NuHAG Numerical Harmonic Analysis Group

Next Semester: Guest Professor at ETH Zürich

Invariant function spaces as double modules illustrated via Fourier Standard Spaces

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu

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The Double Module Diagram

The key object of this talk is a certain diagram which can be associated to most of the usual function spaces arising in Fourier analysis over LCA groups.
For the sake of convenience we restrict our attention here to the choice $G = \mathbb{R}^d$ and the realm of *tempered distributions*, as a widely known setup. The general results are valid for Banach spaces of *ultra-distributions* over LCA groups.
For each such Banach space we will assign a collection of subspaces, all “with the same norm”, and the diagram will express how they are inter-connected by inclusions (as closed subspaces, from low to high in the diagram).

The MAIN DIAGRAM

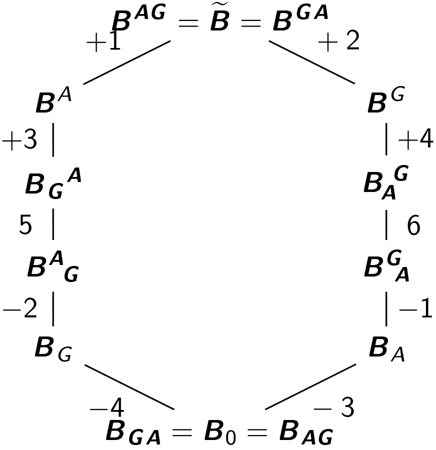

Figure: The MAIN DIAGRAM for double modules

One has to read this diagram by the following rules:
Given any Fourier Standard Space (or more generally a Translation and Modulation Invariant Banach space of (ultra-) distributions) one can form a collection of closed subspaces, based on the two (non-commuting) module structures (which can be unified to *one* such structure making use of the *Schrödinger representation* of the *reduced Heisenberg group*, but this is outside of the scope of this talk.
We rather concentrate on the two module structures developed in [1]. Similar ideas appear in [3] (tempered distributions) and [2].
Abstract version of the results presented have been developed in connection with the compactness criterion in the spirit of Kolmogorov and Riesz in [4].

Banach Algebras and Banach Modules

The presentation will be on *Banach modules* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with respect to different *Banach algebras* $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.
In other words, we call a given Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ a (left/right) Banach module over $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with respect to some action denoted by "$\bullet$" if there is a bilinear operation $(a, b) \mapsto a \bullet b$ which corresponds (essentially) to an embedding of $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ into the operator algebra$\mathcal{L}(\boldsymbol{B})$ of bounded, linear operators. In particular we request

$$a \bullet (a' \bullet b) = (a \cdot a') \bullet b, \quad a, a' \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

Such a Banach module is called *essential* if the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ coincides with all of $\boldsymbol{B}$.

Pointwise Multiplication and Convolution

For us the two prototypical (!commutative) Banach algebras are

1. $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ with respect to pointwise multiplication;
2. $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ with respect to convolution.

In order to make the situation more symmetric with respect to the Fourier transform one can also consider, instead of $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$, the *Fourier algebra* $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$, with respect to the norm

$$\|\widehat{f}\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)} = \|f\|_{\boldsymbol{L}^1(\mathbb{R}^d)}, \; f \in \boldsymbol{L}^1(\mathbb{R}^d).$$

For both $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ and $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ the *Gelfand space* is just $\mathbb{R}^d$ with the usual topology. Moreover, in both cases we do not have a unit element, but a bounded approximate unit (i.e. a bounded family $(e_\alpha)_{\alpha\in I}$ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with

$$\lim_{\alpha\to 0} \|e_\alpha \cdot a - a\|_{\boldsymbol{A}} = 0, \quad \forall a \in \boldsymbol{A}.$$

Pointwise Multiplication and Convolution II

Typical examples of the family of Banach spaces we want to study are the spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p \leq \infty$. **They are BOTH Banach modules over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ with respect to convolution and over $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ under pointwise multiplication.**

Moreover, the essential part $\boldsymbol{B}_\rceil = \boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^\infty(\mathbb{R}^d) = \boldsymbol{C}_{ub}(\mathbb{R}^d)$, the space of uniformly continuous, bounded functions (closed subalgebra in $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$!). The essential part of $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ with respect to the pointwise action of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is the space of functions "vanishing at infinity". The intersection of these two subspaces is just $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$! It is also obvious that they are essential Banach modules if and only if $p < \infty$! On the other hand, these spaces are dual Banach spaces if and only of $1 < p \leq \infty$.

Pointwise Multiplication and Convolution III

Taking the essential part of such a Banach algebra (closed linear span if $\boldsymbol{A} \bullet \boldsymbol{B}$ inside $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$) is in fact the same as just taking the set $\boldsymbol{A} \bullet \boldsymbol{B}$, is a consequence of the *Cohen-Hewitt factorization Theorem* for Banach modules over Banach algebras with BAIs!
We will write $\boldsymbol{B}_{\boldsymbol{A}}$ for this essential part in the sequel, and observe that of course $(\boldsymbol{B}_{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}$, and also we will write (for simplicity of symbols) $\boldsymbol{B}_{\boldsymbol{G}}$ instead of $\boldsymbol{B}_{\boldsymbol{L}^1(G)}$.

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^p(\mathbb{R}^d) \subseteq \boldsymbol{L}^p(\mathbb{R}^d),$$

the convolution inequality together with the norm inequality

$$\|g * f\|_{\boldsymbol{L}^p} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{L}^p}, g \in \boldsymbol{A} = \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B} = \boldsymbol{L}^p(\mathbb{R}^d)$$

results from the fact that translation is isometric on $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ and continuous, in the sense of

$$\lim_{x \to 0} \|T_x f - f\|_{\boldsymbol{L}^p(\mathbb{R}^d)} = 0, \quad \forall f \in \boldsymbol{L}^p(\mathbb{R}^d),$$

The Banach module completion

While taking the "essential part" with respect to a given Banach algebra is an operation comparable to taking the *interior* of a nice set (say a ball), there is a complementary operation, the so-called module completion: $\boldsymbol{B} \mapsto \boldsymbol{B}^{\boldsymbol{A}}$.
Let us start with examples, and recall the concept of a *Banach module homomorphism*, which we denote by $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$.

Definition

A bounded, linear mapping between two Banach $\boldsymbol{A}$-modules $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, with actions $\bullet_1$ and $\bullet_2$ respectively, is called a *Banach module homomorphism* if

$$T(a \bullet_1 b) = a \bullet_2 T(a), \quad a \in \boldsymbol{A}, b \in \boldsymbol{B}^1.$$

Pointwise multipliers

It is not hard to find out that the $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ module homomorphism of $\boldsymbol{B}^1 = \boldsymbol{C}_0(\mathbb{R}^d) = \boldsymbol{B}^2$ are just pointwise multipliers, thus

$$\boldsymbol{H}_{\boldsymbol{C}_0}(\boldsymbol{C}_0(\mathbb{R}^d), \boldsymbol{C}_0(\mathbb{R}^d)) = \boldsymbol{C}_b(\mathbb{R}^d),$$

in fact in the sense of an isometric isomorphism (the operator norm of $M_h : f \to h \cdot f$ coincides with $\|h\|_\infty$ for any $h \in \left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$).
In a similar way we can enlarge $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ to $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, by observing that

$$\boldsymbol{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1(\mathbb{R}^d), \boldsymbol{L}^1(\mathbb{R}^d)) \approx (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$$

according to *Wendel's Theorem*.

Module Completion II

The abstract definition of $\mathbf{B}^{\mathbf{A}}$ is via

Definition

$$\mathbf{B}^{\mathbf{A}} := \mathbf{H}_{\mathbf{A}}(\mathbf{A}, \mathbf{B}).$$

It is easy to check that one has

$$(\mathbf{B}^{\mathbf{A}})^{\mathbf{A}} = \mathbf{B}^{\mathbf{A}}, (\mathbf{B}^{\mathbf{A}})_{\mathbf{A}} = \mathbf{B}_{\mathbf{A}}, (\mathbf{B}_{\mathbf{A}})^{\mathbf{A}} = \mathbf{B}^{\mathbf{A}}, (\mathbf{B}_{\mathbf{A}})_{\mathbf{A}} = \mathbf{B}_{\mathbf{A}}.$$

Note that we can replace any Banach algebra that we have by a smaller Banach algebra, as long as it is dense and still has bounded approximate units of its own. So we could replace $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ by *Beurling algebras* $\left(\mathbf{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$, and we will substitute $\left(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ by the *Fourier algebra* $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$! We could use *Beurling-Fourier algebras* $(\mathcal{F}\mathbf{L}^1_w(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1_w(\mathbb{R}^d)})$!

Non-Commutativity

So far we have two commutative Banach algebras (without units, but with BAIs, both embedded into some larger Banach algebra with unit, namely $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ resp.).
BUT THESE TWO MODULE ACTIONS do NOT commute!

$$h \cdot (g * f) \neq g * (h \cdot f), \quad g \in \boldsymbol{L}^1(\mathbb{R}^d), h \in \boldsymbol{C}_0(\mathbb{R}^d)!$$

Recall that convolution results from translation and multiplication by elements of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ from modulation (by approximation)

$$[T_z f](x) = f(x-z), \quad x, z \in \mathbb{R}^d,$$

$$[M_s f](x) = e^{2\pi i \langle s,x \rangle} f(x), \quad x, s \in \mathbb{R}^d,$$

and translation and modulation operators commute up to *phase factors* there is some hope that the two module structures show some compatibility.

Iterated constructions

For the rest of this section we will consider the very concrete situation of Banach spaces of tempered distributions with a double module structure, which I like to call **Fourier Standard Spaces**.

What is important to know is the following: In the situation described above each of the new spaces allows to apply one of the discussed operations again.
Hence for these space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ not only the symbols

$$\boldsymbol{B}_{\boldsymbol{A}}, \boldsymbol{B}_{\boldsymbol{G}}, \boldsymbol{B}_{\boldsymbol{AG}}, \boldsymbol{B}_{\boldsymbol{GA}}, \boldsymbol{B}^{\boldsymbol{A}}, \boldsymbol{B}^{\mathbf{G}}, \boldsymbol{B}^{\boldsymbol{AG}}, \boldsymbol{B}^{\boldsymbol{GA}}$$

are well defined, but has to think of the mixed symbols

$$\boldsymbol{B}_{\boldsymbol{A}}^{\boldsymbol{G}}, \boldsymbol{B}_{\boldsymbol{G}}{}^{\boldsymbol{A}}, \boldsymbol{B}^{\boldsymbol{G}}{}_{\boldsymbol{A}}, \boldsymbol{B}^{\boldsymbol{A}}{}_{\boldsymbol{G}}$$

and longer chains, such as $(((\boldsymbol{B}_{\boldsymbol{A}})^{\boldsymbol{G}})_{\boldsymbol{A}})_{\boldsymbol{G}}$.

The basic rules

1. Any higher order chain of symbols can be replaced by a chain of order two, which is given by the appearance of the last two symbols of type $\boldsymbol{A}$ and $\boldsymbol{G}$ respectively;
2. $\boldsymbol{B}^{\boldsymbol{AG}} = \boldsymbol{B}^{\boldsymbol{GA}} = \widetilde{\boldsymbol{B}}$, relative completion of $\boldsymbol{B}$;
3. $\boldsymbol{B}_{\boldsymbol{AG}} = \boldsymbol{B}_{\boldsymbol{GA}} = \boldsymbol{B}_0$, the closure of $\mathcal{S}(\mathbb{R}^d)$ in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
4. The main diagram shows a chain of inclusions, labelled by the numbers $1, 2, 3, 4, 5, 6$. Equality appears at $+n$ if and only if appears at $-n$, for $n \in \{1, 2, 3, 4\}$.
5. Minimal spaces show equality at $3, 4$;
6. Dual spaces are maximal and show equality at $1, 2$;

Fourier Standard Spaces, the Idea

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, sandwiched between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions, i.e. with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \quad (1)$$

is called a **Fourier Standard Space** on $\mathbb{R}^d$ (or a **FouSS**) if it has a double module structure over $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution and over the (Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d))$ with respect to pointwise multiplication.

Essentially we require that in addition to (1) one has:

$$\boldsymbol{L}^1 * \boldsymbol{B} \subseteq \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1 \cdot \boldsymbol{B} \subseteq \boldsymbol{B}. \quad (2)$$

Minimal Fourier Standard Spaces

Typically the situation arises in the following context:
Assume in addition to the sandwiching property, that $\mathcal{S}(\mathbb{R}^d)$ is a *dense subspace* of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and that translation is isometric on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ as well as modulation, i.e.

$$\|M_y T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall f \in \boldsymbol{B}, x, y \in \mathbb{R}^d. \tag{3}$$

Then $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a (minimal) Fourier standard space. The dual space of such a minimal standard space is also a FouSS (in fact a maximal one).
Methods from time-frequency analysis allow to show that there is a smallest and a largest member in this family, namely the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (also known as modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$), and its dual, $\boldsymbol{S}_0'(\mathbb{R}^d)$ (or $\boldsymbol{M}^{\infty,\infty}(\mathbb{R}^d)$).

Constructions within the FouSS Family

As a starting point you can take $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$*!*

1. Taking **Fourier transforms**;
2. Conditional dual spaces, i.e. the **dual space** of the closure of $\boldsymbol{S}_0(G)$ within $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
3. With two spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$: take **intersection or sum**
4. forming **amalgam spaces** $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$; e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$;
5. defining pointwise or convolution **multipliers**;
6. using complex (or real) **interpolation methods**, so that we get the spaces $\boldsymbol{M}^{p,p} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)$ (all Fourier invariant);
7. any **metaplectic** image of such a space, e.g. the **fractional Fourier transform**.

Further properties

Theorem

1. *If FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a minimal FouSS, then the dual space belongs to the family and is maximal;*
2. *Conversely, any maximal space is a dual space, with the predual $((\boldsymbol{B}_0)^*)_0$.*

Remark: We all know that $\boldsymbol{B} = \left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a dual space. The predual can be obtained as follows: $\boldsymbol{B}_0 = \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, the dual space is $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. The closure of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is just $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$!

Corollary

A FouSS is reflexive if and only if both $\boldsymbol{B}$ and $\boldsymbol{B}^*$ are minimal.

Minimal Spaces Diagram

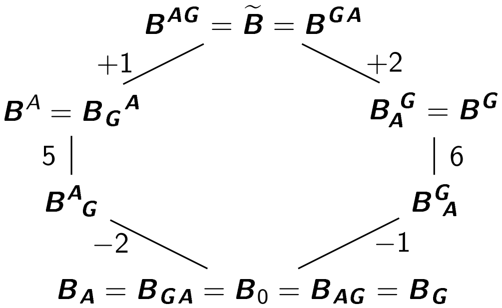

Figure: MINIMAL Spaces Diagram

Maximal Spaces Diagram

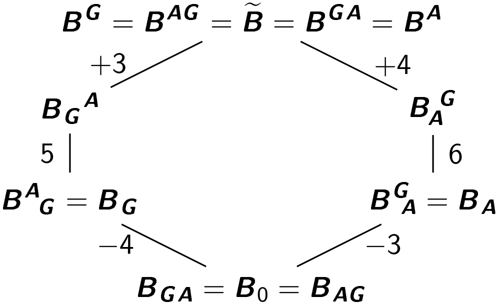

Figure: MAXIMAL Spaces Diagram

sup-norm cases

All the spaces in one of these diagrams have “the same norm” and form FouSS (in particular Banach).

Starting with $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (minimal!) one obtains actually 6 different spaces, with $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ on top.

On the other hand one could start from any of the spaces in that diagram, e.g. $\boldsymbol{B} = \left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and obtain the same collection of spaces, but with a different list of non-collapsing connections (because $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a maximal space)!
For the case $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (minimal) the diagram consists only of 2 spaces, namely $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, with e.g. $\boldsymbol{L}^1(\mathbb{R}^d) = (\boldsymbol{M}_b)_{\boldsymbol{G}}$ (continuous shift!).

The case p=1: $L^1(\mathbb{R}^d)$ and $M_b(\mathbb{R}^d)$

$$\boldsymbol{B}^{G} = \boldsymbol{B}^{AG} = \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{GA} = \boldsymbol{B}^{A}! = \boldsymbol{B}_{A}$$

$$\Big| \, 3$$

$$\boldsymbol{B}_{G} = \boldsymbol{B}_0 = \boldsymbol{B}_{G}{}^{A}$$

Figure: $p = 1$: $\boldsymbol{B} = \boldsymbol{M}_b$

For the case $\boldsymbol{B} = (\boldsymbol{M}_b(\mathbb{R}^d), \| \cdot \|_{\boldsymbol{M}_b})$ we have a collapse of the diagram at $1, 2, 4, 5, 6$. $\left(\boldsymbol{L}^1(\mathbb{R}^d), \| \cdot \|_1\right)$ is a minimal space, and $\boldsymbol{M}_b$ is a maximal space. Hence we just have to spaces. So the distinction occurs at level 3: The elements within $\boldsymbol{M}_b(\mathbb{R}^d)$ with continuous shift are exactly those from $\boldsymbol{L}^1(\mathbb{R}^d)$.

Further interesting cases

Convolutors: Space of convolution kernels from $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ into itself.
The cases $p = 1$ and $p = \infty$ have been discussed already.
In the first case one has only two spaces $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ and $\mathcal{F}\boldsymbol{M}_b(\mathbb{R}^d)$, while for $p = 2$ one has exactly 6 different spaces.
What about $1 < p < \infty$??

Characterization via BAIs,I

For the rest take (e_α) a Dirac sequence and (h_β) plateau functions.

1. $$\lim_\alpha \|e_\alpha * f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{G}}.$$

2. $$\lim_\beta \|h_\beta \cdot f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{A}}.$$

3. $$\lim_{(\alpha,\beta)} \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\alpha \lim_\beta \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\alpha \lim_\beta \|h_\beta \cdot (e_\alpha * f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\beta \lim_\alpha \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

Characterization via BAIs, II

1.

$$\sup_{\alpha} \sup_{\beta} \|h_\beta \cdot (e_\alpha * f)\|_{\boldsymbol{B}} < \infty \quad \Leftrightarrow \quad f \in \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{AG} = \boldsymbol{B}^{GA}.$$

$$\sup_{\alpha} \sup_{\beta} \|e_\alpha * (h_\beta \cdot f)\|_{\boldsymbol{B}} < \infty \quad \Leftrightarrow \quad f \in \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{AG} = \boldsymbol{B}^{GA}.$$

2.

$$\forall \beta \ \lim_{\alpha} \|e_\alpha * (h_\beta \cdot f) - h_\beta \cdot f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in {\boldsymbol{B}^{G}}_{A};$$

3.

$$\forall \alpha \ \lim_{\beta} \|h_\beta \cdot (e_\alpha * f) - e_\alpha * f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in {\boldsymbol{B}^{A}}_{G}.$$

4.

$$\forall \alpha \quad \lim_{\beta} \|e_\alpha * (h_\beta \cdot f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in {\boldsymbol{B}_{G}}^{A}.$$

5.

$$\forall \beta \quad \lim_{\alpha} \|h_\beta \cdot (e_\alpha * f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in {\boldsymbol{B}_{A}}^{G}.$$

Concrete BAIs using dilations

The typical family $(e_\alpha)_{\alpha\in I}$ is a Dirac net (or sequence), obtained by $\boldsymbol{L}^1(\mathbb{R}^d)$ isometric compression of any function $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = \hat{g}(0) = 1$, with

$$[\mathrm{St}_\rho g](x) = \rho^{-d} g(x/\rho), \quad \rho > 0.$$

So $I = (0,1]$ and $\alpha \to \infty$ corresponds to $\rho \to 0$.
Of course one obtains approximate units in $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ (or also $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$) by taking such approximate units to the Fourier side, which corresponds to ordinary dilation. This is the situation of *summability kernels*, with $h(0) = 1$. Again $\beta = \rho$ and $\rho \to 0$ is the relevant limit.

$$[\mathrm{D}_\rho h](x) = h(\rho x), \quad \rho > 0.$$

Typically one takes $g = h = g_0$, the standard Gaussian (which is Fourier invariant), or h is a trapezoidal function (corresponding to the De-la-Vallee Poussin kernel in Fourier Analysis).

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