NuHAG Numerical Harmonic Analysis Group

Discrete Hermite Functions and Computational Aspects of the Fractional Fourier Transform

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official abstract I

It is well known, that the effect of the Fourier transform on a spectrogram (the absolute value squared of a STFT) is just rotation by 90 degrees. Hence one may ask whether there are operators which correspond to a rotation by an arbitrary angle. This is in fact possible and well known for decades, in many different appearances, under the name of a Fractional Fourier transform. There are many different approaches, starting in different communities, which have lead to studies in the last 50 years (engineers, physicists, mathematicians). The most powerful (to our mind) is the view-point of Andre Weil, who introduced the metaplectic group (containing the group of Fractional FTs) in his famous paper in Acta Mathematica ([7]).

Fractional Fourier Transforms

There are different ways to describe the *fractional Fourier transform*, corresponding to *rotations* in the time-frequency plane. The ordinary Fourier transform corresponds to a rotation by 90 degrees, which is reflected by the well known formula in [4], (3.10):

$$|V_{\widehat{g}}\widehat{f}(s,-t)| = |V_g f(t,s)|, \quad t,s \in \mathbb{R}^d.$$

This makes plausible that $\mathcal{F}(\mathcal{F}f)(x) = f(-x)$ and $\mathcal{F}^4(f) = f$. The best way to understand the basic properties of the group of fractional Fourier transform $\mathcal{F}_\alpha$, $\alpha \in \mathbb{R}$, with $\mathcal{F}_{\pi/2} = \mathcal{F}$, is to view it as an operator which has a diagonal matrix representation with respect to the Hermite ONB $(h_n)_{n\geq 0}$ in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.

Basic properties of the Hermite ONB

The family $(h_n)_{n\geq 0}$ forms an orthonormal basis for $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, with the property that

$$\mathcal{F}(h_n) = (-i)^n\, h_n, \quad n \geq 0, \tag{1}$$

i.e. consisting of eigenvectors of $\mathcal{F}$.
Taking diagonal matrices of the form $(e^{i\alpha n})_{n\geq 0}$ produces a group of unitary operators $\mathcal{F}_\alpha$, which for obvious reasons is isomorphic to the torus group, and with

$$\mathcal{F} = \mathcal{F}_{\pi/2},$$

named the fractional Fourier transform.
This group is a commutative subgroup of the larger group of *metaplectic transformations* (see [7], [6], or the work of Maurice de Gosson).

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L. B. Almeida.
The fractional Fourier transform and time-frequency representations.
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H. Ozaktas and D. Mendlovic.
Fourier transforms of fractional order and their optical interpretation.
Optics Communications, 101(3-4):163 – 169, August 1993.

H. M. Ozaktas, Z. Zalevsky, and M. Alper Kutay.
The Fractional Fourier Transform, with Applications in Optics and Signal Processing. John Wiley and Sons, 2001.

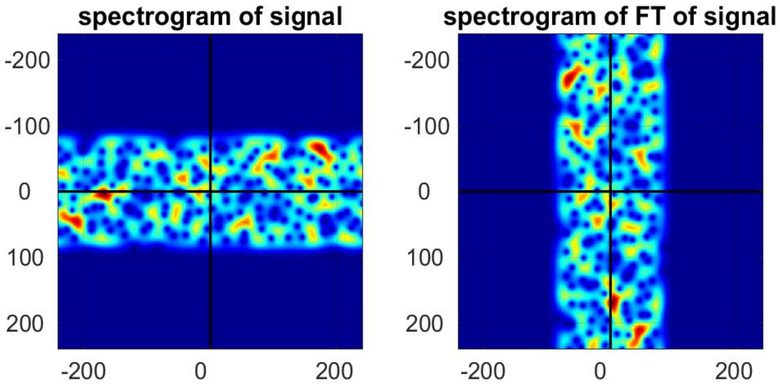
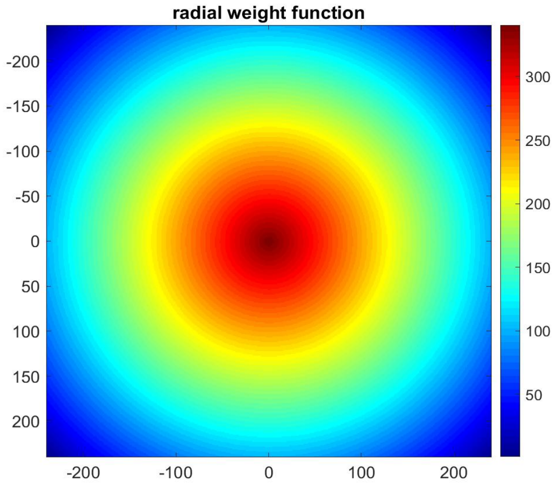


Figure: spectroFTeff1.jpg

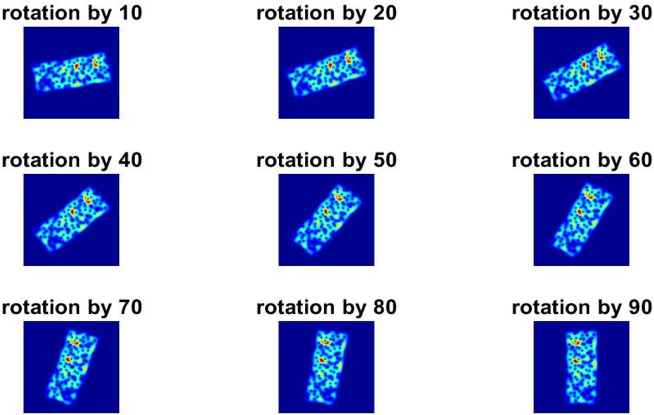
MATLAB Code

```
function [HERM,hermeig,GMW,W] = hermf(n);
RW = radwgh(n); MRW = max(RW(:)); W = 1 + MRW - RW;

[HH, hermeig] = eigsort(gabmulhf(W,g,1,1));

HERM = twtoreal(HH.').';
if norm(imag(HH(:))) < 1000*eps;
HERM = real(HH); end;
for jj=1:n; if sum(HERM(jj,1:round(n/2))) < 100*eps;
HERM(jj,:)=-HERM(jj,:); end; end;

function radM = radwgh(m,n); dm=min(0:m-1,m:-1:1);
dn=min(0:n-1,n:-1:1);
radM= 1+sqrt((ones(m,1)*dn).^2+((dm(:)*ones(1,n)).^2));
```

Radial weight function on phase space



The localized Fourier transform (spectrogram)



Discrete Hermite Functions

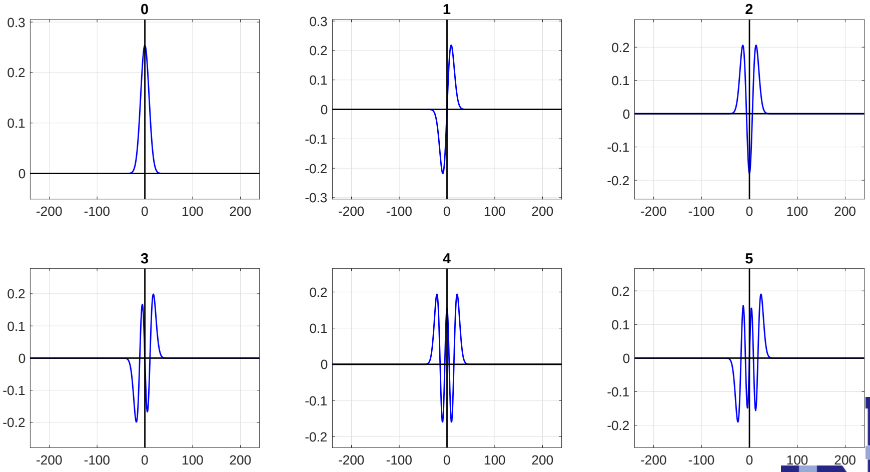
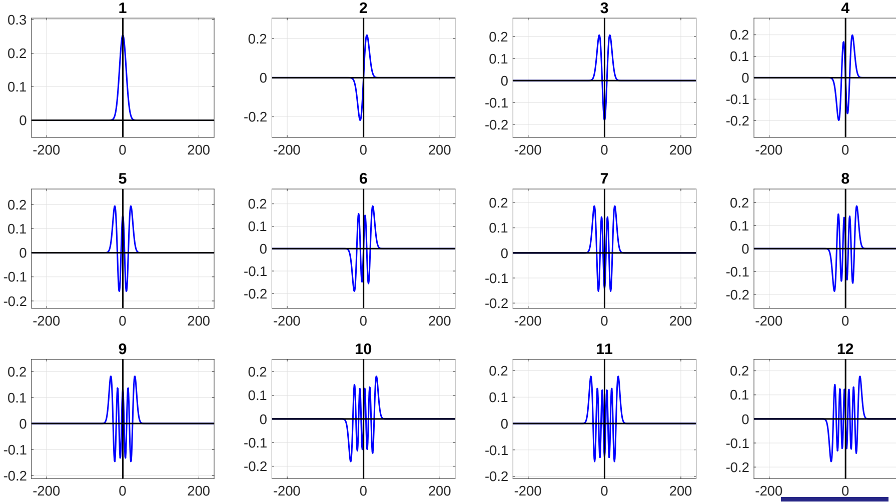
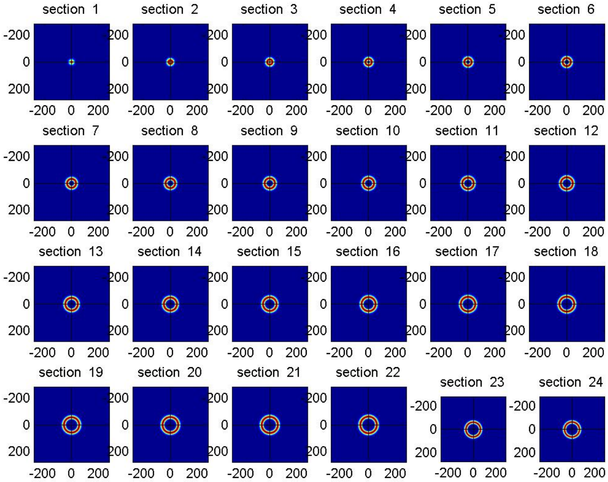


Figure: herm0to5.eps

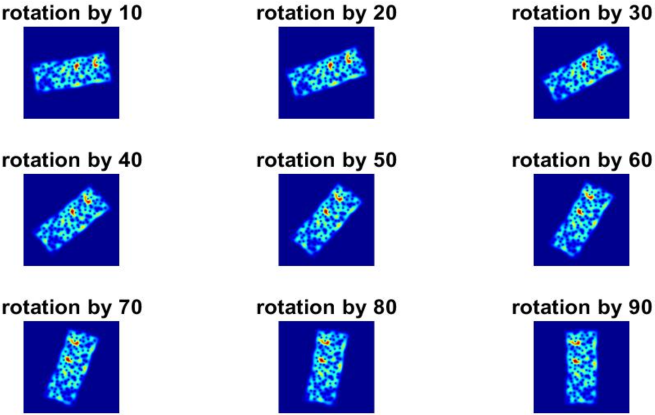
Discrete Hermite Functions



Spectrograms of Hermite Functions



The localized Fourier transform (spectrogram)



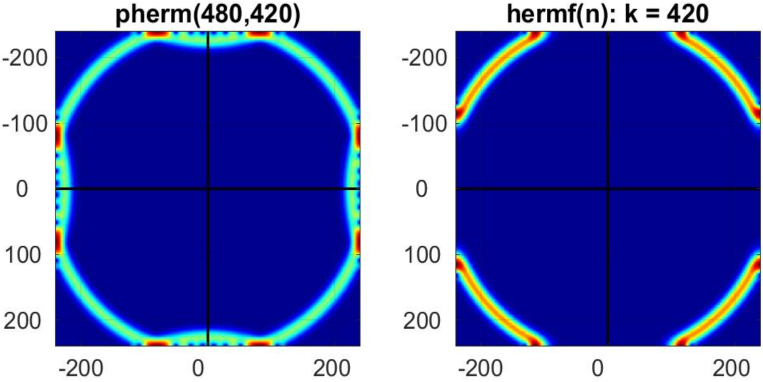


Figure: hermf480a.jpg

Fractional FT applied to linear chirps

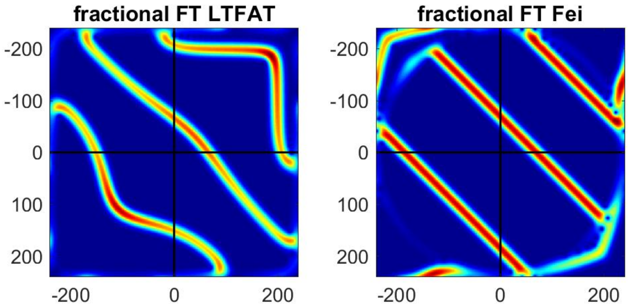


Figure: fracFTpurfr50.jpg

Fractional FT applied to linear chirps

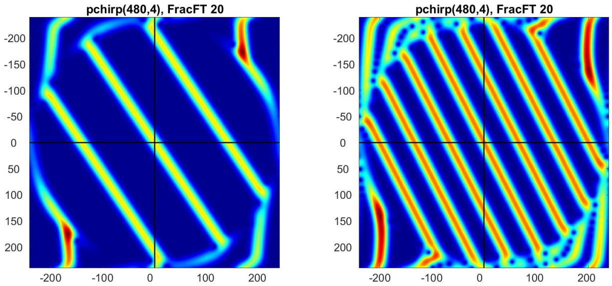
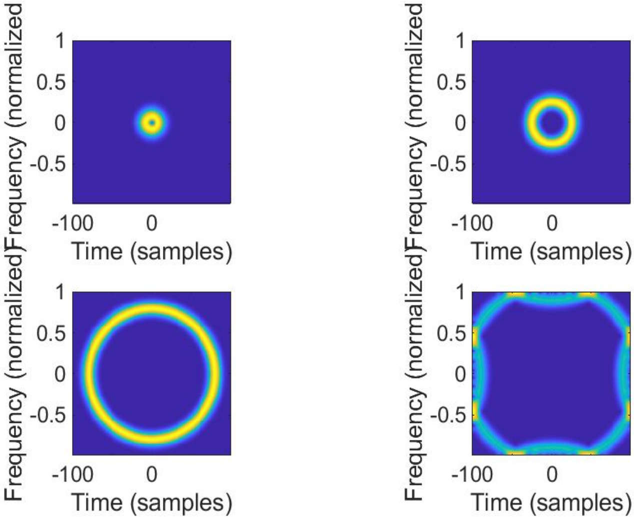


Figure: FracFTchirp01b.jpg

Fractional FT applied to chirp signals



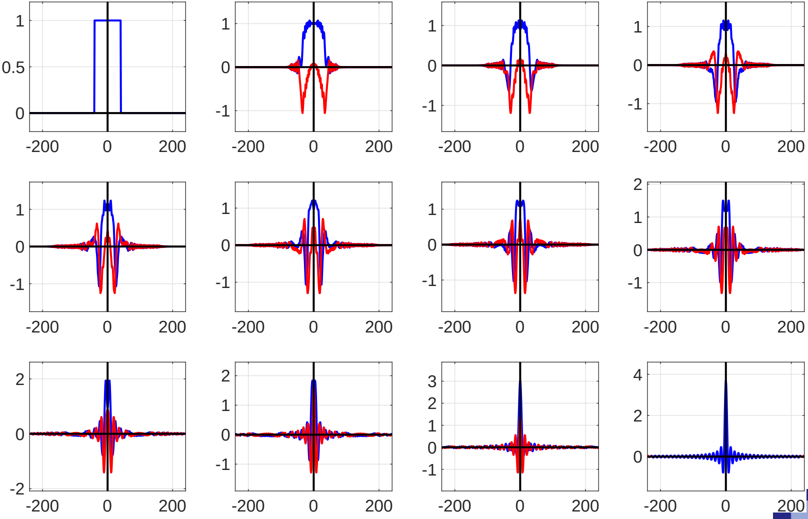


Figure: fracbox.eps

Difference between two types of discrete Hermite Fcts.

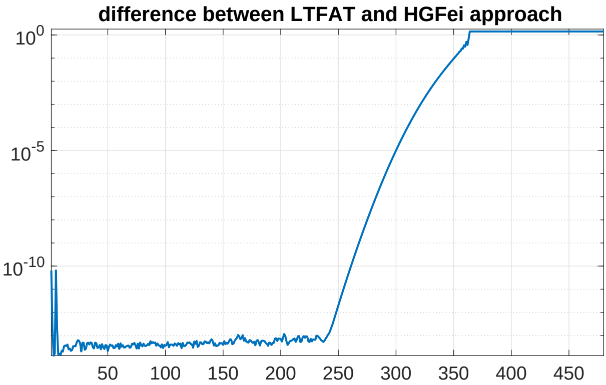
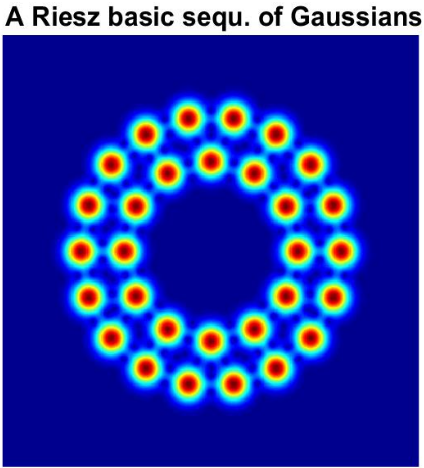


Figure: hermdiff1a.eps

An example of a Riesz basic sequence



Sampling, Approximation by finite sequences

A number of papers covers the problem of connecting the discrete theory with the continuous (via sampling and periodization).

N. Kaiblinger.
Approximation of the Fourier transform and the dual Gabor window.
J. Fourier Anal. Appl., 11(1):25–42, 2005.

H. G. Feichtinger.
Gabor expansions of signals: computational aspects and open questions.
In *Landscapes of Time-Frequency Analysis*, volume ATFA17, pages 173–206. Birkhäuser/Springer, 2019.

J. Fischer.
Four particular cases of the Fourier transform.
Mathematics, 12(6):335, 2018.

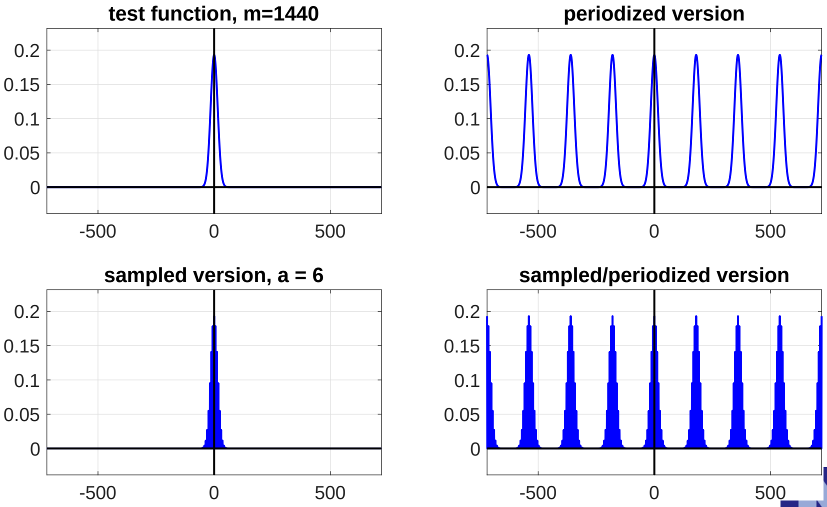


Figure: sampperiod2.pdf

How can one compare vectors of different length

When we work with finite sequences which are *meant to represent a continuous functions*, so typically with equidistant samplings of a smooth functions k, taken from a relatively dense arithmetic progression (over a region containing the support of $k \in \boldsymbol{C}_c(\mathbb{R})$) we expect to *represent* k (or its periodized continuation) well, resp. recover it with small error from the samples (e.g. by piecewise linear interpolation or quasi-interpolation using cubic splines, or other BUPUs!).
This can be compared to the approximation of irrational numbers by rational numbers. But it is hard to check that $53/17$ is in fact a better approximation to π than $41/13$, while it is clear that 3.1415926535897 is a much better approximation than 3.1415. In fact, by adding one decimal term after the other one gets better and better approximations.

In this sense we will look closer at sequences of finite length, by checking the relation between a sequence of length n and a sequence of length $4n$, e.g. $n = 1024$ and $4n = 4096$.
This corresponds to a replacement of the period by a factor of two and the *sampling rate* is also doubled, resp. the sampling distance h is replaced $h/2$.
In our example we have $n = 2^{1}0 = (2^5)^2$, hence it is natural to think of a period of 32 and a sampling rate of $1/32$, which is then doubled to 64 and $1/64$. Obviously a quarter of the samples of the new sampling sequence corresponds to positions used in the coarse scheme. **But it is not obvious, what this means in terms of MATLAB indices!**

Good behaviour of Discrete Hermite Functions

When we study the behaviour of our discrete Hermite functions they do not only show the proper behaviour of discrete analogues of the fractional Fourier transform (transforming linear chirps into linear chirps), **but also very good behaviour with respect to this transition between n to $4n$.**

Following the use of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ for the computation of the Fourier transforms (in the sense of the integral transform) via using the discrete version (FFT) as worked out with N. Kaiblinger on can expect that the application of the DISCRETE FRACTIONAL FT to a sequence of properly sampled functions in $\left(\mathbf{S}_0(\mathbb{R}), \|\cdot\|_{\mathbf{S}_0}\right)$, followed by piecewise linear interpolation, should give a good approximation of $\mathcal{F}_\alpha(f)$, as $n \to \infty$.

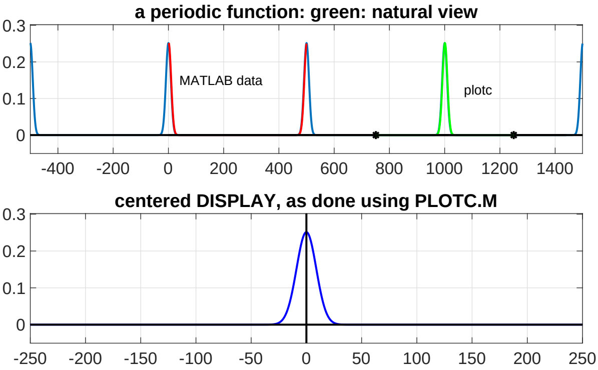
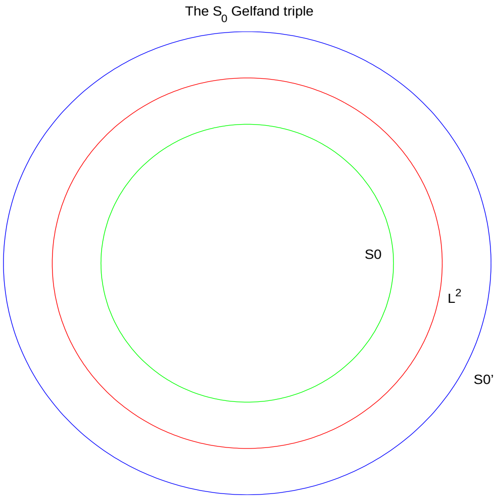


Figure: smpvaldem4a.eps

The S_0-Banach Gelfand Triple



Invariance properties of $\left(S_0(\mathbb{R}), \|\cdot\|_{S_0}\right)$

When it comes to the description of the fractional Fourier transform via "kernels" it is often stated, that the inversion formula holds true **if** $\mathcal{F}_\alpha(f) \in \boldsymbol{L}^1(\mathbb{R})$, but when can one be sure that this is the case (except under the very strong assumption that $f \in \mathcal{S}(\mathbb{R})$!? Here the long established fact (often used in the work of Maurice de Gosson and others) that $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is invariant under the full metaplectic group, is quite remarkable. In fact, this property results from the minimality property of $\left(S_0(G), \|\cdot\|_{S_0}\right)$ (among all isometrically translation and modulation invariant spaces on G).

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A. Weil: , book of K. Gröchenig: [4],
H. Reiter (see , using the Segal algebra $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}))$, and M. DeGosson and coauthors ([1],[2], etc.).
The Segal algebra $\boldsymbol{S}_0(G)$ is described in [3], or [**?**].

M. De Gosson.
Maslov indices on the metaplectic group $Mp(n)$.
Ann. Inst. Fourier (Grenoble), 40(3):537–555, 1990.

M. De Gosson.
On the Weyl representation of metaplectic operators.
Lett. Math. Phys., 72(2):129–142, 2005.

H. G. Feichtinger.
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Monatsh. Math., 92:269–289, 1981.

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M. S. Jakobsen.
On a (no longer) New Segal Algebra: a review of the Feichtinger algebra.
J. Fourier Anal. Appl., 24(6):1579–1660, 2018.

H. J. Reiter.
Metaplectic Groups and Segal Algebras.
Lect. Notes in Mathematics. Springer, Berlin, 1989.

A. Weil.
Sur certains groupes d'opérateurs unitaires.
Acta Math., 111:143–211, 1964.

Some comment on Tauberian Theorems

I would like to mention some work on Tauberian Theorem which is very little known, the so-called Third Tauberian Theorem of Norbert Wiener.

N. Wiener. *The Fourier Integral and certain of its Applications*. Cambridge University Press, Cambridge, 1933.

H. G. Feichtinger.
An elementary approach to Wiener's third Tauberian theorem for the Euclidean *n*-space.
In *Symposia Math.*, volume XXIX of *Analisa Armonica*, pages 267–301, Cortona, 1988.