Numerical Harmonic Analysis Group

An ELEMENTARY APPROACH to MILD DISTRIBUTIONS based on the use of "Feichtinger's Algebra"

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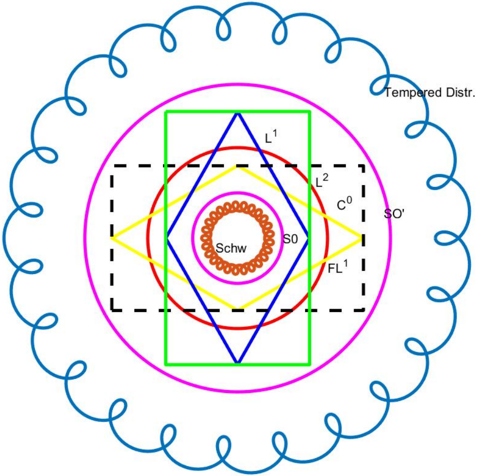
A Zoo of Banach Spaces for Fourier Analysis



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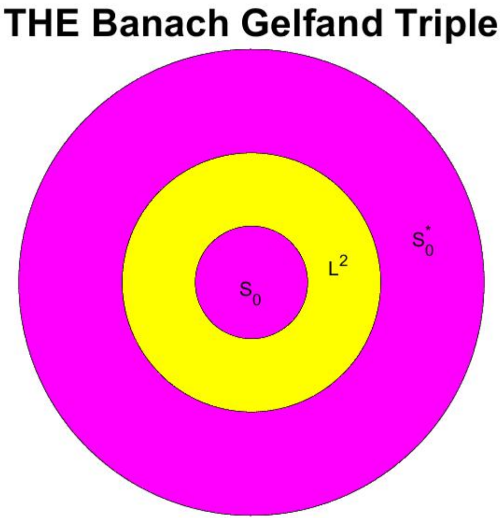
THE Banach Gelfand Triple
S_0^*
L^2
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Abstract versus Conceptual Harmonic Analysis I I

Abstract for the ISAAC talk 2017:
The idea of "Conceptual Harmonic Analysis" grew out of the attempt to make objects arising in Fourier Analysis or Gabor Analysis (such as norms of functions, their Fourier transforms, dual Gabor atoms, etc.) computable. Using suitable function spaces such as the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ it should be possible to find concrete algorithms which allow to compute approximations to the desired on real hardware in finite time, up to (at least potentially) arbitrary requested precision.
Going beyond the ideas of Abstract Harmonic Analysis, which only allows to identify the analogies between objects on different LCA (locally compact Abelian) groups G,

Abstract versus Conceptual Harmonic Analysis II

the idea of **Conceptual Harmonic Analysis** wants to see the connection between these settings to be used, for example, in order to use methods from discrete, periodic Gabor analysis (which are computationally realizable using MATLAB).
The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which can also be seen as a "rigged Hilbert space", allows to describe and justify a number of such procedures providing also a tool to deal with periodic, but also of continuous or discrete signals in a unified way. As it turns out the so-called w-convergence is crucial for the description, and fine partitions of unity as well as regularizing operators play a crucial role in this context.

The need for suitable function spaces

The connection between different settings such as discrete versus continuous, or numerical versus analytical, requires to have a suitable collection of function spaces, for me ideally Banach spaces of distributions, which allow to describe convergence (in the norm sense or in the w^*-sense) and to study robustness of implementable algorithms with respect to precision of the available data (so a highly practical question).
We might have simple linear equation of the form $T(f) = g$ for given right hand side, and we would like to be able to guarantee that a *realizable, constructive algorithm* (this is more than just *constructive approximation!) can achieve an approximate solution within a given ε-error, for a norm which is relevant for the problem at hand. Better approximation should be achievable at a hight computational cost.*

The TRUE SCANDAL

What is for me the true scandal that we have to fight is the situation that Fourier Analysis is perceived and understood completely differently by Engineers and by Mathematicians.
And this is not the fault of either group, but the consequence of a natural development. But it is not fate and unavoidable, and we should do something about it!
Do we have answers for the engineering students, why

- why they should be careful with **integrals** in certain cases, and ignore those rules in other cases (!Dirac!)?
- why there are so different rules for **convolution**, depending on the context (e.g. periodic or non-periodic functions)?
- how the different types of **Fourier transforms** are related to each other?

The Current Course at ETH

Mathematical Methods for Signal Processing

The current course continues to explore ideas and material prepared in the last years, mostly for lectures given at TU Muenich (2017,2018) and ETH Zuerich (2015,2020) or Charles University Prague (2018).

More and more it is possible to connect the ideas of “conceptual harmonic analysis” with the literature and methods used by engineers.

It is not only a matter of terminology or techniques used (e.g. Lebesgue integration, Schwartz distribution theory) which has to be understood, but also a lot of *psychological components*!

Main LINK: `www.nuhag.eu/ETH20` and `https://www.univie.ac.at/nuhag-php/home/skripten.php`

The setting

The goals for this course (and lecture notes prepared for it) are:

- Propose a concrete way to work mathematically correct with a framework relevant for engineers;
- Avoid the use of Lebesgue integration or topological vector spaces;
- Show that the new approach is actually covering most of the important aspects of Applied Fourier Analysis
- Propose new mathematical questions relevant for applications.
- Providing clear concepts concerning the *functional analysis* required (from scratch), such as norm or w^*-approximation.

THIS REQUIRES A LOT OF WORK!

Various function space Pictograms

It appears indispensable to use a number of function spaces, mostly Banach spaces of continuous or measurable functions, dual spaces may consist of (Radon or bounded) measures, or generalized functions, also called *distributions*, in the spirit of *tempered distributions*.

We think that the use of certain distribution spaces is important in order to deal properly with *Dirac measures* or *Dirac Combs*.

In the last few years the collection of illustrating pictograms for function spaces have been developed in great detail, so that within this zoo of function spaces inclusion results can be easily visualized.In particular we have: $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is the smallest, and $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ the biggest among the so-called Fourier standard spaces, but this will be discussed elsewhere!

Operators

Operators		
T_z	$T_z f(x) = f(x - z)$	translation by z
M_s	$M_s f(x) = e^{2\pi i s \cdot x} f(x)$	modulation operator
St_ρ	$\mathrm{St}_\rho f(x) = \rho^{-d} f(x/\rho)$	stretching operator
D_ρ	$\mathrm{D}_\rho f(x) = f(\rho x)$	dilation operator
	$f^{\checkmark}(x) = f(-x)$	flip operator
	$f^*(x) = \overline{f(-x)}$	$\boldsymbol{L}^1$-involution
	$\overline{f}(x) = \overline{f(x)}$	conjugation operator

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Personal Motivation

This talk is an attempt to push further some general ideas, related to the ideas developed in the last 10 to 20 years, based on the *experience* in the field between pure and applied mathematics, between the engineering[1] view (and formalism) and the mathematical literature available to better understand what is needed by the engineers, to *learn to understand their terminology* and to translate it into mathematical terminology.
Often the meaning of their symbols is different from how it is interpreted by mathematicians proverb (whose is it?, W. Thirring):

It doesn't have to be sloppy, just because it is applied!

These attempts are part of a *long term project to renew Fourier Analysis*, with the idea of improving the situation.

[1]Or the symbols used by physicists in quantum theory, using *rigged Hilbert spaces*.

Observations made in the last decades I

During the last decades I have made the following experiences:

- The use of Fourier Analysis in the applied and in the mathematical community drift apart, more and more, and one community does not care about the other (also physicists do not care too much about engineers...).
- A typical view is: mathematicians work with *high precision on problems which are not so relevant for applications, AND engineers get things to work (like mobile communication) despite questionable simplifications in the justification of their algorithms (for example).*
- *It is not even well known in the "other community" what the goals are. Meanwhile (due to the age of the field) local habits (using symbols, teaching the material) get frozen.*

What we need, according to my understanding

Thus, at the psychological and practical level, we need, as part of the scientific community, more of

- communication across disciplines;
- reading papers from the "other side";
- (as mathematicians): develop tools which are really useful for the applications
- (as applied person): discuss with mathematicians what the effective goal is in a given situation

Concepts that might help

Just as a *wordy answer* to these questions I can point to some of the concepts and the vocabulary that has been developed in the last years, because it is not only a technical problem, but also a problem of *communication*.

I have suggested in the last years to go **beyond Abstract Harmonic Analysis** (AHA), which considers Fourier Analysis over different groups G (such as $G = \mathbb{R}, \mathbb{Z}, \mathbb{R}^d, \mathbb{T}, \mathbb{Z}_N, \mathbb{Z}_N \times \mathbb{Z}_N$) is parallel universes with *analogue structures* (group, dual group, characters, Fourier transform $\mathcal{F}_G$, etc.) to introduce (and make use of it in our daily work!) of Conceptual Harmonic Analysis, meaning the integration of Abstract and Computational Harmonic Analysis. We need to understand how the different groups are related to each other, how the FFT-algorithm can be used to compute the FT of a smooth function, or how to compute dual Gabor atoms for irrational lattices.

The current ETH course

I am convinced, that we need to take a *fresh approach to Fourier Analysis*, in order to be able to fill this abstract idea with life. In fact, it was the chance to contribute to some of the practical questions (irregular sampling, Gabor Analysis, etc.) in the last decades, and the experience with the tools which turned out to be helpful compared to other, often well established mathematical principles which did not make life easier, but rather more and more involved as *"scientific progress"* was made.

For this fresh approach *Time-Frequency Methods* certainly will play an important role, and the minimal tool-set is what I have called THE Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, consisting if "Feichtinger's Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$", the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and the dual space of *mild distributions.*

See `www.nuhag.eu/ETH20` *for the current course, also with a lot of material for download and YouTube recording links.*

Some historical perspectives

The original approach (On a New Segal Algebra, 1981) to $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ in aimed at the introduction of a function space that can be defined on general LCA groups G, which is a Segal algebra (with functorial properties, as V. Losert was showing very soon), and minimal among all those Segal algebras which allow isometric TF-shifts (or *strong character-invariance*).
Contacts with the engineers (in particular F. Hlawatsch) and further studies led to the theory of *modulation spaces*, the *coorbit theory* (jointly with K. Gröchenig), and *decomposition spaces*.
The usefulness of the triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ became apparent with the contributions to the first Gabor book in 1998 (work with W. Kozek and G. Zimmermann), the term *Banach Gelfand Triple* was introduced later, see S. Bannert (2010, master thesis, Vienna).

Where to start from? (the overall landscape)

While the original approach started from $(\boldsymbol{L}^2(G), \|\cdot\|_2)$ over LCA group G was based on the theory of Wiener amalgam the more elegant approach, using the STFT (Short-Time FT) is well described in the book of K. Gröchenig from 2001.
There one starts from $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ (well defined via the Lebesgue integral), known to be a Hilbert space, which contains $\boldsymbol{C}_c(\mathbb{R}^d)$ (compactly supported, continuous, complex-valued functions on $\mathbb{R}^d$) as a *dense subspace*, also satisfying $\boldsymbol{C}_0(\mathbb{R}^d) \cdot \boldsymbol{L}^2(\mathbb{R}^d) \subset \boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^2(\mathbb{R}^d) \subset \boldsymbol{L}^2(\mathbb{R}^d)$, together with corresponding norm estimates.
In a compact terminology we would say, that $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ is a *Double Banach module*, namely with respect to the Banach algebra $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ (and pointwise multiplication), and also with respect to $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$, with respect to convolution.

The terminology of Banach Modules

If we see this setting then it is very convenient to use the terminology of *Banach modules*, as popularized by Mark Rieffel (in the late 60th), and the use of different types of approximate units in the algebras involved.
In fact both $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (or its Fourier image, the Fourier algebras $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$, with respect to pointwise multiplication and the norm $\|hatf\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_{\boldsymbol{L}^1}, f \in \boldsymbol{L}^1(\mathbb{R}^d))$ we have *commutative algebras* which do not have a unit, but they do have *bounded approximate identities (BAIs)*, namely "huge plateu-like functions" or "Dirac sequences" r espectively.
Then $\boldsymbol{L}^2(\mathbb{R}^d)$ is an *essential Banach module* with respect to both algebra structures, because these BAIs act on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ in the expected way: $\|\mathrm{St}_\rho g * f - f\|_{\boldsymbol{L}^2} \to 0$, for $\rho \to 0, f \in \boldsymbol{L}^2(\mathbb{R}^d)$.

But do we need to learn Lebesgue Theory first?

It was an interesting insight to me in the last two decades to slowly *realize, that - unlike I was told by my advisor - Lebesgue's integration theory it not the only door to Fourier Analysis!*
It is true, that the integrals defining the Fourier transform in the usual way, is very well justified for the Lebesgue integral. It is also true that convolution, viewed as a kind of abstract multiplication between functions, is also well defined in this context, because the usual convolution integral for $f * g(x)$ is well defined almost everywhere whenever $f, g \in \boldsymbol{L}^1(\mathbb{R}^d)$ (due to the Fubini Theorem). Again, $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$ is a good additional condition for the pointwise Fourier inversion theorem!
But such a view-point *does not allow* to show that $f \to \text{chirp} * f$, with $\text{chirp}(t) = e^{i\pi t^2}$ is the so-called *chirp function*, because $\text{chirp} * \text{SINC}(x)$ does NOT exist for any $x \in \mathbb{R}$.

Why do engineers care about convolution?

The reasons, why engineers care about convolution (and thus for the Fourier transform), is the fact that one can described (and manipulate!) TILS (translation-invariant linear systems) T as convolution operators "by something".

They would say: We assume that $T : \mathbf{x} \to \mathbf{y}$ (input to output) is linear, and *respects delays, i.e.* $T_t(\mathbf{x}) \to T_t(\mathbf{y})$

Starting from the sifting property of the Dirac delta-function

$$f(.) = \int_{-\infty}^{\infty} f(u)\delta(. - u)du = \int_{-\infty}^{\infty} f(u)T_u(\delta)du. \qquad (1)$$

one comes up with (more or less, using $T \circ T_u = T_u \circ T$*) with*

$$T(f) = \int_{-\infty}^{\infty} f(u)T_u[T(\delta)]du = \mu * f, \text{ for } \mu = T(\delta). \qquad (2)$$

Here $T(\delta)$ *is called the impulse response of the system* T.

Impulse response and transfer function

The story goes on by demonstrating (via the convolution theorem) that a Fourier version of the convolution representation $T(f) = \mu * f$ should be

$$\widehat{T(f)} = \widehat{\mu} \cdot \widehat{f}. \tag{3}$$

Thus $\widehat{T}$ is a multiplication operator by $\widehat{\mu}$ (the *transfer function*). This is "kind of true" and a valid heuristic, which can be verified in a strict mathematical sense, whenever all the expressions are justified. In the context of $\boldsymbol{S}_0'(\mathbb{R}^d)$ we would argue for the problem of convolution with the chirp-function:

We know (by Plancherel's) Theorem that the Fourier transform is well defined on $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, and thus we have to just recall that $\mathcal{F}(\mathrm{SINC}) = \mathbf{1}_{[-1/2,1/2]}$ and $\mathcal{F}(\mathrm{chirp}) = \mathrm{chirp} \in \boldsymbol{C}_b(\mathbb{R}^d)$ (in the $\boldsymbol{S}_0'$-sense), thus defining a bounded multiplication operator.

Lots of problems with this viewpoint

The list of arguments, why the above *description is shaky and should not be "sold" as a justification (it can be viewed at best as a heuristic argument) his not so much to do with the problem that it would require tedious and long clarification of details done by mathematicians to justify the approach, it is rather formal in nature and cannot be justified without making serious, additional assumptions.*
One could give a whole talk on why there is no justification possible, and how different parts of the argument experience obstacles for the case that one tried to justify them mathematically. In short, it is NOT a valid claim to say: We are doing it informally, but mathematicians now how to justify things (!) (interchange of integrals etc.).
HOW should the sifting property, saying essentially that the function f has the value $f(u)$ at $u \in \mathbb{R}$ be a "strong tool"?

BIBOs Systems

Still the consideration of BIBO-systems (Bounded Input - Bounded Output) allows to form a correct mathematical model for a situation covering already many case. This is done in the ETH course in great detail:
If we model a BIBO-TILS as bounded linear operators on the space $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, thus assuming that

$$T_y \circ T = T \circ T_y, \forall y \in \mathbb{R}^d \quad \|Tf\|_\infty \leq C\|f\|_\infty, \forall f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Such an operator can be shown to be a convolution operator by a uniquely determined bounded measure, for us just a bounded linear functional on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, we write $\mu \in (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$:

$$[Tf](y) = \mu(T_y f^{\checkmark}), \quad \text{with} \quad f^{\checkmark}(z) = f(-z), f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Convolution by transfer of structure

One can show then that this identification is isometric, i.e. the functional norm

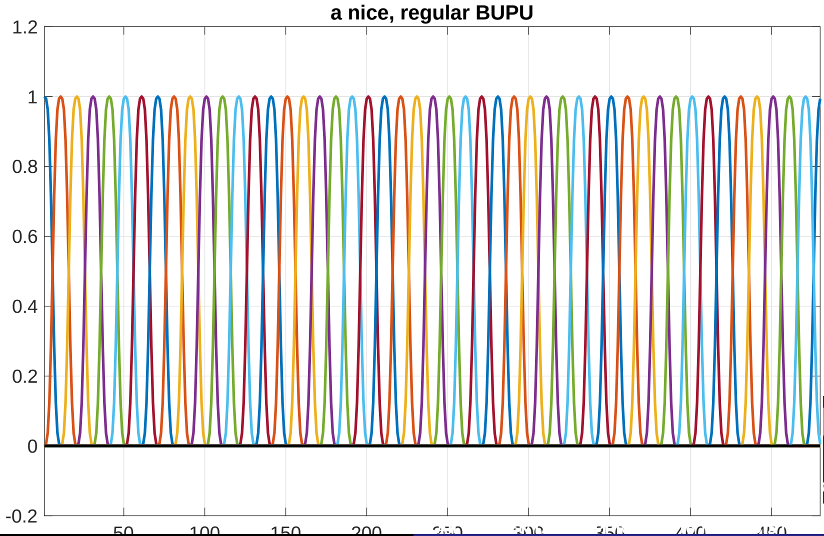
$$\|\mu\|_{\mathbf{M}_b} = \sup_{\|f\|_\infty \leq 1} |\mu(f)|$$

is equal to the operator norm of the associated system

$$|||T|||_{\mathbf{C}_0(\mathbb{R}^d)} = \sup_{\|f\|_\infty \leq 1} \|T(f)\|_\infty .$$

Since obviously the composition of two TILS is another TILS we can transfer the structure and turn $(\mathbf{M}_b(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}_b})$ into a Banach algebra with respect to what is CALLED *convolution*. The justification comes from the observation that the ordinary shift operator, i.e. $T = T_z$ corresponds to $\delta_z \in \mathbf{M}_b(\mathbb{R}^d)$ and thus the convolution defined is just discrete convolution.
Using BUPUs $\Psi = (\psi_k)_{k\in\mathbb{Z}^d}$ we can even approximate general bounded measures by discrete measures $D_\Psi\mu = \sum_{k\in\mathbb{Z}^d} \mu(\psi_k)\delta_{\alpha k}$.

A typical BUPU (Bounded Uniform Partition of Unity)



frame

This still does not justify the definition of the impulse response as $T(\delta_0)$, or the transfer function, because the domain at first hand is just $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$.
So we first extend the action of $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ to all of $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$, by the observation that one has for any BUPU Ψ and any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$:

$$\sum_{k \in \mathbb{Z}^d} \|\mu\psi_k\|_{\boldsymbol{M}_b} = \|\mu\|_{\boldsymbol{M}_b}.$$

This allows to define for $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ the action of μ by $\mu(h) := \sum_{k \in \mathbb{Z}^d} \mu(\psi_k h)$.
With this we extend T to all of $\boldsymbol{C}_b(\mathbb{R}^d)$ and find that the pure frequencies $\chi_s(t) = e^{2\pi i s \cdot t}$ are *eigen-vectors* in $\boldsymbol{C}_b(\mathbb{R}^d)$:

$$\mu * \chi_s = \mu(\chi_{-s})\chi_s := \widehat{\mu}(s)\chi_s, \quad s \in \mathbb{R}^d. \tag{4}$$

Justifying the Impulse Response I

Part of the work to be done in the course was to also justify the commutativity of convolution (associativity is easy in our context), and to verify that different notions of convolution, such as pointwise convolution, or extended linear operators from dense domains are in fact *not leading to confusion*, due to suitable identification and continuity properties.

In order to see that it is meaningful to extend the convolution operator $f \to \mu * f$ from $\mathbf{C}_c(\mathbb{R}^d)$, viewed as subspace of $\mathbf{M}_b(\mathbb{R}^d)$, via

$$\mu_k(f) = \int_{\mathbb{R}^d} f(x)\, k(x) dx, \quad \forall f \in \mathbf{C}_0(\mathbb{R}^d), \tag{5}$$

with $\|k\|_{\mathbf{L}^1} = \int_{\mathbb{R}^d} |k(x)| dx = \|\mu_k\|_{\mathbf{M}_b}$!) to all of $\mathbf{M}_b(\mathbb{R}^d)$.

We observe, that of course the adjoint operator

$$(\nu *_{adj} \mu)(f) := \mu(\nu * f)$$

Justifying the Impulse Response II

can be identified with an internal convolution operator:

$$\nu *_{adj} \mu = \nu^{\checkmark} *_{\boldsymbol{M}_b} \mu.$$

But $\nu^{\checkmark}(f) := \nu(f^{\checkmark}), f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is another bounded measure with $\|\nu^{\checkmark}\|_{\boldsymbol{M}_b} = \|\nu\|_{\boldsymbol{M}_b}$ and thus we can take any Dirac sequence which is bounded in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and hence in $\boldsymbol{M}_b(\mathbb{R}^d)$, which is w^*-convergent to δ_0 (easy to prove) and $T(\mathrm{St}_\rho g)$ will then be w^*-convergent to μ, because

$$T(\delta_0) = T(w^*\text{-}\lim_{\rho \to 0} \mathrm{St}_\rho g) =$$

$$w^*\text{-}\lim_{\rho \to 0} T(\mathrm{St}_\rho g) = w^*\text{-}\lim_{\rho \to 0} \mu * \mathrm{St}_\rho g = \mu.$$

But this argumentation is only valid for BIBO-systems!

From $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$ to $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$

The problem of the current setting is (among others) that

- The signal space $C_0(\mathbb{R}^d)$ and $M_b(\mathbb{R}^d)$ are not contained on in the other;
- Neither $C_0(\mathbb{R}^d)$ nor $M_b(\mathbb{R}^d)$ are invariant under the Fourier transform

This suggests to introduce the Wiener algebra $W(\mathbb{R}^d)$ via BUPUs:

$$W(C_0, \ell^1)(\mathbb{R}^d) := \{f \in C_0(\mathbb{R}^d) \,|\, \|f\|_{W(\mathbb{R}^d)} := \sum_{k \in \mathbb{Z}^d} \|f \psi_k\|_\infty < \infty\}$$

Then $W(\mathbb{R}^d) := W(C_0, \ell^1)(\mathbb{R}^d)$ is of course continuously embedded into its dual space (the space of *translation-bounded Radon measures*), but it is still not Fourier invariant. Any $f \in W(\mathbb{R}^d)$ is Riemann integrable, and one can prove the Fourier Inversion Theorem using R-integrals for $W(\mathbb{R}^d) \cap \mathcal{F}W(\mathbb{R}^d)$.

Considerations concerning $\mathcal{F}L^1(\mathbb{R}^d)$ I

Trying to avoid Lebesgue integration theory (which is alluded regularly in the ETH course) it is possible to introduce $(L^1(\mathbb{R}^d), \|\cdot\|_1)$ as a closed ideal of $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$, by defining it as the norm closure in $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$ of

$$\{\mu_k \mid k \in C_c(\mathbb{R}^d)\},$$

and of course in this way it is clear that it can be identified with the abstract completion of $(C_c(\mathbb{R}^d), \|\cdot\|_1)$, which (by the general uniqueness principle for completions) is isometrically isomorphic (as a subspace of $M_b(\mathbb{R}^d)$, usually called the subspace of *absolutely continuous measures*) to the Lebesgue space $\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

Considerations concerning $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ II

Hence we can define $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$, the Fourier algebra, which IS a pointwise algebra, as it follows easily from the convolution theorem, which is an easy consequence of (4), via

$$\mathcal{F}\mathbf{L}^1(\mathbb{R}^d) = \{\widehat{\mu} \mid \mu \in \mathbf{L}^1(\mathbb{R}^d)\}. \tag{6}$$

Once it is clear (to be shown) that $\mu \to \widehat{\mu}$ is injective we can also define $\|\widehat{f}\|_{\mathcal{F}\mathbf{L}^1} := \|f\|_{\mathbf{L}^1}, f \in \mathbf{L}^1(\mathbb{R}^d)$ and prove the Fourier Inversion Theorem for $(\mathbf{L}^1 \cap \mathcal{F}\mathbf{L}^1)(\mathbb{R}^d)$ (dense in $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$!) and derive

$$\|\mu * h\|_{\mathcal{F}\mathbf{L}^1} \leq \|\mu\|_{\mathbf{M}_b} \cdot \|h\|_{\mathcal{F}\mathbf{L}^1}, \quad h \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d). \tag{7}$$

$W(A, \ell^1)(\mathbb{R}^d)$ and atomic decompositions

The key definition (corresponding to the original approach) is the:

Definition

$$\mathbf{S}_0(\mathbb{R}^d) := \{f \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d) \mid \sum_{k\in\mathbb{Z}^d} \|f\psi_k\|_{\mathcal{F}\mathbf{L}^1} < \infty\}. \qquad (8)$$

We also write $\mathbf{S}_0(\mathbb{R}^d) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d)$, with natural norm.

The current status of the course (early Nov. 2020) is that we have derived various equivalent norms, especially the so-calld *atomic description* (using compactly supported atoms which are absolutely convergent in $(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}))$ or the continuous version of the norm, which allows to establish the connection to the approach in Gröchenig's book via the STFT (Short-Time Fourier transform).

Basic Properties

We already have verified basic properties of this newly defined Wiener amalgam space, with *local component* $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ and global ℓ^1-behaviour, such as

- completeness, equivalence of various norms;
- $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d), \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$.
- Fourier invariance property: $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d)$!
- Validity of Poisson's formula:

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

- norm equivalence with $\|V_g(f)\|_{\boldsymbol{L}^1(\mathbb{R}^{2d})}$ (STFT).

Dual spaces, w^*−convergence I

Next the dual space, simple denoted by $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is introduced. It contains obviously $\boldsymbol{M}_b(\mathbb{R}^d)$, and hence thus

$$\boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{W}(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{M}_b(\mathbb{R}^d).$$

Since it is clear that norm convergence in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is often too strict (recalling that we have $\|\delta_x - \delta_y\|_{\boldsymbol{M}_b} = 2$ for $x \neq y$), we talk a lot about w^*-convergence in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
In particular we realize that $\boldsymbol{S}_0(\mathbb{R}^d)$ is w^*−dense in $\boldsymbol{S}_0(\mathbb{R}^d)$.
We also have already shown $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ enjoys a lot of invariance properties, e.g. translation and dilation invariance, and that for any lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}$ (for a non-singular $d \times d$-matrix $\boldsymbol{A}$) the corresponding Dirac comb $\sqcup\!\sqcup_\Lambda$ belongs to $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Fourier invariance of $S_0'(\mathbb{R}^d)$

One of the most important (now easy) claims is the definition of the *extended Fourier transform* in the context of $S_0'(\mathbb{R}^d)$, via

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), \quad \sigma \in S_0'(\mathbb{R}^d), f \in S_0(\mathbb{R}^d). \tag{9}$$

This is truly an extension of the classical FT (defined on $S_0(\mathbb{R}^d)$ via Riemann integrals) and due to the w^*-density of $S_0(\mathbb{R}^d)$ in $S_0'(\mathbb{R}^d)$ it is the unique w^*-w^*--continuous extension!
Using the properties of the Dirac combs it is not easy to harvest classical principles (for engineers at least), such as:

- sampling on the time side goes to periodization on the frequency side.
- Shannon's Sampling Theorem: recovery of f from the samples is possible of the periodic repetition of the spectrum does not show overlaps (aliasing problem).

Just ONE Fourier transform

It is important to observe that the general setting of $\boldsymbol{S}_0'(\mathbb{R}^d)$ allows to define the Fourier transform of discrete signals supported on lattice $\Lambda \lhd \mathbb{R}^d$ (weigted Dirac combs) as long as they belong to $\ell^\infty(\Lambda)$ (in partcular for $\ell^2(\Lambda)$, but also the theory of almost periodic functions is included in this setting and the computations of (non-regular) Fourier coefficients can be explained.
Above all, the **periodic and discrete signals**, which can be written as a *finite linear combination of Dirac combs* turn out (as a consequence of $\mathcal{F}(\sqcup\!\sqcup) = \sqcup\!\sqcup$) are mapped onto corresponding finite linear combinations of finite linear combinations (of equal cardinality) Dirac combs in the frequency domain, AND the transition from the coefficients **a** (for the time domain) to the coefficients **b** (in the frequency domain) is provided by the FFT/DFT. See also the papers by Jens Fischer.

Classical results

There are many places where in the literature the space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ can be used or should be used.
More or less all the classical summability kernels belong to $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, as F. Weisz has shown.
The subclass of *periodic* functions or measures can be treated also as a subclass of the extended Fourier transform.
Of course the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ is highly useful in the context of Gabor Analysis, but this is well described in the literature!

Thank for your attention

`Details on the course: see www.nuhag.eu/ETH20`
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