Numerical Harmonic Analysis Group

Function Spaces for Fourier Analysis A Fresh Approach and Banach Gelfand Triples & CONCEPTUAL HARMONIC ANALYSIS

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The beginning: 1969 - 1979

- As as (teacher) student of mathematics and physics I was (luckily) confronted with Lebesgue integration in the first year, with Fourier analysis in the second year (1970) by L. Schmetterer, before H. Reiter arrived in Vienna, introducing us to "Harmonic Analysis on LC Groups";
- I was then interested much in Function Spaces, especially Besov-Triebel-Lizorkin spaces (H. Triebel, J. Peetre);
- From the publications of P. Butzer I learned about approximation theory and *sampling theory*.
- J. Cigler introduced us to Banach modules;
- I did my PhD in 1974 on *convolution Banach algebras*;

It was the time of BOURBAKI!

New Areas

In this period J. Dieudonne has been visiting H. Reiter and told us that "abstract harmonic analysis is off-stream" (so to say old-fashioned, not important anymore!). I also felt that is is hard to find new problems that could be attacked. Obviously the results of L. Carleson (a.e. convergence of F.S. in $\boldsymbol{L}^2$) were out of reach. But this was before we saw the rise of

- wavelet theory
- time frequency analysis, Gabor analysis
- frame theory, coorbit theory
- modulation spaces, ...

Actually I had done Gabor Analysis before knowing anything about the work of D. Gabor (from 1946).

What I have done afterwards

In all the years I have spent a lot of time in order to understand a little bit better what the possible applications of Fourier Analysis outside of mathematics could be, and there are many (*even if you try to avoid PDEs, as I had to do due to my poor education in this direction.*).
I was trying to develop various function spaces, I learned how to run numerical experiments with MATLAB (since 1989), I am nowadays using GEOGEBRA for illustrations of mathematical content, I am doing a live presentation of STX (downloadable from the ARI homepage, the director there is Peter Balazs, my former PhD student), or I point to the Gaborator.
I also have done many applied projects with engineers, with people from musicology or medicine, and spent time on on public awareness (workshops, internships, pupils).

Relation to Applied Scientists I

Some words about *Going applied*.

I consider myself an application oriented mathematicians, whose toolbox arose from Harmonic Analysis. I view meanwhile Harmonic Analysis as a branch of Functional Analysis, because all the signals or functions or distributions live in not-finite-dimensional vector spaces, and operators are continuous or rather unbounded, depending on our choice of norms on the domain or target space. Even for numerical work, which of course has to be done efficiently, we would like to know how good an approximation we can achieve with a certain computational effort, and hopefully we get much better results (or faster, or more accurate results) if we spend more time, more memory, or longer computation times. But unlike numerical integration methods, where we just go for one "real number" it is much more challenging if we talk about operators (e.g. $f \mapsto \widehat{f}$, realized via an appropriate FFT).

Relation to Applied Scientists II

Having spent a lot of time with applied scientist from different areas I have learned myself (and can only *reommend to others, especially younger ones*) to connect with them, to understand to some extent, what they are going for, and which formulas they use, This process is most often time-consuming, sometimes frustrating, but *occasionally very fruitful.*.
Thus I learned from the engineers (F. Hlawatasch, Inst. of Communication Theory, TU Vienna, early 80th) that the function space $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ I had at that time is closely connected with the work of D. Gabor (1946) and his own work on the Wigner distribution. This was the basis for a number of joint projects over the years, which even lead up to two patents in the area of mobile communication, thanks to a PhD student (S. Das) who was able to put IEEE specifications into C-code, base on our theory.

Relation to Applied Scientists III

It is also a general experience, that it needs *a long time of contact and interaction*, to grasp well enough the problem arising from the applications, and not just to be helpful in computing some integral. Moreover it helps to understand the setting, for which mathematical theory is helpful. We all know, that something like a point-mass is physically not feasible in the strict sense, but still mechanical problems (like forces of a rotating object) can be easier understood of we think that all the mass is concentrated in the center of mass, instead of simulating the situation, using the fact that they physical object is just composed from finitely many atoms, closely connected.

In this sense, the use of the "continuous basis" of Dirac measures (e.g. $(\delta_x)_{x\in\mathbb{R}}$ for functions on the real line $\mathbb{R}$) in physics is a natural, but only symbolic representation of a function, usually described by the collection of values $(f(x))_{x\in\mathbb{R}}$.

Relation to Applied Scientists IV

But it is (psychogically *and* practically) not helpful to tell the engineers, that a formula such as

$$\int_{-\infty}^{\infty} e^{2\pi i s t} ds = \delta(s)$$

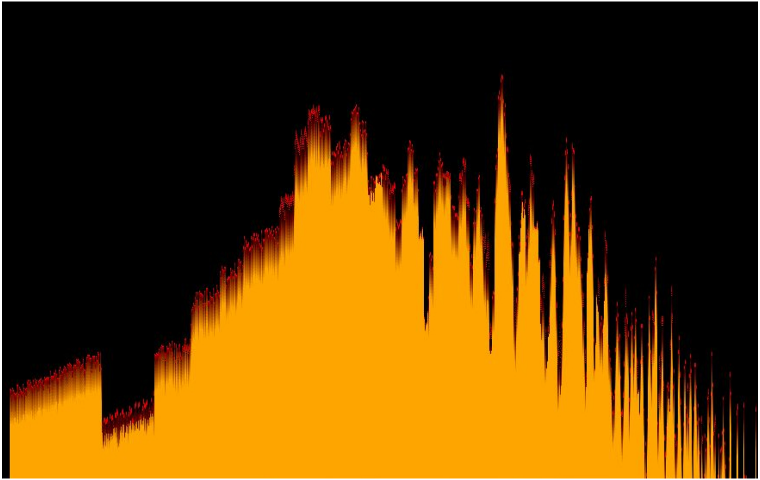
is NOT meaningful, because they will continue to use it anyway. It is much better to find out what they mean (namely $\mathcal{F}(\mathbf{1})$, which is obviously δ_0, because $\mathcal{F}(\delta_0) = \mathbf{1}$!) and then providing a mathematical correct interpretation of the situation. This also helps to avoid formal manipulations which are not meaningful (and consequently cannot be justified mathematically). An there are always new cases, where on finds out, that something *has to be checked*, and as long as we cannot do it we must say: It might be true, but please do not rely on the formal argument, we might finally find a counter-example.

Where do we use Fourier Analysis in our Daily Life?

Perhaps you have to think a bit? But there are MANY opportunities, and few activities do not involve the use of FFT-based technology.

1. You use your mobile phone to communicate?
2. You listen to music? (MP3 or WAV-files);
3. You download images? (JPEG format);
4. Your computer communicates with your printer;
5. You watch digital videos?

Gabor Analysis in our kid's daily live (MP3)



The GABORATOR: webpage, allowing to visualize music

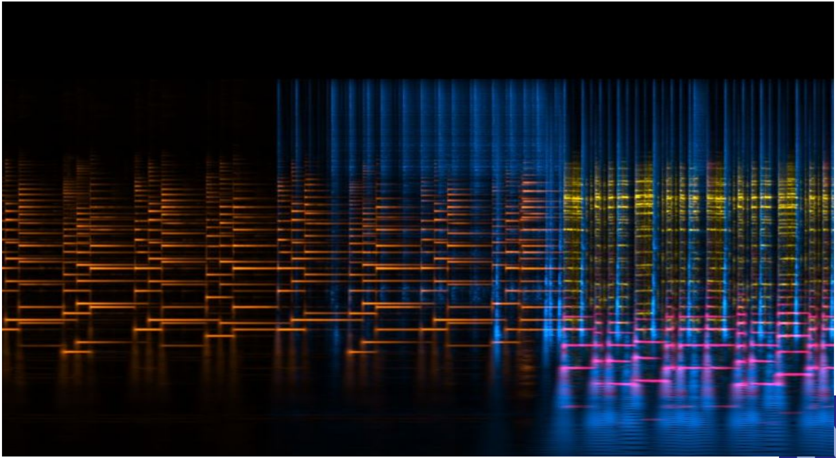
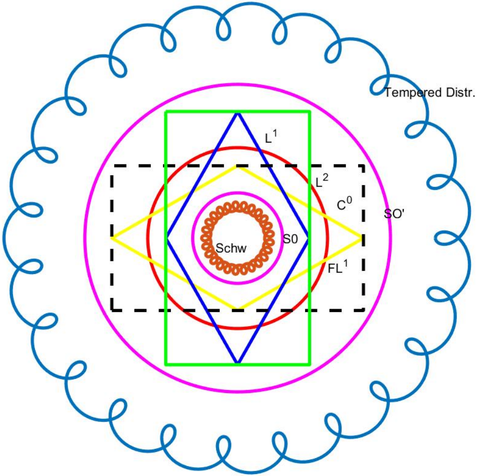
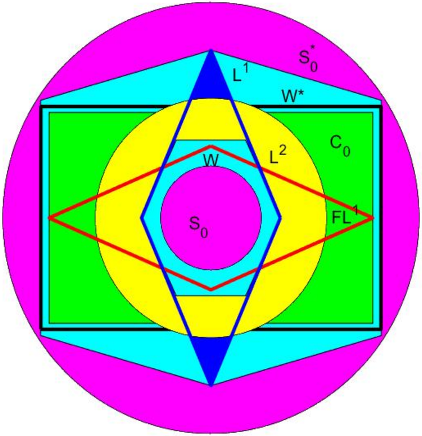


Abbildung: Gaborator3.jpg

A Zoo of Banach Spaces for Fourier Analysis





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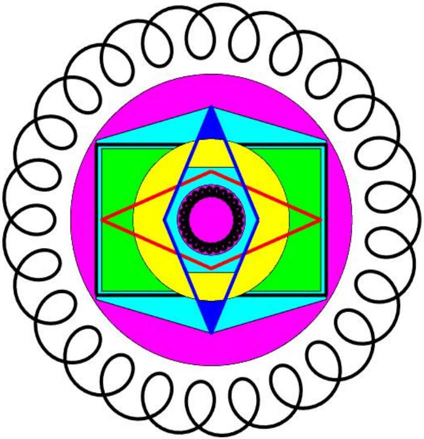

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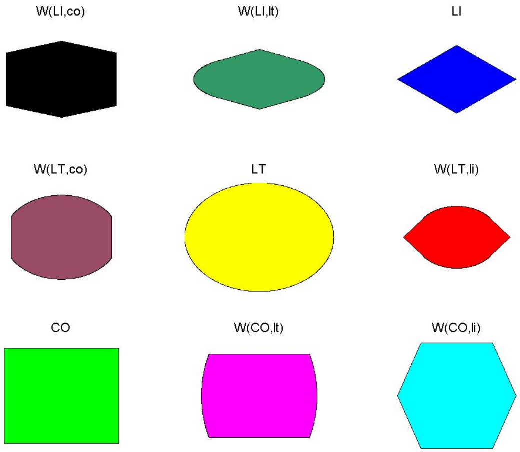
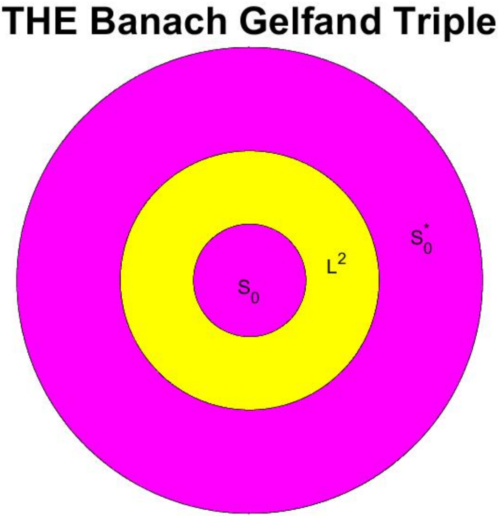
Comparing Wiener Amalgam Spaces




THE Banach Gelfand Triple
S0
L2
S0*

Progress by simplification

The simple world of the BANACH GELFAND TRIPLE, just consisting of the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ as a space of "nice test functions" and the dual space, as ambient vector space of "mild distributions", containing everything which is needed for a discussion of digital signal processing, and finally, in the middle, sandwiched by the two, the most beautiful Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$.

The setting is also known as "rigged Hilbert space" in the literature (rather physics), and appears in the context of theoretical physics (mostly quantum theory, coherent frames, etc.), i.e. a Hilbert space *decorated with additional structure*.

The prototypical example is $(\ell^1, \ell^2, \ell^\infty)$, with the obvious inclusions $\ell^1 \hookrightarrow \ell^2 \hookrightarrow \ell^\infty$ and $\ell^\infty = {\ell^1}^*$.

Related papers by the author: www.nuhag.eu/BIBTEX I

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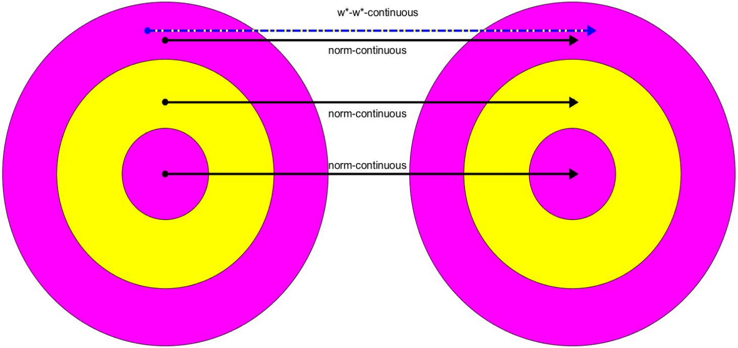
Different Aspects of the New Approach

This is a PERSPECTIVE talk

not a technical presentation, explaining *how to do things*, but rather explaining *how to look at things*?

- How can we introduce these spaces?
- What are the properties is these spaces?
- How can these spaces be used?
- Where are they most useful?
- How can one introduce the alternative spaces?
- How are the new spaces related to the traditional ones?

Banach-Gelfand-Tripel-Homomorphisms



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ (hence w^*-dense there) is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is a unitary isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

Banach Gelfand Triples

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\mathbb{Z})$.

Sufficient conditions for BGT-automorphisms

A couple of basic facts resulting direct from the definition, combined with the general fact that the space of w^*-continuous linear functionals on a dual space $\boldsymbol{B}'$ coincides naturally with the original Banach space, gives the following facts.
We apply them to the concrete Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.

- For every BGTr-mapping there is an adjoint BGTr-mapping T^*, and $T^{**} = T$ for each of them;
- If a bounded linear mapping on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ leaves $\boldsymbol{S}_0(\mathbb{R}^d)$ invariant and so does T^* (the Hilbert adjoint), then T (and also T^*) define BGTr-homomorphism.

Among others this principle can be used to show that the even *fractional Fourier transforms* define (unitary) BGTr-automorphisms on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

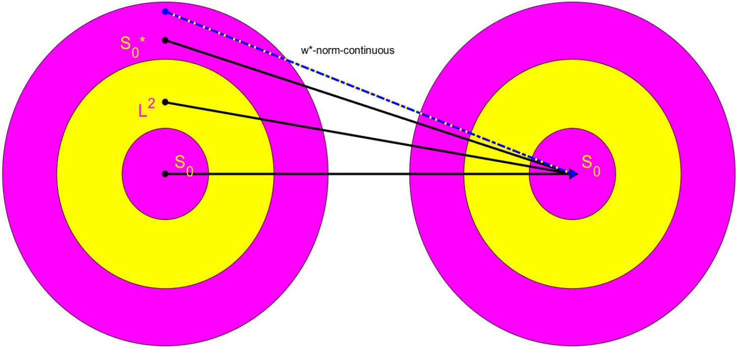
Some general scenarios arising in Gabor Analysis

There are situations, where one is forced to view even more general situations than BGTr-operators (BGOs), even if one is a priori interested only in the Hilbert space behaviour:
Here are some examples:

1. Let $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ be given, and any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the coefficient operator
$$C : f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$$
only maps $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\mathrm{cosp}(\Lambda)$, but $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\ell^2(\Lambda)$.
2. similar situation for the synthesis operator, which is its adjoint.
3. Finally the frame-operator is a priori an operator which maps $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.

But this is good enough to show that it has a Janssen representation.

Regularizing Operators



Regularizers help to approximate distributions

An important family of regularizing operators are those bounded families of BGOs, where each of them maps $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ into $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, and which form an approximating the identity. Abstractly speaking they form bounded nets of BGOs which converge strongly (e.g. in their action on $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$) to the identity. Since we have

$$(S_0' * S_0) \cdot S_0 \subset S_0 \quad \text{and} \quad (S_0' \cdot S_0) * S_0 \subset S_0$$

this can be product-convolution (or convolution-product) operators.
A similar property is shared by Gabor multipliers with finitely many symbols (using lattice points in a bounded domain).

Traditional Mathematical Look at Fourier Analysis I

If you look into books on Fourier Analysis in the last century (since the times of Lebesgue), you get the impression that one first has to study integration theory (Lebesgue), *group theory*, and *topology* before even starting to discuss questions of *Harmonic Analysis*.
I agree with the view (going back to Andre Weil, who was considered his real teacher by my advisor Hans Reiter) that the natural setting for *Fourier Analysis* is that of LCA groups (locally compact Abelian groups). This *unified approach* allows to deal with questions of Fourier analysis in the different settings (discrete versus continuous, periodic versus non-periodic, with the discrete/periodic case covered by the DFT/FFT!)
But does it help us to understand how we can obtain the FT of a decent function with the help of the FFT?
Typically such books follow the *historical path*:

Traditional Mathematical Look at Fourier Analysis II

1. First comes the theory of Fourier Series (periodic/continous);
2. Then comes the Fourier Transform on $\mathbb{R}$;
3. Then the theory is extended to $\mathbb{R}^d$, $d \geq 1$;
4. Perhaps multiple Fourier series are mentioned;
5. "On the computers we are using the DFT/FFT!"
6. Occasionally the usefulness for PDEs is demonstrated, and some numerical schemes are demonstrated.

THESIS: it is natural to start with $\boldsymbol{L}^1(\mathbb{R}^d)$ adn $\boldsymbol{L}^2(\mathbb{R}^d)$ in order to derive e.g. Plancherel's Theorem, but $\boldsymbol{L}^p$-spaces and the FT do not go well along with each other (despite Hausdorff-Young)!

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

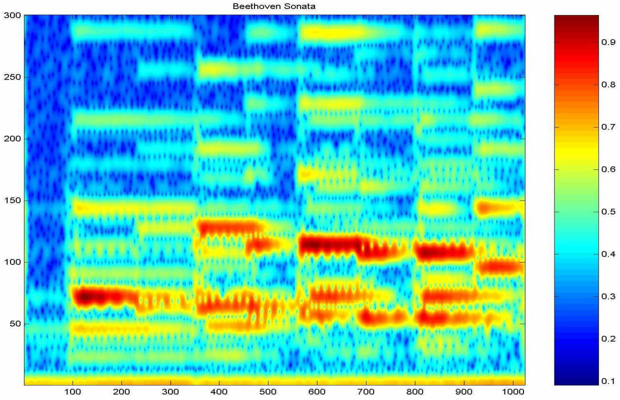
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Various Function Spaces

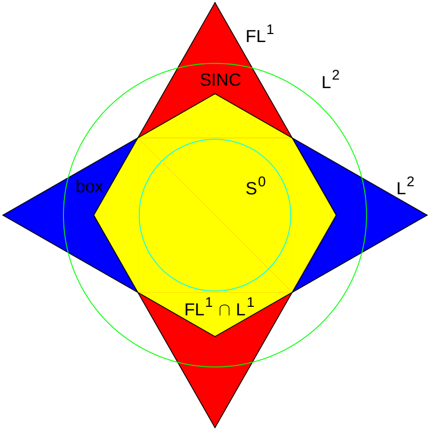


Abbildung: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

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Definition

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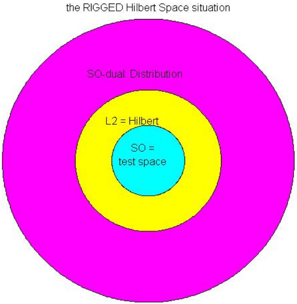
1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
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The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{1}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Why do engineers care about convolution?

The reasons, why engineers care about convolution (and thus for the Fourier transform), is the fact that one can described (and manipulate!) TILS (translation-invariant linear systems) T as convolution operators "by something".
They would say: We assume that $T : \mathbf{x} \to \mathbf{y}$ (input to output) is linear, and *respects delays, i.e.* $T_t(\mathbf{x}) \to T_t(\mathbf{y})$
Starting from the sifting property of the Dirac delta-function

$$f(.) = \int_{-\infty}^{\infty} f(u)\delta(. - u)du = \int_{-\infty}^{\infty} f(u)T_u(\delta)du.$$

one comes up with (more or less, using $T \circ T_u = T_u \circ T$*) with*

$$T(f) = \int_{-\infty}^{\infty} f(u)T_u(T(\delta))du = \mu * f, \text{ for } \mu = T(\delta).$$

Here $T(\delta)$ *is called the impulse response of the system* T.

Impulse response and transfer function

The story goes on by demonstrating (via the convolution theorem) that a Fourier version of the convolution representation $T(f) = \mu * f$ should be

$$\widehat{T(f)} = \widehat{\mu} \cdot \widehat{f}. \qquad (2)$$

Thus $\widehat{T}$ is a multiplication operator by $\widehat{\mu}$ (the *transfer function*). This is "kind of true" and a valid heuristic, which can be verified in a strict mathematical sense, whenever all the expressions are justified. In the context of $\boldsymbol{S}_0'(\mathbb{R}^d)$ we would argue for the problem of convolution with the chirp-function:
We know (by Plancherel's) Theorem that the Fourier transform is well defined on $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, and thus we have to just recall that $\mathcal{F}(\text{SINC}) = \mathbf{1}_{[-1/2,1/2]}$ and $\mathcal{F}(\text{chirp}) = \text{chirp} \in \boldsymbol{C}_b(\mathbb{R}^d)$ (in the $\boldsymbol{S}_0'$-sense), thus defining a bounded multiplication operator.

Lots of problems with this viewpoint

The list of arguments, why the above *description is shaky and should not be "sold" as a justification (it can be viewed at best as a heuristic argument) his not so much to do with the problem that it would require tedious and long clarification of details done by mathematicians to justify the approach, it is rather formal in nature and cannot be justified without making serious, additional assumptions.*
One could give a whole talk on why there is no justification possible, and how different parts of the argument experience obstacles for the case that one tried to justify them mathematically. In short, it is NOT a valid claim to say: We are doing it informally, but mathematicians now how to justify things (!) (interchange of integrals etc.).
HOW should the sifting property, saying essentially that the function f has the value $f(u)$ at $u \in \mathbb{R}$ be a "strong tool"?

BIBOs Systems

Still the consideration of BIBO-systems (Bounded Input - Bounded Output) allows to form a correct mathematical model for a situation covering already many case. This is done in the ETH course in great detail:
If we model a BIBO-TILS as bounded linear operators on the space $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, thus assuming that

$$T_y \circ T = T \circ T_y, \forall y \in \mathbb{R}^d \quad \|Tf\|_\infty \leq C\|f\|_\infty, \forall f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Such an operator can be shown to be a convolution operator by a uniquely determined bounded measure, for us just a bounded linear functional on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, we write $\mu \in (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$:

$$[Tf](y) = \mu(T_y f^{\checkmark}), \quad \text{with} \quad f^{\checkmark}(z) = f(-z), f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Convolution by transfer of structure

One can show then that this identification is isometric, i.e. the functional norm

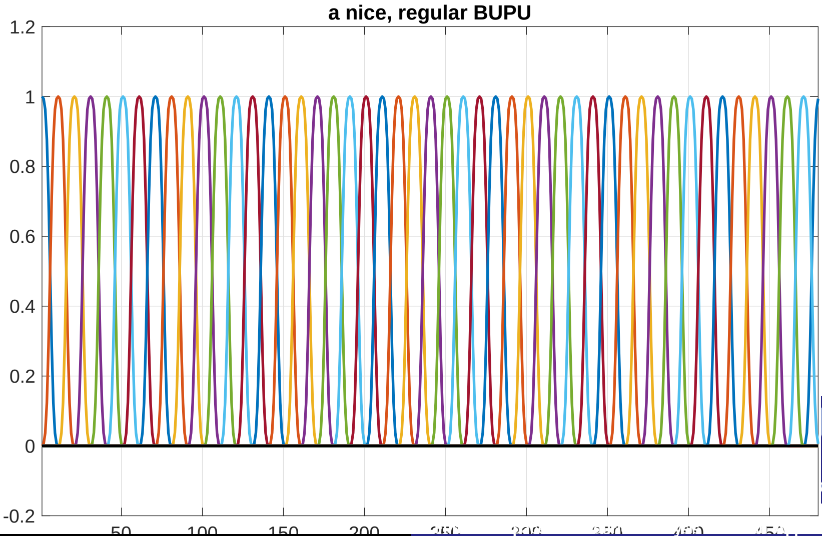
$$\|\mu\|_{\mathbf{M}_b} = \sup_{\|f\|_\infty \leq 1} |\mu(f)|$$

is equal to the operator norm of the associated system

$$|||T|||_{\mathbf{C}_0(\mathbb{R}^d)} = \sup_{\|f\|_\infty \leq 1} \|T(f)\|_\infty.$$

Since obviously the composition of two TILS is another TILS we can transfer the structure and turn $(\mathbf{M}_b(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}_b})$ into a Banach algebra with respect to what is CALLED *convolution*. The justification comes from the observation that the ordinary shift operator, i.e. $T = T_z$ corresponds to $\delta_z \in \mathbf{M}_b(\mathbb{R}^d)$ and thus the convolution defined is just discrete convolution.
Using BUPUs $\Psi = (\psi_k)_{k\in\mathbb{Z}^d}$ we can even approximate general bounded measures by discrete measures $D_\Psi\mu = \sum_{k\in\mathbb{Z}^d} \mu(\psi_k)\delta_{\alpha k}/$.

A typical BUPU (Bounded Uniform Partition of Unity)



frame

This still does not justify the definition of the impulse response as $T(\delta_0)$, or the transfer function, because the domain at first hand is just $\left(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$.
So we first extend the action of $\mu \in \mathbf{M}_b(\mathbb{R}^d)$ to all of $\left(\mathbf{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$, by the observation that one has for any BUPU Ψ and any $\mu \in \mathbf{M}_b(\mathbb{R}^d)$:

$$\sum_{k \in \mathbb{Z}^d} \|\mu\psi_k\|_{\mathbf{M}_b} = \|\mu\|_{\mathbf{M}_b}.$$

This allows to define for $h \in \mathbf{C}_b(\mathbb{R}^d)$ the action of μ by $\mu(h) := \sum_{k \in \mathbb{Z}^d} \mu(\psi_k h)$.
With this we extend T to all of $\mathbf{C}_b(\mathbb{R}^d)$ and find that the pure frequencies $\chi_s(t) = e^{2\pi i s \cdot t}$ are *eigen-vectors* in $\mathbf{C}_b(\mathbb{R}^d)$:

$$\mu * \chi_s = \mu(\chi_{-s})\chi_s := \widehat{\mu}(s)\chi_s, \quad s \in \mathbb{R}^d. \tag{3}$$

Justifying the Impulse Response I

Part of the work to be done in the course was to also justify the commutativity of convolution (associativity is easy in our context), and to verify that different notions of convolution, such as pointwise convolution, or extended linear operators from dense domains are in fact *not leading to confusion*, due to suitable identification and continuity properties.

In order to see that it is meaningful to extend the convolution operator $f \to \mu * f$ from $\boldsymbol{C}_c(\mathbb{R}^d)$, viewed as subspace of $\boldsymbol{M}_b(\mathbb{R}^d)$, via

$$\mu_k(f) = \int_{\mathbb{R}^d} f(x)\, k(x) dx, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d), \tag{4}$$

with $\|k\|_{\boldsymbol{L}^1} = \int_{\mathbb{R}^d} |k(x)| dx = \|\mu_k\|_{\boldsymbol{M}_b}$!) to all of $\boldsymbol{M}_b(\mathbb{R}^d)$.

We observe, that of course the adjoint operator

$$(\nu *_{adj} \mu)(f) := \mu(\nu * f)$$

Justifying the Impulse Response II

can be identified with an internal convolution operator:

$$\nu *_{adj} \mu = \nu^{\checkmark} *_{\boldsymbol{M}_b} \mu.$$

But $\nu^{\checkmark}(f) := \nu(f^{\checkmark}), f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is another bounded measure with $\|\nu^{\checkmark}\|_{\boldsymbol{M}_b} = \|\nu\|_{\boldsymbol{M}_b}$ and thus we can take any Dirac sequence which is bounded in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and hence in $\boldsymbol{M}_b(\mathbb{R}^d)$, which is w^*-convergent to δ_0 (easy to prove) and $T(\mathrm{St}_\rho g)$ will then be w^*-convergent to μ, because

$$T(\delta_0) = T(w^*\text{-}\lim_{\rho\to 0} \mathrm{St}_\rho g) =$$

$$w^*\text{-}\lim_{\rho\to 0} T(\mathrm{St}_\rho g) = w^*\text{-}\lim_{\rho\to 0} \mu * \mathrm{St}_\rho g = \mu.$$

But this argumentation is only valid for BIBO-systems!

From $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$ to $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$

The problem of the current setting is (among others) that

- The signal space $C_0(\mathbb{R}^d)$ and $M_b(\mathbb{R}^d)$ are not contained on in the other;
- Neither $C_0(\mathbb{R}^d)$ nor $M_b(\mathbb{R}^d)$ are invariant under the Fourier transform

This suggests to introduce the Wiener algebra $W(\mathbb{R}^d)$ via BUPUs:

$$W(C_0,\ell^1)(\mathbb{R}^d) := \{f \in C_0(\mathbb{R}^d) \mid \|f\|_{W(\mathbb{R}^d)} := \sum_{k\in\mathbb{Z}^d} \|f\psi_k\|_\infty < \infty\}$$

Then $W(\mathbb{R}^d) := W(C_0,\ell^1)(\mathbb{R}^d)$ is of course continuously embedded into its dual space (the space of *translation-bounded Radon measures*), but it is still not Fourier invariant. Any $f \in W(\mathbb{R}^d)$ is Riemann integrable, and one can prove the Fourier Inversion Theorem using R-integrals for $W(\mathbb{R}^d) \cap \mathcal{F}W(\mathbb{R}^d)$.

Considerations concerning $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ I

Trying to avoid Lebesgue integration theory (which is alluded regularly in the ETH course) it is possible to introduce $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as a closed ideal of $\left(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}\right)$, by defining it as the norm closure in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ of

$$\{\mu_k \mid k \in \boldsymbol{C}_c(\mathbb{R}^d)\},$$

and of course in this way it is clear that it can be identified with the abstract completion of $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$, which (by the general uniqueness principle for completions) is isometrically isomorphic (as a subspace of $\boldsymbol{M}_b(\mathbb{R}^d)$, usually called the subspace of *absolutely continuous measures*) to the Lebesgue space $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

Considerations concerning $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ II

Hence we can define $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$, the Fourier algebra, which IS a pointwise algebra, as it follows easily from the convolution theorem, which is an easy consequence of (3), via

$$\mathcal{F}\mathbf{L}^1(\mathbb{R}^d) = \{\widehat{\mu} \,|\, \mu \in \mathbf{L}^1(\mathbb{R}^d)\}. \quad (5)$$

Once it is clear (to be shown) that $\mu \to \widehat{\mu}$ is injective we can also define $\|\widehat{f}\|_{\mathcal{F}\mathbf{L}^1} := \|f\|_{\mathbf{L}^1}, f \in \mathbf{L}^1(\mathbb{R}^d)$ and prove the Fourier Inversion Theorem for $(\mathbf{L}^1 \cap \mathcal{F}\mathbf{L}^1)(\mathbb{R}^d)$ (dense in $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$!) and derive

$$\|\mu * h\|_{\mathcal{F}\mathbf{L}^1} \le \|\mu\|_{\mathbf{M}_b} \cdot \|h\|_{\mathcal{F}\mathbf{L}^1}, \quad h \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d). \quad (6)$$

$W(A, \ell^1)(\mathbb{R}^d)$ and atomic decompositions

The key definition (corresponding to the original approach) is the:

Definition

$$S_0(\mathbb{R}^d) :== \{f \in \mathcal{F}L^1(\mathbb{R}^d) \,|\, \sum_{k \in \mathbb{Z}^d} \|f\psi_k\|_{\mathcal{F}L^1} < \infty\}. \qquad (7)$$

Alternatively we write $S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$, with natural norm.

The current status of the course (early Nov. 2020) is that we have derived various equivalent norms, especially the so-calld *atomic description* (using compactly supported atoms which are absolutely convergent in $(\mathcal{F}L^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}L^1}))$ or the continuous version of the norm, which allows to establish the connection to the approach in Gröchenig's book via the STFT (Short-Time Fourier transform).

Basic Properties

We already have verified basic properties of this newly defined Wiener amalgam space, with *local component* $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ and global ℓ^1-behaviour, such as

- completeness, equivalence of various norms;
- $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d), \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$.
- Fourier invariance property: $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d)$!
- Validity of Poisson's formula:

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

- norm equivalence with $\|V_g(f)\|_{\boldsymbol{L}^1(\mathbb{R}^{2d})}$ (STFT).

Dual spaces, w^*-convergence I

Next the dual space, simple denoted by $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is introduced. It contains obviously $M_b(\mathbb{R}^d)$, and hence thus

$$S_0(\mathbb{R}^d) \subset W(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \subset M_b(\mathbb{R}^d).$$

Since it is clear that norm convergence in $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is often too strict (recalling that we have $\|delta_x - delta_y\|_{M_b} = 2$ for $x \neq y$), we talk a lot about w^*-convergence in $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$. In particular we realize that $S_0(\mathbb{R}^d)$ is w^*-dense in $S_0(\mathbb{R}^d)$. We also have already shown $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ enjoys a lot of invariance properties, e.g. translation and dilation invariance, and that for any lattice $\Lambda = A * \mathbb{Z}$ (for a non-singular $d \times d$-matrix A) the corresponding Dirac comb $\sqcup\!\sqcup_\Lambda$ belongs to $S_0'(\mathbb{R}^d)$.

Fourier invariance of $S_0'(\mathbb{R}^d)$

One of the most important (now easy) claims is the definition of the *extended Fourier transform* in the context of $S_0'(\mathbb{R}^d)$, via

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), \quad \sigma \in S_0'(\mathbb{R}^d), f \in S_0(\mathbb{R}^d). \tag{8}$$

This is truly an extension of the classical FT (defined on $S_0(\mathbb{R}^d)$ via Riemann integrals) and due to the w^*-density of $S_0(\mathbb{R}^d)$ in $S_0'(\mathbb{R}^d)$ it is the unique w^*-w^*--continuous extension!
Using the properties of the Dirac combs it is not easy to harvest classical principles (for engineers at least), such as:

- sampling on the time side goes to periodization on the frequency side.
- Shannon's Sampling Theorem: recovery of f from the samples is possible of the periodic repetition of the spectrum does not show overlaps (aliasing problem).

Just ONE Fourier transform

It is important to observe that the general setting of $\boldsymbol{S}_0'(\mathbb{R}^d)$ allows to define the Fourier transform of discrete signals supported on lattice $\Lambda \lhd \mathbb{R}^d$ (weigted Dirac combs) as long as they belong to $\boldsymbol{\ell}^\infty(\Lambda)$ (in partcular for $\boldsymbol{\ell}^2(\Lambda)$, but also the theory of almost periodic functions is included in this setting and the computations of (non-regular) Fourier coefficients can be explained.
Above all, the **periodic and discrete signals**, which can be written as a *finite linear combination of Dirac combs* turn out (as a consequence of $\mathcal{F}(\sqcup\!\sqcup) = \sqcup\!\sqcup$) are mapped onto corresponding finite linear combinations of finite linear combinations (of equal cardinality) Dirac combs in the frequency domain, AND the transition from the coefficients **a** (for the time domain) to the coefficients **b** (in the frequency domain) is provided by the FFT/DFT.

Classical results

There are many places where in the literature the space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ can be used or should be used.
More or less all the classical summability kernels belong to $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, as F. Weisz has shown.
The subclass of *periodic* functions or measures can be treated also as a subclass of the extended Fourier transform.
Of course the Banach Gelfand Triple $(S_0, L^2, S_0')(\mathbb{R}^d)$ is highly useful in the context of Gabor Analysis, but this is well described in the literature!

Book References

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H.G. Feichtinger and T. Strohmer: Gabor Analysis, 1998.
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A. Benyi, K. A. Okoudjou Modulation Spaces., Birkh. 2020.
E. Cordero, L. Rodino: Time-Frequency Analysis of Operators De Gruytes Studies in Mathematics. 2020.

See also www.nuhag.eu/talks.

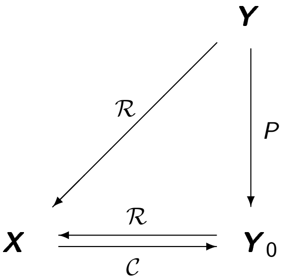
Retracts in Interpolation Theory I

Definition

A linear mapping $\mathcal{C}$ is defining a retract from $\boldsymbol{X}$ into $\boldsymbol{Y}$ if there exists a left inverse to it, i.e. a mapping $\mathcal{R}$ from $\boldsymbol{Y} into \boldsymbol{X}$ such that $\mathcal{R} \circ \mathcal{C} = Id_{\boldsymbol{X}}$.

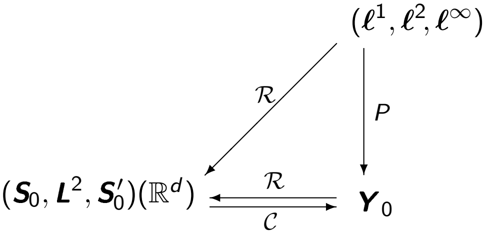
In other words, we have the following commutative diagram, with $\boldsymbol{Y}_0$ being the range of the analysis mapping $\mathcal{C}$. Moreover, it is clear that the mapping $\mathcal{C} \circ \mathcal{R}$ is an idempotent, and for this reason $\boldsymbol{Y}_0$ is a closed an complemented subspace of $\boldsymbol{Y}$. Moreover, T establishes and isomorphism from $\boldsymbol{X}$ onto $\boldsymbol{Y}_0$. Since $\mathcal{R}$ is also surjective, we find that $\boldsymbol{X}$ is not only identified with a closed subspace of $\boldsymbol{Y}$, but also isomorphic to the quotient $\boldsymbol{Y} / \boldsymbol{Y}_0$.

Retracts in Interpolation Theory II



The method of retracts is often used to push results known for vector-valued $\boldsymbol{L}^p$-spaces to the setting of Besov- or modulation spaces.

Gabor (Banach) Frames for $(S_0, L^2, S_0')(\mathbb{R}^d)$

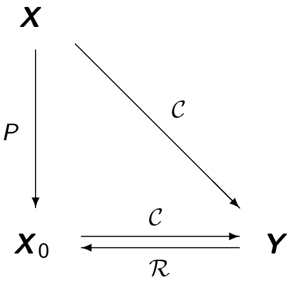


This diagram is valid for $g \in S_0(\mathbb{R}^d)$, describing the coefficient mapping$\mathcal{C} : f \mapsto V_g f|_\Lambda$ and the reconstruction $\mathcal{R}$ by the dual atom $\widetilde{g} \in S_0(\mathbb{R}^d)$: $\mathbf{c} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\tilde{g}$, which turns into an *isomorphism* using *Wilson bases*.

Riesz Basis case: E.g. $X_0 \subset X = L^p$, and $Y = \ell^p$:

The Riesz basis case

Here we just of the reflected picture. Instead of embedding the Banach space (typically a space of functions or distributions) into a sequence space we do the opposite embedding, e.g. spline-type spaces inside of $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$.



Kernel Theorems I

The so-called Kernel Theorem for $\boldsymbol{S}_0$-spaces allows to establish a number of further unitary BGTr-isomorphism. It involves certain types of operators, their (integral) kernels, but also their representation as *pseudo-differential* operators, via the Weyl or Kohn-Nirenberg symbol, or (important for applications in mobile communication) their spreading distribution.

These situations allow to make use of the general principles. In order to understand the transformation one can start from the core spaces, where the analogy with the finite dimensional case is valid in a very natural sense, e.g. if one has $K(x,y) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, then red $K(x,y) = T(\delta_y)(x)$, in analogy to the matrices $a_{n,k} = [T(\mathbf{e}_k)]_n$.

The Hilbert space case of the well-known characterization

Kernel Theorems II

Theorem

There is a unitary BGTr isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and* $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))(\mathbb{R}^d)$*, which is a unitary mapping between* $(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2)$ *and* $(\mathcal{HS}, \|\cdot\|_{\mathcal{HS}})$*, with*

$$\langle T_1, T_2 \rangle_{\mathcal{HS}} = \operatorname{trace}(T_1 \circ T_2^*).$$

Alternative unitary BGTr describe operators via *Kohn-Nirenberg* symbol resp. their spreading representation, such as $T \in \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} H(\lambda)\pi(\lambda), \quad H \in \boldsymbol{S}_0(\mathbb{R}^{2d}).$$

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
Gabor Book 1998.

Wilson Bases

For the case $G = \mathbb{R}^d$ one can derive the kernel theorem also from the description of operators mapping ℓ^1 to ℓ^∞ or vice versa (in a w^*-to-norm continuous way).
The key is the fact, that local Fourier basis, but in particular the so-called Wilson bases are suitable for modulation spaces. In our situation we can formulat the following

Theorem

Any ON Wilson basis (obtained by a smart pairwise recombination of the elements of a tight Gabor frame of redundancy 2) establishes a unitary BGTr between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ *and* $(\ell^1, \ell^2, \ell^\infty)$.

H. G. Feichtinger, K. Gröchenig, and D. F. Walnut.
Wilson bases and modulation spaces.
Math. Nachr., 155:7–17, 1992.

Personal Motivation

This talk is an attempt to push further some general ideas, related to the ideas developed in the last 10 to 20 years, based on the *experience* in the field between pure and applied mathematics, between the engineering[1] view (and formalism) and the mathematical literature available to better understand what is needed by the engineers, to *learn to understand their terminology* and to translate it into mathematical terminology.
Often the meaning of their symbols is different from how it is interpreted by mathematicians proverb (whose is it?, W. Thirring):

It doesn't have to be sloppy, just because it is applied!

These attempts are part of a *long term project to renew Fourier Analysis*, with the idea of improving the situation.

[1] Or the symbols used by physicists in quantum theory, using *rigged Hilbert spaces*.

Observations made in the last decades I

During the last decades I have made the following experiences:

- The use of Fourier Analysis in the applied and in the mathematical community drift apart, more and more, and one community does not care about the other (also physicists do not care too much about engineers...).
- A typical view is: mathematicians work with *high precision on problems which are not so relevant for applications, AND engineers get things to work (like mobile communication) despite questionable simplifications in the justification of their algorithms (for example).*
- *It is not even well known in the "other community" what the goals are. Meanwhile (due to the age of the field) local habits (using symbols, teaching the material) get frozen.*

What we need, according to my understanding

Thus, at the psychological and practical level, we need, as part of the scientific community, more of

- communication across disciplines;
- reading papers from the "other side";
- (as mathematicians): develop tools which are really useful for the applications
- (as applied person): discuss with mathematicians what the effective goal is in a given situation

Concepts that might help

Just as a *wordy answer* to these questions I can point to some of the concepts and the vocabulary that has been developed in the last years, because it is not only a technical problem, but also a problem of *communication*.

I have suggested in the last years to go **beyond Abstract Harmonic Analysis** (AHA), which considers Fourier Analysis over different groups G (such as $G = \mathbb{R}, \mathbb{Z}, \mathbb{R}^d, \mathbb{T}, \mathbb{Z}_N, \mathbb{Z}_N \times \mathbb{Z}_N$) is parallel universes with *analogue structures* (group, dual group, characters, Fourier transform $\mathcal{F}_G$, etc.) to introduce (and make use of it in our daily work!) of Conceptual Harmonic Analysis, meaning the integration of Abstract and Computational Harmonic Analysis. We need to understand how the different groups are related to each other, how the FFT-algorithm can be used to compute the FT of a smooth function, or how to compute dual Gabor atoms for irrational lattices.

The current ETH course

I am convinced, that we need to take a *fresh approach to Fourier Analysis*, in order to be able to fill this abstract idea with life. In fact, it was the chance to contribute to some of the practical questions (irregular sampling, Gabor Analysis, etc.) in the last decades, and the experience with the tools which turned out to be helpful compared to other, often well established mathematical principles which did not make life easier, but rather more and more involved as *"scientific progress"* was made.

For this fresh approach *Time-Frequency Methods* certainly will play an important role, and the minimal tool-set is what I have called THE Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, consisting if "Feichtinger's Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$", the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and the dual space of *mild distributions.*

See www.nuhag.eu/ETH20 *for the current course, also with a lot of material for download and YouTube recording links.*

Abstract versus Conceptual Harmonic Analysis I I

Abstract for the ISAAC talk 2017:
The idea of "Conceptual Harmonic Analysis" grew out of the attempt to make objects arising in Fourier Analysis or Gabor Analysis (such as norms of functions, their Fourier transforms, dual Gabor atoms, etc.) computable. Using suitable function spaces such as the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ it should be possible to find concrete algorithms which allow to compute approximations to the desired on real hardware in finite time, up to (at least potentially) arbitrary requested precision.
Going beyond the ideas of Abstract Harmonic Analysis, which only allows to identify the analogies between objects on different LCA (locally compact Abelian) groups G,

Abstract versus Conceptual Harmonic Analysis II

the idea of **Conceptual Harmonic Analysis** wants to see the connection between these settings to be used, for example, in order to use methods from discrete, periodic Gabor analysis (which are computationally realizable using MATLAB).
The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which can also be seen as a "rigged Hilbert space", allows to describe and justify a number of such procedures providing also a tool to deal with periodic, but also of continuous or discrete signals in a unified way. As it turns out the so-called w-convergence is crucial for the description, and fine partitions of unity as well as regularizing operators play a crucial role in this context.

The need for suitable function spaces

The connection between different settings such as discrete versus continuous, or numerical versus analytical, requires to have a suitable collection of function spaces, for me ideally Banach spaces of distributions, which allow to describe convergence (in the norm sense or in the w^*-sense) and to study robustness of implementable algorithms with respect to precision of the available data (so a highly practical question).
We might have simple linear equation of the form $T(f) = g$ for given right hand side, and we would like to be able to guarantee that a *realizable, constructive algorithm* (this is more than just *constructive approximation!) can achieve an approximate solution within a given ε-error, for a norm which is relevant for the problem at hand. Better approximation should be achievable at a hight computational cost.*

Thank you and further links

Thank for your attention!

... and please think about application oriented analysis!

A number of talks by the speaker are found at

`www.nuhag.eu/talks : access via visitor//nuhagtalks`

`Details on the course: see www.nuhag.eu/ETH20`
`Direct question to: hans.feichtinger@univie.ac.at`