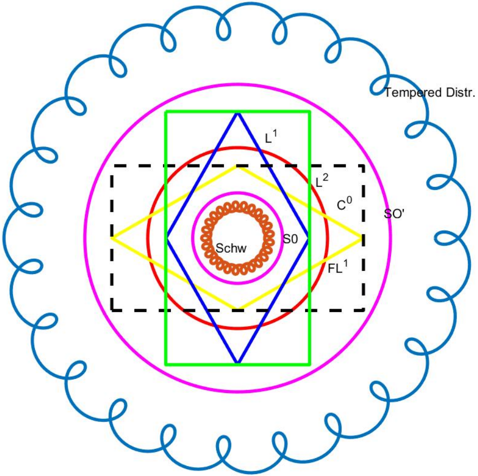
Wiener amalgams and product-convolution operators

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at
www.nuhag.eu TALKS at www.nuhag.eu/talks

GF2020, Ghent, via ZOOM
Talk held 03.09.2020

A Zoo of Banach Spaces for Fourier Analysis



R. C. Busby and H. A. Smith.
Product-convolution operators and mixed-norm spaces.
Trans. Amer. Math. Soc., 263:309–341, 1981.

H. G. Feichtinger.
Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.

H. G. Feichtinger.
Modulation spaces on locally compact Abelian groups.
Technical report, University of Vienna, January 1983.

J. J. F. Fournier and J. Stewart.
Amalgams of L^p and ℓ^q.
Bull. Amer. Math. Soc. (N.S.), 13:1–21, 1985.

Features of Wiener Amalgams

We treat $1 \leq p, q \leq \infty$, the Banach space case.
The guiding principles for the use of Wiener amalgams is the fact that they allow to control in some sense local and global properties of a function (? *separately*).
For step functions or smooth functions membership of $f \in \boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$ is the same as membership in $\boldsymbol{L}^q(\mathbb{R}^d)$.
For compactly supported functions $f \in \boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$ one has just another norm, equivalent to the (local) $\boldsymbol{L}^p$-norm.
A BETTER description is: The norm of a function in a Wiener amalgam space describes the global behaviour (described by the sequence space ℓ^q) of a local quantity, or norm, or quality, expressed by the *local* $\boldsymbol{L}^p$-norm.

Obvious generalization

Instead of using local $\boldsymbol{L}^p$-norm, one may take local Lorentz or Orlicz space norms, and instead of global ℓ^q-norms one may take weighted sequence space norms, this is quite easy to do.
For such simple spaces it is easy to establish the expected duality or (complex) interpolation results!
It is also true what $\boldsymbol{W}(\boldsymbol{L}^p, \ell^p) = \boldsymbol{L}^p(\mathbb{R}^d)$ with equivalence of norms, for $1 \leq p \leq \infty$, and there is a *Hausdorff-Young Theorem*

Lemma

For $1 \leq p, q \leq 2$ one has

$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)) \subset \boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'}).$$

This is consistent with the general fact that *local properties* of f correspond to *global* properties of $\widehat{f}$.

Nice inclusion results

The classical norms for Wiener amalgams decompose $\mathbb{R}^d$ into cubes, obtained by shifting the *fundamental domain* of $\mathbb{R}^d$ with respect to the standard lattice $\mathbb{Z}^d$, and thus takes the $\boldsymbol{L}^p$-norm over all these cubes and then takes a global ℓ^q-sum.
The extreme cases are to take a local sup-norm and a global ℓ^1-norm. The space (of continuous functions inside of) $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)(\mathbb{R})$ consists of all continuous functions with finite upper Riemannian sum (we write $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R})$).
Correspondingly the largest of these classical spaces is $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)(\mathbb{R})$ which consists of all locally integrable functions of "uniform density" in the sense of boundedness of the local integrals.
This space is a closed subspace of the dual of $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$, which is the space of *translation bounded* Radon measures, we write $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R})$.

The relevance of Wiener Amalgam norms for Sampling

The Wiener amalgam spaces, especially those of the form $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)(\mathbb{R}^d)$, are of particular importance for the theory of irregular sampling and for the derivation of convergence results for iterative methods which allow to recover band-limited functions from their irregular samples (as long as the sampling density is high enough, compared to the size of the spectrum of the sampled function).
The other ingredient is of course that a smooth function will not deviate too much from e.g. a piecewise interpolation obtained from the given irregular samples.
One of the key steps in the proof of such theorems (you can download various talks on this subject at `www.nuhag.eu`) is the norm equivalence between the usual $\boldsymbol{L}^p$-norm and the $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$-norm, for a given spectrum Ω.

Discrete versus continuous norm

On the positive side, all the spaces $\boldsymbol{W}(\boldsymbol{L}^p, \boldsymbol{\ell}^q)(\mathbb{R}^d)$ are translation invariant, with a uniform bound (depending mostly on d) on the norm of translation operators.
However, especially the norm in $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ (now for $d \geq 1$) shows a slightly annoying behaviour. Given a bump function supported by say (we look at $d = 2$) one fundamental domain may be split into 4 different pieces, so that the norm is multiplied by $4 = 2^d$ in such a case. One can thus define a new norm, by taking

$$\|f\|_{new} := \sup_{x \in \mathbb{R}^d} \|T_x f\|_{old}$$

which is not really computable and more of theoretical interest. For example, with such a norm $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ is a so-called *Segal algebra* (see H. Reiter's book of 1968).

Discrete versus continuous norms

One of the early ideas introduced into the area was based on a simple consideration: Why not "measures" a function in a continuous way, and describe the continuous behaviour of the local property. Then, instead of the indicator function $\mathbf{1}_Q$, for the cubic domain $Q = [0,1)^d$, one can take any bump function.
This we look (for $q < \infty$) at the *continuous norm*

$$\|f\|_{cont} := \left(\int_{\mathbb{R}^d} \|T_x\varphi \cdot f\|^q_{\boldsymbol{L}^p(\mathbb{R}^d)} \right)^{1/q} .$$

IT IS an EASY EXERCISE to verify that this norm is *equivalent* to the discrete norm, and strictly translation invariant:

$$\|T_x f\|_{cont} = \|f\|_{cont}, \quad \forall x \in \mathbb{R}^d.$$

More general Local Components

The idea behind the spaces now called *Wiener amalgam spaces* with *more general local components* (like Lipschitz or Besov norms) can be easier expressed by the continuous norm: We just replace the local $\mathbf{L}^p$-norm by such a more general norm. However, then the question of the independence of the norm (up to equivalence) on the bump-function arises. It turns out that there is a rather simple condition for a local component $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$:

$$\|T_x\varphi \cdot f\|_{\mathbf{B}} \leq C\|f\|_{\mathbf{B}}, \quad \forall f \in \mathbf{B}. \tag{1}$$

If the norm on $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is translation invariant this simply means that φ is a pointwise multiplier for $\mathbf{B}$, or that φ is sufficiently smooth (and compactly supported).

Basic Facts concerning Wiener Amalgams

- Can one replace the continuous norm by a discrete one?
- Are there natural duality results? interpolation?
- What about pointwise multiplication operators?
- What about convolution operators?

More or less all these questions have been answered in the early papers on the subject, between 1980 (time of writing) and 1985 (publication). At the same time the theory of modulation spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ was developed, with the idea that they should be viewed as $\mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q{}_{v_s}))$.
Nowadays the more elegant approach using the STFT is prevalent in the literature and often the only known variant.

Further references I I

H. G. Feichtinger.
Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, 1983.
(BUPU characterization, multiplication and **convolution** results)

H. G. Feichtinger and P. Gröbner.
Banach spaces of distributions defined by decomposition methods. I.
Math. Nachr., 123:97–120, 1985. (providing duality results)

H. G. Feichtinger.
Banach spaces of distributions of Wiener's type and interpolation.
In P. Butzer, S. Nagy, and E. Görlich, editors, *Proc. Conf. Functional Analysis and Approximation, Oberwolfach*, 153–165. Birkhäuser Boston, Basel, 1981.
interpolation results

T. Dobler.
Wiener Amalgam Spaces on Locally Compact Groups.
Master's thesis, University of Vienna, 1989.
completely continuous derivations

Which symbols to use?

First of all let us recall that there is a continuous and a discrete description of the classical Wiener amalgams. This suggest to use the symbols $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ (thinking of the discrete norm) and $\boldsymbol{W}(\boldsymbol{L}^p, \boldsymbol{L}^q)(\mathbb{R}^d)$, if one has the continuous norm in mind.
We prefer to use the discrete version of the norm most of the time, because then it is clear that $\boldsymbol{W}(\boldsymbol{L}^{p_1}, \ell^{q_1}) \subset \boldsymbol{W}(\boldsymbol{L}^{p_2}, \ell^{q_2})$ if and only if $\boldsymbol{L}^{p_1}(Q) \subset \boldsymbol{L}^{p_2}(Q)$ for compact sets, i.e. $p_2 \leq p_1$, and $\ell^{q_1} \subset \ell^{q_2}$, or $q_1 \leq q_2$.
As a consequence spaces are equal if and only if their parameters are equal, i.e. $p_1 = p_2$ and $q_1 = q_2$.

The Magic Square for Wiener Amalgams

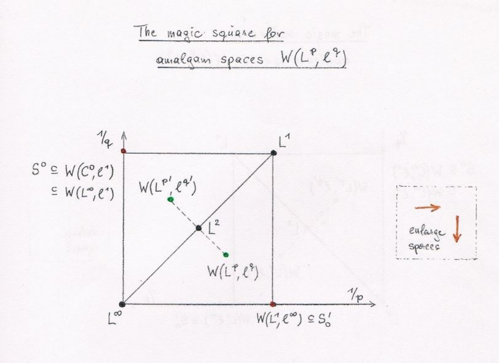


Abbildung: The inclusion relations: magic square

BUT overall classical Wiener Amalgams do not behave well under the Fourier transform!

The Hausdorff-Young Result for Amalgams

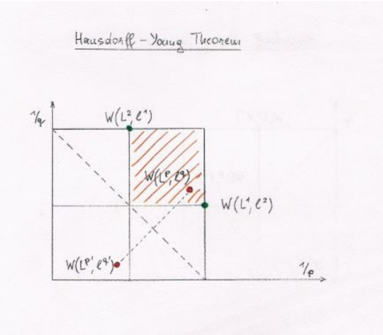


Abbildung: Hausdorff-Young theorem for Wiener amalgams

The Wiener algebra $W(C_0, \ell^1)(\mathbb{R}^d)$ and $S_0(\mathbb{R}^d)$

One of the early observations has been that the so-called *Wiener algebra*, the closure of $\mathcal{S}(\mathbb{R}^d)$ or $C_c(\mathbb{R}^d)$ in $W(L^\infty, \ell^1)(\mathbb{R}^d)$, or just $W(C_0, \ell^1)(\mathbb{R}^d)$ (having in mind that $C_0(\mathbb{R}^d)$ is the closure of $\mathcal{S}(\mathbb{R}^d)$ resp. $C_c(\mathbb{R}^d)$ in $\left(L^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)!$), is the smallest among all translation invariant spaces which allow pointwise multiplication with $C_0(\mathbb{R}^d)$.

In a similar way the Segal algebra $S_0(\mathbb{R}^d) := W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$ has been introduced (in 1979) as the smallest translation invariant Banach space of functions which allows pointwise multiplication with the Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$.

By duality the dual space of $S_0(\mathbb{R}^d)$ is just $W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of tempered distributions which are locally pseudomeasures, uniformly bounded over $\mathbb{R}^d$.

Pointwise multiplication and convolution

We do not formally describe the abstract relations, which hold in the most general form, but instead illustrate them by examples.

Lemma

The spaces $\left(\boldsymbol{M}^p(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^p}\right) := \left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p), \|\cdot\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)}\right)$ *are Banach spaces of distributions which are invariant under the Fourier transform, with*

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^{p_1}(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^{p_2}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$$

if and only if $1 \leq p_1 \leq p_2 \leq \infty$.

Proof: $p = 1$ direct, $p = \infty$ by duality, then interpolation.

Hausdorff-Young Theorem and Sobolev algebras

The proof of Sobolev embedding's theorem can be realized as follows using the Cauchy inequality, whenever $s > d/2$:

$$\mathcal{H}_s(\mathbb{R}^d) = \mathcal{F}^{-1}(\boldsymbol{L}^2_{v_s}(\mathbb{R}^d)) = \mathcal{F}^{-1}(\boldsymbol{W}(\boldsymbol{L}^2, \ell^2_{v_s})) \hookrightarrow \mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^1))$$

But since obviously $1 \leq 2$ we can apply the Hausdorff- Young theorem and obtain

$$\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^2) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \ell^2) \hookrightarrow \boldsymbol{L}^2 \cap \boldsymbol{C}_0.$$

In a similar way one find $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{H}_s, \ell^2)$

Pointwise multipliers of Sobolev spaces

In order to verify that not only $(\mathcal{H}_s(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_s}) \hookrightarrow (\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ for $s > d/2$, but that it is in fact closed under pointwise multiplication we just have to find out that $\boldsymbol{L}^2 * \boldsymbol{L}^2 \subset \boldsymbol{C}_0$ and that $\ell^2_{v_s} = \ell^1 \cap \ell^2_{v_s}$ is closed under convolution, which can be easily derived using the WSA (*weakly sub-additive*) property of $v_s(x) = (1+|x|)^s$:

$$v_s(x+y) \leq C_s(v_s(x) + v_s(y)), \quad x, y \in \mathbb{R}^d.$$

Since $\mathcal{F}\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^2_{v_s})$ we also see that $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2_{v_s}, \ell^2) = \boldsymbol{W}(\boldsymbol{H}, \ell^2)$. As a consequence it is not surprising that one has

$$\mathcal{M}(\mathcal{H}_s(\mathbb{R}^d)) = \mathcal{M}(\boldsymbol{W}(\mathcal{H}_s, \ell^2)) = \boldsymbol{W}(\mathcal{H}_s, \ell^\infty).$$

Product-Convolution Operators

Next we discuss the role of Wiener amalgam for the study of product-convolution and convolution-product operators, i.e. operators of the form

$$f \mapsto g * (h \cdot f) \quad \text{or} \quad f \mapsto h \cdot (g * f).$$

Such operators are often as regularization operators, because convolution produces smoothness and pointwise multiplication produces decay.
It is well known that

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) * (\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)) \subset \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$$

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot (\boldsymbol{\mathcal{S}}(\mathbb{R}^d) * \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)) \subset \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

Regularization of Mild Distributions

We have similar results for the pair $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$

$$\boldsymbol{S}_0(\mathbb{R}^d) \cdot (\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d)) \subset \boldsymbol{S}_0(\mathbb{R}^d),$$

$$\boldsymbol{S}_0(\mathbb{R}^d) * (\boldsymbol{S}_0(\mathbb{R}^d) \cdot \boldsymbol{S}_0'(\mathbb{R}^d)) \subset \boldsymbol{S}_0(\mathbb{R}^d).$$

Derivation (of the first inclusion) via:

$$\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty).$$

using the fact that $\mathcal{F}\boldsymbol{L}^1 * \mathcal{F}\boldsymbol{L}^\infty \subset \mathcal{F}\boldsymbol{L}^1$ and $\boldsymbol{\ell}^1 * \boldsymbol{\ell}^\infty \subset \boldsymbol{\ell}^\infty$, and

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) \cdot \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d),$$

using the fact that $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ is a pointwise algebra.

Recent regularization results

Let us recall that

$$M^1_{v_s}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \,|\, \langle z \rangle^s \cdot V_{g_0} f \in L^1(\mathbb{R}^{2d})\}$$

with $\langle z \rangle^s = (1 + x^2 + y^2)^{1/2}$ and natural norm

$$\|f\|_{M^1_{v_s}} := \|\langle z \rangle^s \cdot V_{g_0} f\|_{L^1}.$$

Theorem

For any $s \geq 0$ we have:

$$M^1_{v_{3s}} \cdot (M^1_{v_{3s}} * M^\infty_{v_{-s}}) \subset M^1_{v_s}. \tag{2}$$

$$M^1_{v_{3s}} * (M^1_{v_{3s}} \cdot M^\infty_{v_{-s}}) \subset M^1_{v_s}. \tag{3}$$

Using inclusion results

Based on the comparison between tensor product weights and radial symmetric weights one obtains the following inclusions:

Lemma

For any $s \geq 0$ one has the continuous (proper) embeddings:

$$\boldsymbol{M}^1_{v_{2s}} \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_s}) \subset \boldsymbol{M}^1_{v_s} \tag{4}$$

$$\boldsymbol{M}^1_{v_{3s}} \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_{2s}}, \ell^1_{v_s}) \cap \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_{2s}}) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_s}) \subset \boldsymbol{M}^1_{v_s} \tag{5}$$

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_{2s}}, \ell^1_{v_s}) + \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_{2s}}) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_s}) \subset \boldsymbol{M}^1_{v_s} \tag{6}$$

Proof.

(I) Let us start with the proof of (2). It can be proved using first the convolution relation (cf. **fe83 : [1]**)

$$\boldsymbol{M}^1_{v_{3s}} * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty_{v_{-s}}, \ell^\infty_{v_{-s}}) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^\infty_{v_{-s}}) \qquad (7)$$

and then the pointwise multiplication result

$$\boldsymbol{M}^1_{v_{3s}} \cdot \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^\infty_{v_{-s}}) \subset \boldsymbol{M}^1_{v_s}. \qquad (8)$$

Recalling that we have $\boldsymbol{M}^1_{v_{3s}} \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_{2s}}, \ell^1_{v_s})$ we observe that (7) can be obtained from the standard convolution relations for Wiener amalgams via

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_{2s}}, \ell^1_{v_s}) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty_{v_{-s}}, \ell^\infty_{v_{-s}}) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^\infty_{v_{-s}}). \qquad (9)$$

□

Proof.

(II) We also need a pointwise multiplier result for Wiener amalgams, using the fact that the Fourier Beurling algebra $\mathcal{F}\boldsymbol{L}^1_{v_s}$ is closed under pointwise multiplication:

$$\boldsymbol{M}^1_{v_{3s}} \cdot \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^\infty_{v_{-s}}) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_{2s}}) \cdot \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^\infty_{v_{-s}}) \subset \tag{10}$$

$$\subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_{v_s}, \ell^1_{v_s}) \subset \boldsymbol{M}^1_{v_s}. \tag{11}$$

OBSERVE that (similar claims are valid for $\boldsymbol{M}^p_{v_s}(\mathbb{R}^d)$, $1 \leq p \leq \infty$):

$$\bigcap_{s>0} \boldsymbol{M}^1_{v_s}(\mathbb{R}^d) = \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \quad \text{and} \quad \bigcup_{s>0} \boldsymbol{M}^\infty_{v_{-s}}(\mathbb{R}^d) = \boldsymbol{\mathcal{S}}'(\mathbb{R}^d).$$

□

TMIBs and DTMIBs

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *minimal tempered standard space* (or a TMIB) if

1. One has the following sandwiching property:

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d); \tag{12}$$

2. $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (minimality);
3. $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is translation invariant, and for $n_1 \in \mathbb{N}$ and $C_1 > 0$:

$$\|T_x f\|_{\boldsymbol{B}} \leq C_1 \langle x \rangle^{s_1} \|f\|_{\boldsymbol{B}} \quad \forall x \in \mathbb{R}^d; \tag{13}$$

4. $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is modulation invariant, and for $n_2 \in \mathbb{N}$ and $C_2 > 0$

$$\|M_y f\|_{\boldsymbol{B}} \leq C_2 \langle y \rangle^{s_2} \|f\|_{\boldsymbol{B}} \quad \forall y \in \mathbb{R}^d. \tag{14}$$

Double modules

Any such TMIB (or its dual) is a double modules (!not a bimodule) in the following sense:

1. a Banach module over some Beurling algebra with convolution
2. a pointwise Banach module over some Fourier-Beurling algebra

Among others one can show that (for example) the *projective tensor product* of two such spaces (over $\mathbb{R}^d$) is another TMIB (over $\mathbb{R}^{2d}$). Typical examples are the space $\boldsymbol{L}^p(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{L}^q(\mathbb{R}^d)$, which can be built as absolute convergent series of elementary tensors (product functions)

$$f \otimes g(x,y) = f(x)g(y), \quad x, y \in \mathbb{R}^d.$$

Generalized modulation spaces

It was a surprise to me that the construction of modulation spaces, i.e. of spaces which are described via the behaviour of the STFT of a tempered distribution in some Banach space $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ of functions over $\mathbb{R}^{2d}$ STILL MAKES SENSE (as pointed out in the recent paper by Stevan Pilipovic, Bojan Prangovski, Pavel Dimovski, and Jason Vindas) for such tensor product spaces. In contrast to the very general abstract approach leaving to coorbit spaces (Fei/Groch, 1989) there is no solidity for such spaces, i.e. one does NOT have:

$$|F(x)| \leq |G(x)| \Rightarrow \|F\|_{\boldsymbol{Y}} \leq \|G\|_{\boldsymbol{Y}}.$$

Generalized modulation spaces II

They define, via the STFT with respect to a Gaussian window:

$$\mathcal{M}^{\boldsymbol{Y}} = \{f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d), V_g(f) \in \boldsymbol{Y}\}.$$

The crucial identity for an identification of some of these space is relying on the adjoint mapping V_g^*, which has the property

$$V_g^*(\phi \otimes \psi) = \mathcal{F}(\psi) \cdot (\phi * g).$$

Since $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_w, \ell^1_w)(\mathbb{R}^d)$ for many (polynomial) weight functions one has to make use of PC-CP mapping properties of amalgam spaces in order to come up with an identification of these new *generalized modulation space*.

Generalized modulation spaces III

Lemma

For $p \in [1,2]$, $\mathcal{M}^{L^p \hat{\otimes}_\pi L^p} = W(\mathcal{F}L^p, L^1)$, with equivalent norms.

The most important special cases are the well-known fact

$$\mathcal{M}^{L^1 \hat{\otimes}_\pi L^1} = \mathcal{M}^{L^1(\mathbb{R}^{2d})} = \boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$$

and the case $p = 2$, which among many others will be part of an upcoming joint paper with Stevan:

Lemma

$\mathcal{M}^{L^2 \hat{\otimes}_\pi L^2} = W(L^2, L^1)$ *with equivalent norms.*

Literature concerning TMIBs I

W. Braun and H. G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

H. G. Feichtinger.
Compactness in translation invariant Banach spaces of distributions and compact multipliers.
J. Math. Anal. Appl., 102:289–327, 1984.

P. Dimovski, S. Pilipovic, and J. Vindas.
New distribution spaces associated to translation-invariant Banach spaces.
Monatsh. Math., 177(4):495–515, 2015.

P. Dimovski, S. Pilipovic, B. Prangoski, and J. Vindas.
Translation–modulation invariant Banach spaces of ultradistributions.
J. Fourier Anal. Appl., 25(3):819–841, 2019.

H. G. Feichtinger and A. Gumber.
Completeness of shifted dilates in invariant Banach spaces of tempered distributions.
submitted, page 12, Aug. 2020. ArXiv.

Further Recent References II I

- E. Cordero, H. G. Feichtinger, and F. Luef.
  Banach Gelfand triples for Gabor analysis.
  In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

- H. G. Feichtinger.
  Choosing Function Spaces in Harmonic Analysis, volume 4 of *The February Fourier Talks at the Norbert Wiener Center, Appl. Numer. Harmon. Anal.*, pages 65–101.
  Birkhäuser/Springer, Cham, 2015.

- H. G. Feichtinger.
  Ingredients for Applied Fourier Analysis.
  In *Sharda Conference Feb. 2018*, pages 1–22. Taylor and Francis, 2020.

- H. G. Feichtinger.
  A sequential approach to mild distributions.
  Axioms, 9(1):1–25, 2020.

- A. Beyni and K. A. Okoudjou.
  Modulation Spaces.
  Springer (Birkhäuser), New York, 2020.

THANKS you for your attention
All the best to Stevan, you go ahead!

To the audience: maybe you visit www.nuhag.eu
and checkout other talks and related material. hgfei
`www.nuhag.eu/talks`, PWD required!