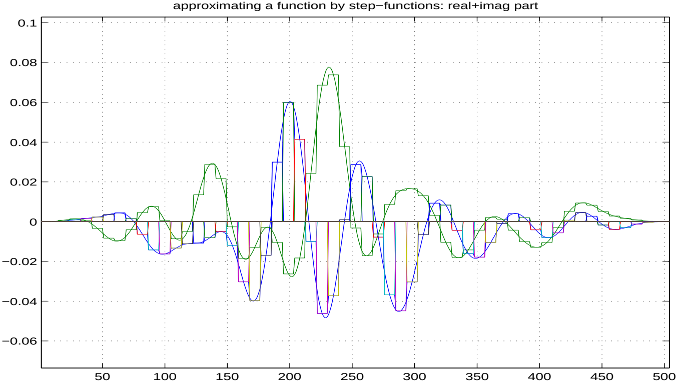


Figure: A typical illustration of an approximation to the input signal of a TILS, named T, preparing for the use of the Dirac impulse.

Questions arising from these pictures

- In which sense does the limit of the rectangular functions exist?
- What kind of argument is given for the transition to integrals? Do we collect (as I learned in the physics course) uncountably many infinitely small terms in order to get the integral?
- In which sense are these step function convergent to the input signal f, and how are the steps determined?
- What has to be assumed about the boundedness properties of the operator T? In other words, which kind of convergence of signals in the domain will guarantee corresponding (or different) convergence in the target domain?

Starting from systems

Let us first look at BIBOS systems, i.e. at systems which convert bounded input in bounded output. If one wants to avoid problems with integration technology and sets of measure zero it is reasonable to assume that the operator has $\left(C_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ as a domain and as target space.

Recall the **scandal in systems theory** observed by I. W. Sandberg.

I. W. Sandberg. A note on the convolution scandal. *Signal Processing Letters, IEEE*, 8(7) (2001) p.210–211.

I. W. Sandberg The superposition scandal. Circuits Syst. Signal Process., 17/6, (1998) p.733-735.

Formulas in engineering books

In one of the books on Fourier Analysis for engineers, at an established US university, I found the following key arguments for the proof of the Fourier inversion theorem:
Primarily the formula (without real proof):

$$\int_{-\infty}^{\infty} e^{2\pi isx} ds = \delta(x)$$

combined with the so-called *sifting property of the Delta Dirac*

$$f(x) = \int_{-\infty}^{\infty} \delta(x-y)f(y)dy.$$

If we write in a slightly more mathematical style this is equivalent to $\delta_0 * f = f$.

TILS: Translation invariant linear systems

A translation invariant system is a linear operator which commutes with translation. In typical situations, i.e. when translation is isometric and continuous on the domain, meaning

$$\lim_{x\to 0} \| T_x f - f \|_{\boldsymbol{B}} \to 0 \quad \forall f \in \boldsymbol{B}.$$

such operators, if they are bounded, satisfy

$$T(g * f) = g * f, \quad \forall f \in \boldsymbol{B}.$$

Then they typical argument is

$$T(f) = T(\delta_0 * f) = T(\delta_0) * f = h * f$$

for the *impulse response* $h := T(\delta_0)$. But does δ_0 belong to the domain of T? And if this is the case, can one say that

$$\widehat{T(f)} = \widehat{h} \cdot \widehat{f}.$$

Taking a closer look

Taking at the closer look and assuming that T is defined on $(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1)$ or $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$, so that δ_0 has to be seen as a (generalized) limit (not in norm) of compressed version of a given function in $\boldsymbol{L}^1(\mathbb{R})$ by dilation, e.g.

$$g_n(x) = 2^{-n} g(x/2^n), n \to \infty,$$

the claim is actually (with various limits involved):

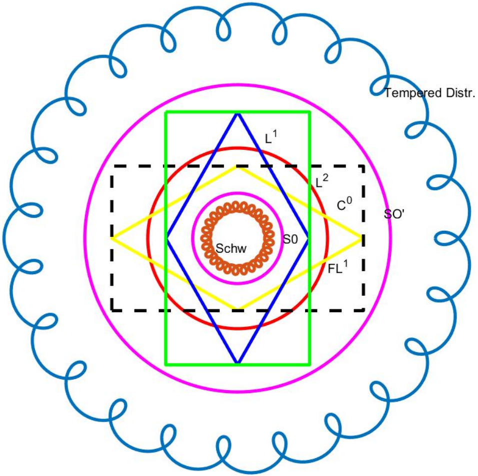
$$T(f) = T(\lim_{n\to\infty} g_n * f) = \lim_{n\to\infty} T(g_n * f) =$$

$$= [\lim_{n\to\infty} T(g_n)] * f = T(\delta_0) * f.$$

No help from the side of mathematics

One might hope that mathematical books on Fourier analysis cover all these topics in a more appropriate fashion and explain all the technical details in order to show that the claims are correct, even if the explanation is just a heuristic one.
BUT CURRENT books in the field cover very little of these problems, but emphasize completely different aspects of Fourier analysis.
One may find details about the famous result of L. Carleson about the almost everywhere convergence of a Fourier series for any periodic function in $\boldsymbol{L}^2$, even if this is not of great *practical importance*. Moreover, very often one may not find a detailed proof of **Shannon's sampling theorem**, which obviously is very important for the understanding of signal processing (of band-limited functions, via their regular samples, taken at or above the Nyquist rate).

A Zoo of Banach Spaces for Fourier Analysis



Something about multipliers I

In a book of R. Larsen one finds some results about so-called "multipliers". For few cases one can completely characterize the multipliers: over $\boldsymbol{L}^1(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\mathbb{R}^d)$.

Theorem

Assume that T is a bounded linear operator on $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ which commutes with translation. Then it is of the form

$$T(f) = \mu * f, \quad f \in \boldsymbol{L}^1(\mathbb{R}^d),$$

for a uniquely determined $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

Something about multipliers II

Theorem

Assume that T is a bounded linear operator on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ which commutes with translation. Then it is of the form

$$T(f) = \mathcal{F}^{-1}(h \cdot \widehat{f})$$

for a uniquely determined, essentially bounded function $h \in L^\infty(\mathbb{R}^d)$.

A typical multiplier would be a chirp function $\chi(t) = e^{it^2}$, which has a (generalized) Fourier transform, but $\chi * SINC$ is not well defined if one uses pointwise (a.e.) defined convolution products.

The MATHEMATICAL viewpoint

1. mathematical formulations are usually precise;
2. sometimes they are viewed as just abstract claims;
3. operators have well-defined domains;
4. formal correctness has priority over relevance;
5. F.A. requires Lebesgue integration;
6. and topological vector spaces (distribution theory);

The ENGINEERING/PHYSICIST's viewpoint

- problems relate to the real world;
- they are motivated by the applications;
- good engineers "know what they do", even if their arguments may be (formally) shaky;
- it is enough to draw an input/output diagram;
- the models may be oversimplified;
- sometime vague terms are used;
- **tradition prevails**, at least in introductory books!

Diagnosis

Overall the diagnosis is a critical one:

- many mathematicians do not care for real applications;
- their results may be correct but useless;
- many engineers do not care to much about formal correctness;
- their results may be correct despite a vague argument;
- both of them ignore the "continuous to discrete/finite" problem, i.e. the transition from continuous signal models to finite dimensional ones as they can be (computationally) realized on a computer (e.g. using MATLAB).

The Problem of Imprecise Domains

While mathematicians are (perhaps too) pedantic about domains and ranges of functions and operators, it is fair to mentioned that physicists or engineers are perhaps sometimes *to generous* (or sloppy) in their derivations of facts (or their teaching of TILS!).
As in real life one has to distinguish between producers and users.
Of course producers have to ensure that their products (e.g. cars, or machines) are *state of the art* and reliable, even if they are not used correctly (according to the instructions).
On the other hand, it may be enough that the teacher in the driving school can make (e.g. how the motor of a car works) plausible *how the car works* without providing a course, starting from the first principles of physics.
Still, inappropriate (over)simplifications may be misleading in *some situations*, and this should be avoided!

Modelling BIBOS systems I

From my point of view the most natural space to start with, when one models TILS, is the space $\boldsymbol{C}_c(\mathbb{R}^d)$ of compactly supported complex-valued functions. It's completion (within $\boldsymbol{C}_b(\mathbb{R}^d)$, with respect to uniform convergence, i.e. with respect to the sup-norm) is just $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, the space of continuous (bounded) functions vanishing at infinity. In fact, it is also an algebra with respect to pointwise multiplication.
A TILS over this space is simply a linear operator from $\boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$ with the two properties

1. $\|Tf\|_\infty \leq C\|f\|_\infty, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d)$,
2. and $T \circ T_x = T_x \circ T, \quad \forall x \in \mathbb{R}^d$.

Modelling BIBOS systems II

With not too much effort one can show, that every such TILS is just a moving average, arising from some functional $\nu \in \boldsymbol{C}_0'(\mathbb{R}^d)$, the space of (bounded) linear functionals on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, usually called the space of bounded (regular, Borel) measures, via

$$Tf(x) = T_x\nu(f) = \nu(T_{-x}f), f \in \boldsymbol{C}_0(\mathbb{R}^d), x \in \mathbb{R}^d.$$

This is an isometric isomorphism, i.e. the operator norm of the system equals the functional norm of the associated function ν

$$\sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty = \sup_{\|f\|_\infty \leq 1} |\nu(f)|.$$

Since a translation operator $\delta_u = \delta(t-u)$ corresponds to the shift T_{-u} one introduces an extra flip in order to come to the usual convolution: $\mu * f(x) := \mu(T_x f^\checkmark)$.

Modelling BIBOS systems III

Then $\delta_x * f = T_x(f)$, hence $\delta_0 * f = f$ (an easy variant of the sifting property), using $\check{f}(z) = f(-z)$ and then

$$\mu * f(x) = \mu(T_x f^{\checkmark}), \quad \text{and} \quad \mu(f) = T(f^{\checkmark})(0).$$

Since the composition and linear combinations of TILS give an algebra we can transfer this structure to the dual space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, by claiming that

$$(\mu_2 *_M \mu_1) * f = \mu_2 * (\mu_1 * f),$$

which is the natural extension of the rule

$$\delta_x * \delta_y = \delta_{x+y}$$

Then one can go on and show that these TILS can in fact extended to TILS on all of $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$, the space of bounded

Modelling BIBOS systems IV

and continuous functions on $\mathbb{R}^d$ with the sup-norm (but this gives only the "reasonable" TILS, not the pathological ones, cf. Sandberg's scandal story) and then one can show that the pure frequencies $\chi_s(t) = exp(2\pi i s \cdot t), s, t \in \mathbb{R}^d$, are *eigenvectors* for $f \mapsto \mu * f$, we call the *eigenvalues* $\widehat{\mu}$, i.e.

$$\mu * \chi_s = \widehat{\mu}(s), \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d), s \in \mathbb{R}^d.$$

This gives a clean approach to the Fourier-Stieltjes transform (without measure theory!), and contains the ordinary Fourier transform for $\boldsymbol{L}^1(\mathbb{R}^d)$-functions (the absolutely continuous measures), via the classical definition

$$\widehat{f}(s) = \langle f, \chi_s \rangle = \int_{\mathbb{R}^d} f(t)\overline{\chi_s(t)}dt = \int_{\mathbb{R}^d} f(t)e^{-2\pi i s t}dt.$$

Alternative interpretation

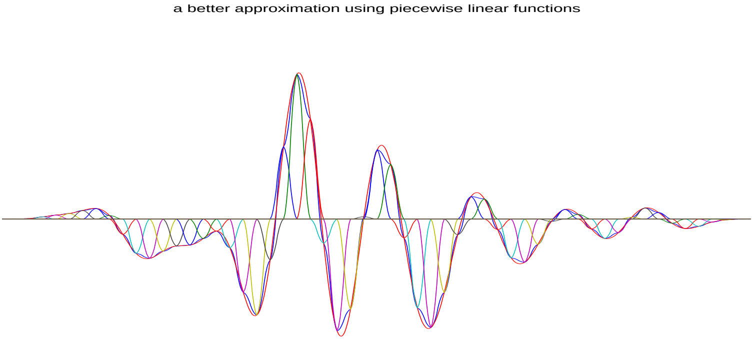


Figure: The piecewise linear functions are special cases of our approximations of the form $\mathrm{Sp}_\Psi(f)$ and thus converge to f in the uniform norm. Hence by the BIBOs assumption we can also ensure that the output $T(\mathrm{Sp}_\Psi(f))$ will also converge to $T(f)$ uniformly.

Impulse response and transfer function

The above picture can be now used in the same way as usual: One has to use a Dirac sequence of compressed triangular functions, which tend to δ_0 in the usual sense, namely

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \Delta_n(t) f(t) = f(0) = \delta_0(f),$$

in order to found that the above limit relations are valid and $\mu = \lim_{n\to\infty} T(\Delta_n)$.
The norm convergence of the piecewise linear interpolation of $f \in \boldsymbol{C}_0(\mathbb{R})$ combined with the assumed boundedness of T justify the other limit exchanges.
For details see my TUM lecture notes (2017 and 2018).

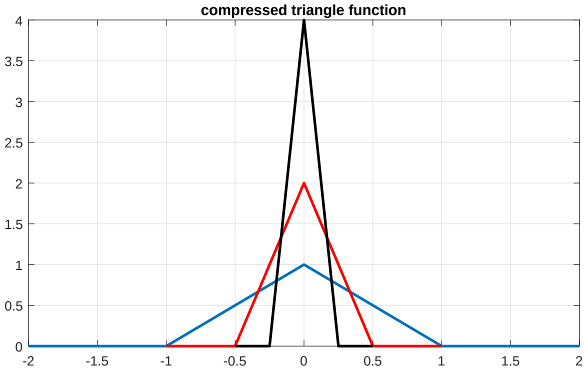


Figure: triplot1.eps

Rigged Hilbert spaces I

There is not enough time to explain the details of THE Banach Gelfand Triple resp. Rigged Hilbert space $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. The short explanation is that it consists of three layers

1. A space of test functions $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$;
2. The Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$;
3. The dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, the space of mild distributions,

with the chain of continuous embeddings

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}).$$

The first embedding is a dense one, the second one, only if one looks at the (important) w^*-convergence:

$$\lim_{n\to\infty} \sigma_n(f) = \sigma_0(f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Rigged Hilbert spaces II

It may be helpful to compare the situation with the embedding of number systems:

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

The idea is: actual computations are done within $\mathbb{Q}$. The real number system $\mathbb{R}$ (with various equivalent realizations, e.g. infinite decimal expressions) is the completion of rationals, with respect to the Euclidean distance $d(q_1, q_2) = |q_1 - q_2|$. And sometimes one understands things better and has more freedom when one works with the complex number.
Also, despite their different appearance the embeddings of one space/field into the other is a natural one, and computations can be carried out at any level, before or after the embedding.

Symbolic and actual computations/integrals I

- $(\frac{3}{4})^{-1} = \frac{4}{3}$
- $(\frac{\sqrt{2}}{\pi})^{-1} = \frac{\pi}{\sqrt{2}}$
- $x(t) = \int_{-\infty}^{\infty} X(\omega) e^{2\pi i \omega t} d\omega$
- $\int_{-\infty}^{\infty} e^{2\pi i s t} ds = \delta_0$
- $\pi \cdot \frac{1}{\pi^2} \cdot \pi = ??$
- $\mathcal{F}(\delta_0) = \mathbf{1}, \mathcal{F}^{-1}(\mathbf{1}) = \delta_0$.

The purpose of this slide is to discuss the discrepancy between actual computation (inversion of numbers, evaluation of a Riemann or Lebesgue integral) and corresponding purely symbolic manipulations. In this sense the equation

$$\pi \cdot 1/\pi = 1$$

cannot be proved, but is a matter of convention concerning the use of the symbol $1/\pi$.

Symbolic and actual computations/integrals II

So what are real numbers: they are the *ideal limits* of their finite approximation (decimal or other number systems), with the idea that one can work at any required level in a concrete situation, as needed.
A similar question is: "What is an image". From a modern point of view I would say: A collection of digital versions of a given scene, with different resolutions, some of the show poor resolution (small number of pixels) or better approximations, suitable for the production of posters. Once we have even better cameras we will be able to get even better approximations.
What is an audio-signal? up to 44100 Hz, or up to 192kHz?

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

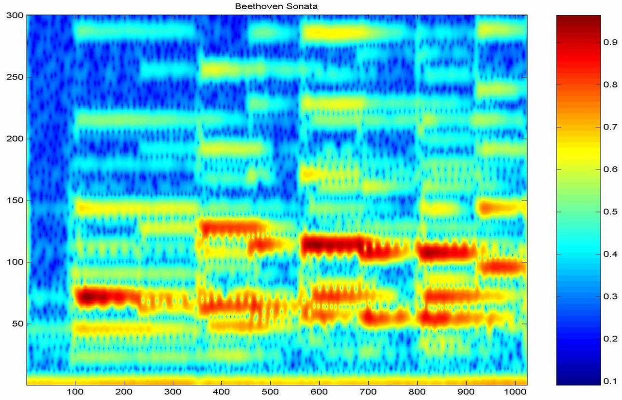
$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t,\omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Various Function Spaces

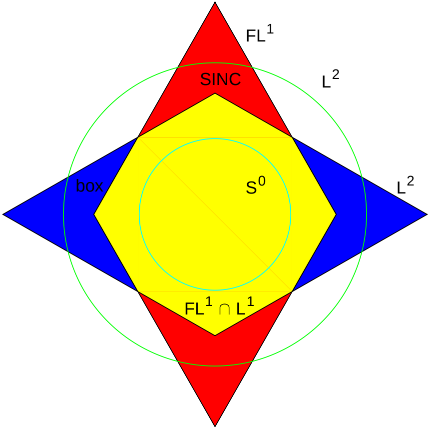


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

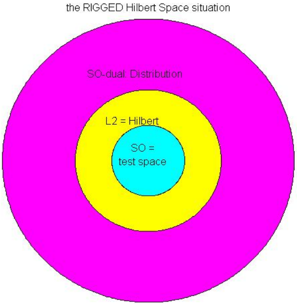
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{1}$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

Characterization of w^*-convergence

Let me shortly mention how convergence of mild distributions (in $\boldsymbol{S}_0'(\mathbb{R}^d)$) can be described in an explicit way:

- Norm convergence is just uniform convergence of the STFT

$$V_g(\sigma_n)(t,\omega) \to V_g(\sigma_0)(t,\omega) \quad \text{uniformly.}$$

- weak-star (w^*) convergence is just pointwise convergence or equivalently uniform convergence over bounded subdomains of phase space!!
  (selling you 3 minutes of music, up to 20 kHz on a CD).

In both cases it is important to prove (or just know) that the notions do NOT depend on the choice of g within $\mathcal{S}(\mathbb{R}^d)$ or even $\boldsymbol{S}_0(\mathbb{R}^d)$.

Multipliers in the BGTr setting

The key result is the identification of the multipliers from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ can be desribed as a convolution operator by a uniquely determined $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$.
In fact

$$Tf(x) = \sigma(T_x f^{\checkmark}) = \sigma * f(x).$$

In fact, any such multiplier maps $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$. From there one gets the above identification, with equivalence of norms.
Using the FT of σ, given by

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in \boldsymbol{S}_0(\mathbb{R}^d)$$

we have the description via transfer distribution $\widehat{\sigma}$:

$$\widehat{Tf} = \widehat{\sigma} \cdot \widehat{f}.$$

Preparing for the Kernel Theorem I

Motivation for the kernel theorem. We write (in the MATLAB spirit) $\mathbf{z}'$ for the *transpose conjugate* of a vector or matrix. Recall that one can describe $\boldsymbol{A} * \mathbf{x}$ coordinatewise, or as a quadratic form through

$$\langle \boldsymbol{A} * \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y} | \boldsymbol{A} | \mathbf{x} \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} x_k \overline{y_j}.$$

But this is of course the same as the scalar product of $\boldsymbol{A}$ with the matrix $\boldsymbol{B} := \mathbf{y}' * \mathbf{x}$ (rank one matrix) in the sense of the Frobenius/Hilbert Schmidt sense:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_{\mathcal{HS}} = trace(\boldsymbol{A} * \boldsymbol{B}') = \langle \boldsymbol{A}(:), \boldsymbol{B}(:) \rangle_{\mathbb{C}^{n^2}} = \sum_{j,k=1}^{n} a_{j,k} b_{j,k}.$$

Preparing for the Kernel Theorem II

Recall that the usual form of the identity operator is given by the unit matrix $\boldsymbol{E}$ (in MATLAB `eye(n)`), resp. by the *Kronecker delta* $\delta_{j,k} = 1$ for $j = k$ and zero else. But this (symmetric) matrix can also be viewed as collection of row-vectors, with $\mathbf{r}_j = \mathbf{e}_j$, the j-th unit vector, so the discrete version of a *Dirac impulse*. The quadratic form is then of course just

$$(\mathbf{x}, \mathbf{y}) \mapsto \langle \mathrm{Id}(\mathbf{x}), \mathbf{y} \rangle \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

resp.

$$\boldsymbol{B} \mapsto \sum_{k=1}^{n} b_{k,k} = \mathrm{trace}(\boldsymbol{B} * \boldsymbol{E}) = \mathrm{trace}(\boldsymbol{B}).$$

Preparing for the Kernel Theorem III

Every linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^n$ is given by a matrix (for a fixed bases), whose columns are the images of the basis vectors, e.g. $T(\mathbf{e}_k), 1 \leq k \leq n$, under the given linear mapping (again described in the coordinates of the same bases, for simplicity). **Matrix multiplication is just made in such a way** (constructive, manipulative) **that the composition of operators** $T_2 \circ T_1$ **corresponds to the matrix multiplication** $\boldsymbol{A}_2 * \boldsymbol{A}_1$, with the usual rules.

On shows that $\boldsymbol{A}_2$ is the (unique) inverse matrix of $\boldsymbol{A}_1$ if $\boldsymbol{A}_2 * \boldsymbol{A}_1 = \boldsymbol{E}$ (unit matrix).

Preparing for the Kernel Theorem IV

As it is clear that not every linear operator between function spaces can be written as a *true integral operator*, defining values pointwise by the continuous analogue of matrix multiplication, namely by some (locally) integrable function $K(x, y)$ of two variables in the form

$$Tf(x) = \int K(x, y) f(y) dy$$

we will use the description through the quadratic form

$$\langle Tf, g \rangle = \int \left(\int K(x, y) f(y) dy \right) g(x) dx.$$

Preparing for the Kernel Theorem V

Writing $B(x,y) = g(x) \cdot f(y) = (g \otimes f)(x,y)$ it is then clear that the identity operator corresponds to the linear function $B \mapsto \int B(x,x)dx$, or a Dirac delta along the main diagonal, or (viewed as a collection of "continuous rows") a collection of Dirac Deltas, resp. the sifting property of the Dirac.

$$f(t) = \int f(s)\delta(t-s)ds.$$

Of course there is a continuous analogue of matrix multiplication for kernels, if they are "nice".

$$K(x,z) = \int K_2(x,y)K_1(y,z)dy.$$

The KERNEL THEOREM for $\mathcal{S}(\mathbb{R}^d)$

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{2}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (2) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)

One of the corner stones for the kernel theorem is: One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert, [1]) is the tensor-product factorization:

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k) \hat{\otimes} \boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \quad (3)$$

with equivalence of the corresponding norms.

The KERNEL THEOREM for S_0 II

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

In analogy to the matrix case, where the entries of the matrix

$$a_{k,,j} = T(\mathbf{e}_j)_k = \langle T(\mathbf{e}_j), \mathbf{e}_k\rangle$$

we have for $K \in \boldsymbol{S}_0$ the continuous version of this principle:

$$K(x, y) = \delta_x(T(\delta_y), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$*, where the first set is understood as the* w^* *to norm continuous operators from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\mathbb{R}^d)$*, the so-called regularizing operators.*

Spreading function and Kohn-Nirenberg symbol

1. For $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$$

The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}d\omega\Big)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta,u)M_\eta T_{-u}f(x)dud\eta.$$

$\widehat{\sigma}$ is called the spreading function of T_σ.

Further details concerning Kohn-Nirenberg symbol

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

The kernel $K(x,y)$ can be described as follows:

$$K(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x)$$
$$= \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y-x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

Kohn-Nirenberg symbol and spreading function II

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x),dt\,d\nu\,,$

Spreading representation and commutation relations

The description of operators through the spreading function and allows to understand a number of commutation relations.
If an operator is a limit (in the strong operator topology) of translation operators it is just a convolution operator with some $\tau \in \boldsymbol{S}_0'(\mathbb{R}^d)$, resp. its spreading representation is just an element concentrated on the *time axis* (more or less representing $\hat{\tau}$, the "individual frequency contributions".
Similarly, multiplication operators require just the use of modulation operators, so their spreading function is concentrated in the frequency axis of the TF-plane.
Finally typical *Gabor frame operators* arising from a family of Gabor atoms (g_λ), where $\lambda \in \Lambda$, some lattice within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ typically commute with TF-shift operators, one can say that they are obtained by periodizing the projection operator $f \mapsto \langle f, g \rangle g$ along the lattice.

The symplectic Fourier transform

The *symplectic Fourier transform* connects the Kohn-Nirenberg symbol with the spreading function, i.e.

$$\mathcal{F}_s(\sigma(T)) = \eta(T) \quad \text{resp.} \quad \mathcal{F}_s(\eta(T)) = \sigma(T). \tag{4}$$

$$(\mathcal{F}_{symp} f)(k,l) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y) e^{-2\pi i(k\cdot y - l\cdot x)}; \quad f \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d). \tag{5}$$

It is completely characterized by its action on elementary tensors:

$$\mathcal{F}_{symp}(f \otimes \hat{g}) = g \otimes \hat{f}, \quad f, g \in \boldsymbol{S}_0(\mathbb{R}^d), \tag{6}$$

and extends from there in a unique way to a $w^* - w^*$ continuous mapping from $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ to $\boldsymbol{S}_0'(\mathbb{R}^{2d})$, also $\mathcal{F}_s{}^2 = Id$.

Understanding the Janssen representation

The spreading representation of operators has properties very similar to the ordinary Fourier expansion for functions!
Periodization at one side corresponds to sampling on the transform side, if we understand "translation" either at the level of ordinary translation of the Kohn-Nirenberg symbol (which is the *symplectic Fourier transform* of the spreading function), OR by conjugation of an operator by the corresponding TF-shifts.
In other words: for any given operator T and $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we can **define** [recall $\pi(x,\omega) = M_\omega T_x$ for $\lambda = (x,\omega)$]

$$\pi \otimes \pi^*(T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \tag{7}$$

providing the important *covariance property* for KNS:

$$\sigma[\pi \otimes \pi^*(\lambda)(T)] = T_\lambda[\sigma(T)], \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{8}$$

Periodization goes over to sampling

If we have a "nice operator" T_0 we can form its periodic version $\sum_{\lambda\in\Lambda} \pi \otimes \pi^*(\lambda)(T_0)$ and it is still a well defined operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$. Its KNS is just the Λ-periodization of T_0. Consequently its spreading function is obtained by sampling of $\eta(T) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, over the *adjoint lattice* Λ° and obtain in this case an ℓ^1-sequence.
The adjoint lattice Λ° can be characterized by the fact that

$$\mathcal{F}_s(\sqcup\!\sqcup_\Lambda) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\circ}. \tag{9}$$

For the projection on the Gabor atom $P_g : f \mapsto \langle f, g\rangle g$ the spreading functions is essentially

$$[\eta(P_g)](\lambda) = Vg(g)(\lambda) = \langle g, \pi(\lambda)g\rangle, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Janssen representation II

An important insight concerning the connection between the Gabor atom g, the TF-lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and the quality of the resulting Gabor frame resp. Gabor Riesz basis (e.g. condition number) clearly comes from the *Janssen representation* of the *Gabor frame operator* for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$S_{g,\Lambda}(f) = \sum_{\lambda \in \Lambda} P_{g_\lambda}(f) = \sum_{\lambda \in \Lambda} \pi \otimes \pi^*(\lambda)[P_g]. \tag{10}$$

The periodization principle gives the Janssen representation

$$S_{g,\Lambda} = \eta^{-1}[\eta(S_{g,\Lambda})] = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g(g))(\lambda^\circ)\pi(\lambda^\circ), \tag{11}$$

as an absolutely convergent sum of TF-shifts from Λ°.

Fourier Standard Spaces of Operators

The kernel theorem allows to identify many spaces of linear operators (with different forms of continuity) with suitable FouSSs over $\mathbb{R}^{2d}$.
For example, there are the so-called *Schatten classes* of operators on the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ which are compact operators with singular values in ℓ^p, for $1 \leq p < \infty$. These spaces are *operator ideals* within $\mathcal{L}(\mathcal{H})$, i.e. they are Banach spaces, continuously embedded into the space of compact operators over the Hilbert space $\mathcal{H}$, as well as two-sided Banach ideals, i.e. whenever one has an operator T in such a space, and two bounded operators S_1, S_2 on $\mathcal{H}$, then $S_1 \circ T \circ S_2$ also belongs to that *operator ideal* and the operator ideal norm is bounded by the operator ideal norm of T multiplied with the operator norms of S_1 and S_2.

Conclusion and Recommendation

1. Application oriented mathematics does not have to be sloppy;
2. Both communities, the pure and the applied of to cooperate on providing a better basis for a modern and mathematically correct approach to questions related to Fourier analysis;
3. We have to look out for best practice examples, or the analogue of *consumer reports*, not only concerning the efficiency of algorithms;
4. Since the computational side is part of the modern digital age, but also provides the possibility to run simulation and numerical experiments at a larger scale such an approach, the study of its relationship to their continuous limit deserves a more detail study.

The Official Abstract I

Compared to the classical theory of Fourier Series and Fourier Transforms over Euclidean Spaces the analysis of signals (i.e. functions or distributions = generalized functions) is a relatively young mathematical discipline. Known under the name TIME-FREQUENCY ANALYSIS, or in its discretized form as GABOR ANALYSIS one starts from the so-called sliding-window or Short-Time Fourier Transform (STFT), also known as spectrogram over phase space. The value of the spectrogram is typically the scalar-product between the signal to be analyzed and a coherent state, i.e. a Gauss-function (or some more general Gabor window), shifted to position t and frequency s. This method is also the basis of the MP3 compression algorithm for digital music.
While this transform, originally defined on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, is isometric, mapping energy distributions in time into (smooth) energy distributions in phase space, it is not clear, which

The Official Abstract II

function spaces on phase space correspond to function spaces (in the sense of Triebel, i.e. Banach spaces of tempered distributions) on the time resp. frequency side.

We will explain that the space $\boldsymbol{S}_0(\mathbb{R}^d)$ of all (integrable and continuous functions) is a good reservoir for such studies. It is densely embedded into the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which in turn is contained in the dual space. While members of $\boldsymbol{S}_0(\mathbb{R}^d)$ are characterized by the integrability of their STFT, the dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$ is characterized by boundedness of the STFT. One can say that any piece of music can be viewed as such an object. The abstract concept of weak*-convergence corresponds to uniform approximation of the signal over a finite rectangle in phase space. This is what a good CD provides: very good reproduction of a piece of music within the duration of the piece of music, in the audible range from zero to 20kHz.

The Official Abstract III

The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ is not only a perfect tool for a clean mathematical description of the Fourier transform (much easier to handle than the space of tempered distributions by L.Schwartz), it is also useful for the description of ideas arising in physics or communication theory. Among others it is possible to derive a so-called kernel theorem, the continuous analogue of the matrix representation of a matrix representation of a linear mapping between finite-dimensional vector spaces.

Added in proof:

Overall we are going for a rigged Hilbert space situation, i.e. a Hilbert space, containing a subspace of nice objects and embedded into the dual space of all linear functionals on that dual space. We would like to provide a setting comparable to $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

V. Losert.
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Ann. Inst. Fourier (Grenoble), 30:129–139, 1980.

THANKS!

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