NuHAG Numerical Harmonic Analysis Group

Time-Frequency Analysis and Suitable Function Spaces

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Harmonic analysis seminar
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G. B. Folland.
Harmonic Analysis in Phase Space.
Princeton University Press, Princeton, N.J., 1989.

H. G. Feichtinger and T. Strohmer.
Gabor Analysis and Algorithms. Theory and Applications.
Birkhäuser, Boston, 1998.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

H. G. Feichtinger and T. Strohmer.
Advances in Gabor Analysis.
Birkhäuser, Basel, 2003.

A. Beyni and K. A. Okoudjou.
Modulation Spaces.
Springer (Birkhäuser), New York, 2020.

E. Cordero and L. Rodino.
Time-frequency Analysis of Operators and Applications.
De Gruyter Studies in Mathematics, Berlin, 2020.

Abstract

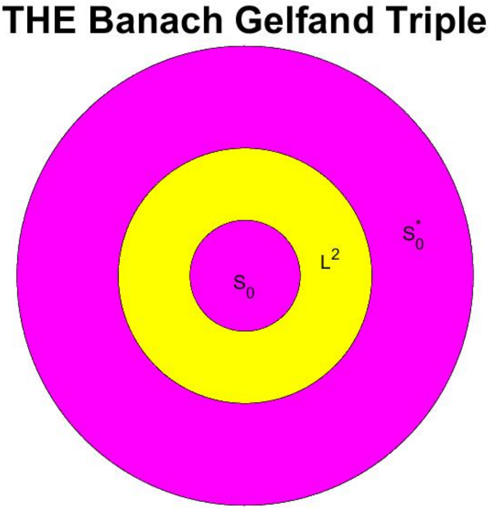
The key-point of this talk will be some exploration of function spaces concepts arising from time-frequency analysis respectively Gabor Analysis. Modulation spaces and Wiener amalgams have proved to be indispensable tools in time-frequency analysis, but also for the treatment of pseudo-differential operators or Fourier integral operators.

Given the limited time we will concentrate on the spaces $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}), (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (the space of *mild distributions*), also known as modulation spaces $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, $(\boldsymbol{M}^2(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^2(\mathbb{R}^d)})$ and $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$ respectively.

This will be a bottom up talk, with applications coming first.

In this sense the talk is an experimental one, trying to test modern tools, both mathematical tools and presentation tools.

THE Banach Gelfand Triple



More precisely, we will recall a short summary of the concepts of Wiener amalgam spaces and modulation spaces, as well as the concept of Banach Gelfand Triples, with the associated kernel theorem (in the spirit of the L.Schwartz kernel theorem). We will indicate in which sense these spaces allow to capture more precisely the mapping properties of operators which may be unbounded in the Hilbert space setting. The subfamily of translation and modulation invariant spaces plays a specific role, with naturally associated regularization operators involving smoothing by convolution and localization by pointwise multiplication.

The presentation will be in the spirit of *Conceptual Harmonic Analysis*, which is more than just the combination of *Abstract Harmonic Analysis* and *Numerical* or *Computational Harmonic Analysis*.

Comparison with integration theory

In order to emphasize the punch line of my presentation let us compare the situation with the more familiar integration theory. One may take different view-points on this subject:

1. The theoretical approach emphasizes that the Lebesgue integral is more general and more powerful than the Riemann integral (e.g. because it allows to integrate more functions);
2. Generalized functions allow to give a meaning to integrals of the form $\int_{-\infty}^{\infty} e^{2\pi ist} ds = \delta(t)$;
3. A computer scientist wants to find an efficient implementation of an existing integration formula;
4. A fine numerical analyst is concerned with efficient realization of an integral for large classes of (sufficiently well-behaved) functions, providing a guarantee of precision of the results.

The role of function spaces

So what is the motivation to use function spaces (and what are function spaces). As a matter of convenience (and *personal conviction!*) I am restricting my attention to (families of) Banach spaces of (generalized) functions, because many of the interesting topological vector spaces (e.g. those used for the definition of ultra-distributions) can be based on the intersection of families of Banach spaces and their topology can be obtained by the family of (semi)norms which arise from these individual Banach spaces.
You may take the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions (with the usual family of seminorms) and you will find that this space can also be viewed as the intersection of (Fourier invariant) modulation spaces, the (Fourier invariant) so-called Shubin classes $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$. The Fourier invariance of $\mathcal{S}(\mathbb{R}^d)$ thus follows easily, but on the other hand the invariance with respect to differentiation is less obvious.

The use of function spaces

Many of the "applications" of function spaces are in the description of operators and their mapping properties. Typically the scale of *Sobolev spaces* is suitable for the description of the mapping properties of the Laplace operator $f \mapsto f''$ (first) and extending it to $\mathbb{R}^d$.

As Yves Meyer once put it (in a conversation with the author):

Function spaces are only good for the description of operators, not in order to study them by themselves!

At that time he had shown how wavelet expansions are a good way to understand the mapping properties of Calderon-Zygmund operators on the classical function spaces, namely the Besov-Triebel-Lizorkin spaces, which include the Sobolev spaces.

I just had *modulation spaces*, and still very few results showing that they are useful. This has changed meanwhile.

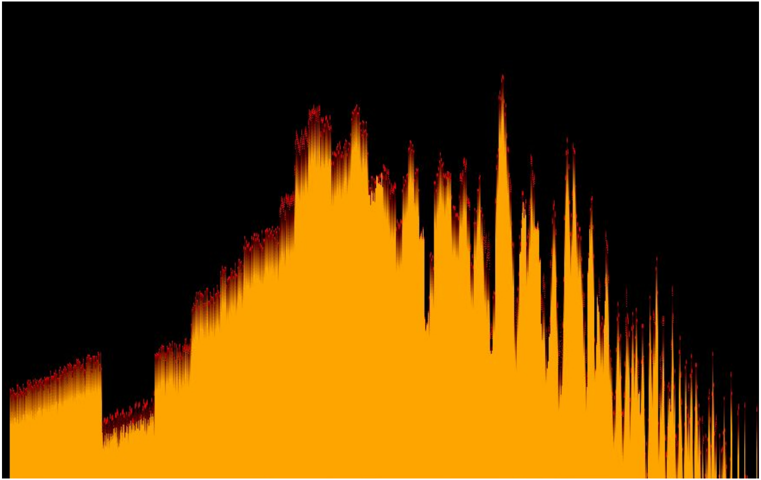
H. G. Feichtinger.
Modulation spaces on locally compact Abelian groups.
Technical report, University of Vienna, January 1983.

H. G. Feichtinger.
Modulation spaces on locally compact Abelian groups.
In R. Radha, M. Krishna, and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications*, pages 1–56, Chennai, January 2002, 2003. New Delhi Allied Publishers.

H. G. Feichtinger.
Choosing Function Spaces in Harmonic Analysis, volume 4 of *The February Fourier Talks at the Norbert Wiener Center, Appl. Numer. Harmon. Anal.*, pages 65–101.
Birkhäuser/Springer, Cham, 2015.

H. G. Feichtinger.
Thoughts on Numerical and Conceptual Harmonic Analysis.
In A. Aldroubi, C. Cabrelli, S. Jaffard, and U. Molter, editors, *New Trends in Applied Harmonic Analysis. Sparse Representations, Compressed Sensing, and Multifractal Analysis*, Applied and Numerical Harmonic Analysis, pages 301–329.
Birkhäuser, Cham, 2016.

Gabor Analysis in our kid's daily live (MP3)



The GABORATOR: webpage, allowing to visualize music

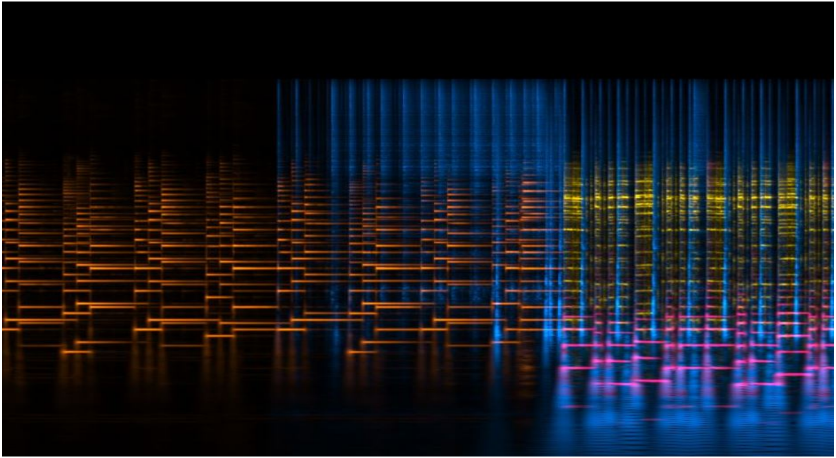
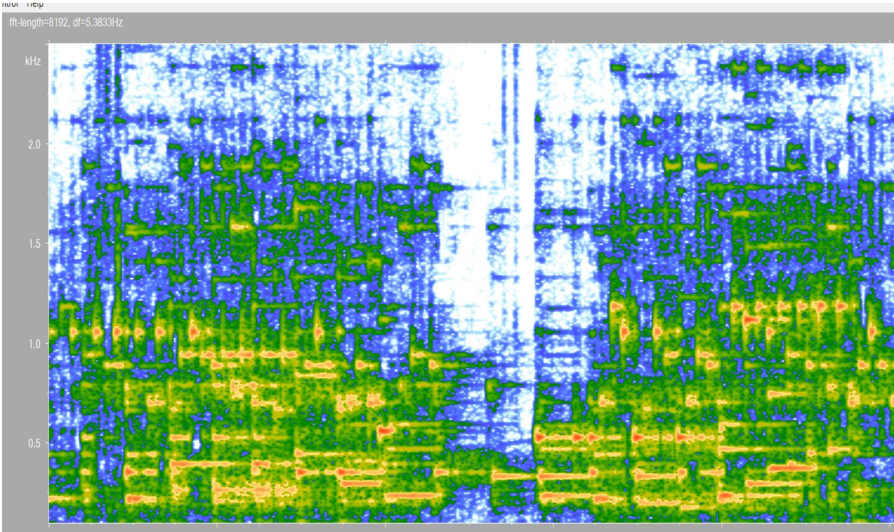


Abbildung: Gaborator3.jpg

Live demonstration: Mozart, Variations KV 54



OUTLINE of this seminar talk:

The concrete plan for this talk is as follows:

- Describe that Gabor Analysis can be naturally formulated over LCA (locally compact Abelian) groups;
- in order to understand the algebraic structure on can work over finite Abelian groups;
- thus the key questions reduce to linear algebra questions;
- properly implemented one can learn a lot from the case $G = \mathbb{U}_N$, the unit roots of order N;
- if one moves on to the case $G = |Rdst$ a number of (mostly functional analytic) questions arise. In fact, leaving the domain of finite dimensional signal spaces one is faced with boundedness properties and the proper choice of norms constituting useful function space
- demonstrating that the Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ allows to explain the transition from the finite case to the continuous, non-periodic case.

H. G. Feichtinger, W. Kozek, and F. Luef.
Gabor Analysis over finite Abelian groups.
Appl. Comput. Harmon. Anal., 26(2):230–248, 2009.

H. G. Feichtinger, F. Luef, and T. Werther.
A guided tour from linear algebra to the foundations of Gabor analysis.
In *Gabor and Wavelet Frames*, volume 10 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 1–49. World Sci. Publ., Hackensack, 2007.

E. Cordero, H. G. Feichtinger, and F. Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

A couple of master theses in Vienna

T. Dobler.
Wiener Amalgam Spaces on Locally Compact Groups.
Master's thesis, University of Vienna, 1989.

K. Schnass.
Gabor Multipliers. A Self-contained Survey.
Master's thesis, University of Vienna, 2004.

S. Paukner.
Foundations of Gabor Analysis for Image Processing.
Master's thesis, 2007.

S. Bannert.
Banach-Gelfand Triples and Applications in Time-Frequency Analysis.
Master's thesis, University of Vienna, 2010.

K. Döpfner.
Quality of Gabor Multipliers for Operator Approximation.
Master's thesis, University of Vienna, 2012.

Recent articles

H. G. Feichtinger.
Elements of Postmodern Harmonic Analysis.
In *Operator-related Function Theory and Time-Frequency Analysis. The Abel Symposium 2012, Oslo, Norway, August 20–24, 2012*, pages 77–105. Cham: Springer, 2015.

H. G. Feichtinger.
Gabor expansions of signals: computational aspects and open questions.
In *Landscapes of Time-Frequency Analysis*, volume ATFA17, pages 173–206. Birkhäuser/Springer, 2019.

H. G. Feichtinger.
Ingredients for Applied Fourier Analysis.
In *Sharda Conference Feb. 2018*, pages 1–22. Taylor and Francis, 2020.

H. G. Feichtinger.
A sequential approach to mild distributions.
Axioms, 9(1):1–25, 2020.

Consequences and benefits

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, 1998.

H. G. Feichtinger and K. Nowak.
A first survey of Gabor multipliers.
In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.

H. G. Feichtinger and N. Kaiblinger.
Varying the time-frequency lattice of Gabor frames.
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.

Really from scratch

I see the difficulties with the current approach (including the academic career paths), but I would like to encourage each person to at least try to deviate from the traditional path and explore interesting (*not necessary traditional*) questions, and try to find answers to *natural questions* instead of inventing complicated terminology or involved questions which can be answered with the current methods.

But I claim that *one can rebuild Fourier Analysis and Gabor Analysis (resp. time-frequency analysis) from scratch with modest technical (functional analytic) tools!* Let us give a try!

What are the key questions

Let us just summarize the key questions of Gabor analysis:

1. Under which conditions can one expand "every function" (or signal, or distribution) as an (infinite double) sum of time-frequency shifted Gabor atoms (despite the fact that such families cannot be an orthonormal basis for $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$): Gabor expansion of functions
2. Under which conditions (on the signal or on the Gabor window) can one have a *stable reconstruction* of the signal from the sampled STFT (the *regular* case covers sampling along a *lattice* $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ in *phase space*): Signal reconstructions from sampled STFT.

Recalling basic facts from linear algebra I

We know that a linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^m$ is realized by matrix multiplication: $T(\mathbf{x}) = \boldsymbol{A} * \mathbf{x}$, for a column vector $\mathbf{x} = [x_1, ..., x_n] \in \mathbb{C}^n$, or

$$T(\mathbf{x}) = \boldsymbol{A} * \mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{a}_k,$$

a linear combination of *columns* $(\mathbf{a}_k)_{k=1}^n$ of the $m \times n$-matrix $\boldsymbol{A}$.
Note that of course $\mathbf{a}_k = T(\mathbf{e}_k)$.
But we can also use the system of columns in order to produce a linear mapping from $\mathbb{C}^n$ into $\mathbb{C}^m$ by doing matrix multiplication with $\boldsymbol{A}' = \overline{\boldsymbol{A}^t}$:

$$\boldsymbol{A}' * \mathbf{y} = (\langle \mathbf{y}, \mathbf{a}_k \rangle_{\mathbb{C}^m})_{k=1}^n .$$

Recalling basic facts from linear algebra II

We call the *range of* T the *column space*, and (slightly misusing the terminology) the range of $\boldsymbol{A}'$ the row-space of $\boldsymbol{A}$.
It is an easy reformulation of terms to verify that the orthogonal complement of the column space is just the *null-space* of the linear mapping induced by $\boldsymbol{A}'$ and correspondingly (since $\boldsymbol{A}'' = \boldsymbol{A}$) we have that the null-space of $\boldsymbol{A}$ (resp. T) is the orthogonal complement of the row-space.
By restricting T to the row space (or quotiening out the null-space of T) we realize that this defines an isomorphism between the column-space of $\boldsymbol{A}'$ to the column-space of $\boldsymbol{A}$, both being r-dimensional spaces (with $r = \text{rank}(\boldsymbol{A})$, computed via Gauss elimination, for example).
The *pseudo-inverse* of $\boldsymbol{A}$ is just undoing what can be undone!, i.e. it projects $\mathbb{C}^m$ onto the column space, and inverts the invertible part of T.

Recalling basic facts from linear algebra III

This geometric way provides a clear understanding of the MNLSQ-problem (normal equations, etc.), i.e. the best solution to the linear problem $\boldsymbol{A} * \mathbf{x} = \mathbf{b}$.
If $\mathbf{b} \in \mathbb{C}^m$ does not belong to the column space of $\boldsymbol{A}$, then $\mathbf{x} = \mathrm{pinv}(\boldsymbol{A}) * \mathbf{x}$ ensures that $\boldsymbol{A} * \mathbf{x}$ minimizes the distance from $\mathbf{b}$ to the column space (i.e. it is the orthogonal projection of $\mathbf{b}$ onto the column space), while (for the case of non-uniqueness) the element $\mathbf{x}$ obtained belongs to the row-space of $\boldsymbol{A}$ and thus is the unique element of minimal norm with this property. The so-called *singular values decomposition* of an $m \times n$-matrix $\boldsymbol{A}$

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[U,S,V] = svd(A); r = rank(A);
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Recalling basic facts from linear algebra IV

provides an ONB $[u_1, ..., u_r]$ for the column space of $\boldsymbol{A}$, while $[v_1, ..., v_r]$ is the corresponding orthonormal basis in $\mathbb{C}^n$, such that

$$T(\mathbf{x}) = \sum_{k=1}^{r} \sigma_k \langle \mathbf{x}, \mathbf{v}_k \rangle \mathbf{u}_k,$$

or in short: $\boldsymbol{A} = \mathbf{U} * \mathbf{S} * \mathbf{V}'$.
Note that the format of $\mathrm{pinv}(\boldsymbol{A})$ is that of $\boldsymbol{A}'$, but if one writes $\widetilde{\boldsymbol{A}} = \mathrm{pinv}(\boldsymbol{A}') = [\mathrm{pinv}(\boldsymbol{A})]'$ one has

$$\mathrm{pinv}(\boldsymbol{A}) * \mathbf{b} = \widetilde{\boldsymbol{A}}' * \mathbf{b}.$$

If the columns of $\boldsymbol{A}$ span all of $\mathbb{C}^m$, then these n scalar products provide the minimal norm coefficients. In a modern terminology the columns of $\boldsymbol{A}$ for a *frame* form $\mathbb{C}^m$ (of redundancy n/m), and $\widetilde{\boldsymbol{A}}$ is the *dual frame.*

Recalling basic facts from linear algebra V

One of the fundamental formulas in this context is the following one:

$$\mathrm{pinv}(\boldsymbol{A}') = \mathrm{pinv}(\boldsymbol{A} * \boldsymbol{A}') * \boldsymbol{A} = \boldsymbol{A} * \mathrm{pinv}(\boldsymbol{A}' * \boldsymbol{A}). \quad (1)$$

If the columns of $\boldsymbol{A}$ generate $\mathbb{C}^m$, then $S = \boldsymbol{A} * \boldsymbol{A}'$ is invertible. It is called the *frame operator* and one has

$$\mathrm{pinv}(\boldsymbol{A}') = (\boldsymbol{A} * \boldsymbol{A}')^{-1} * \boldsymbol{A}, \text{ or } \widetilde{\mathbf{a}_k} = S^{-1}(\mathbf{a}_k).$$

If the columns of $\boldsymbol{A}$ are linear independent, then the *Gram-matrix* $\boldsymbol{A}' * \boldsymbol{A}$ is invertible, and $\mathrm{pinv}(\boldsymbol{A}' * \boldsymbol{A}) = (\boldsymbol{A}' * \boldsymbol{A})^{-1}$. The columns of $\mathrm{pinv}(\boldsymbol{A}')$ represent the *biorthogonal Riesz basis* then.
These are the two cases where the matrix $\boldsymbol{A}$ has *maximal rank*, i.e. $r = \max(m, n)$. Via transposition one can be exchanged for the other and the Gram-matrix corresponds to the frame operator.

Recalling basic facts from linear algebra VI

Applying the inverse square root of the Gram matrix one obtains the optimal approximation (in the sense of the Frobenius norm) of a given system of vectors by an orthonormal system, spanning the same subspace. This method (different from the Gram-Schmidt process!) is called the *Loewdin orthogonalization*.
The analogue, i.e. applying $S^{-1/2}$ to the given frame, generated a *tight frame* $\boldsymbol{C}$ (n vectors in $\mathbb{C}^m$), i.e. a frame with the property that one has:

$$\mathbf{y} = \sum_{k=1}^{n} \langle \mathbf{y}, \mathbf{c}_k \rangle \mathbf{c}_k, \quad \forall \mathbf{y} \in \mathbb{C}^m.$$

In many cases certain structures of the original frame go over to the dual frame (or the biorthogonal system). This is the case for (regular) Gabor families.

Let us give a LIFE demonstration

We are going to show how the elements of a *Gabor system* are obtained, by applying so-called *time-frequency shifts* to a given *Gabor atom*.
Then we inspect the properties of a Gabor family respectively the Gabor frame operator. We observe that $S = S_{g,a,b}$ commutes with certain TF-shifts and that this is equivalent to a specific form of the matrix representing S.
We also verify that sampling the STFT (the Short-Time Fourier transform) is equivalent to obtain the frame coefficients with respect to a given Gabor frame.
We switch to Mojed01.pdf

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t).$$

Behavior under Fourier transform

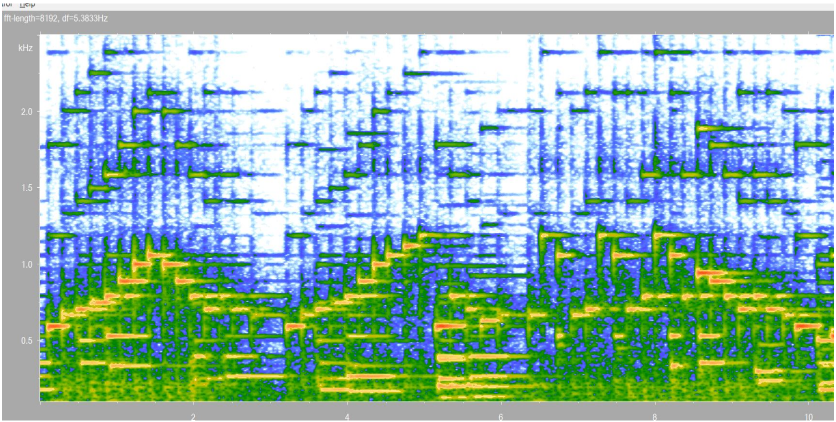
$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

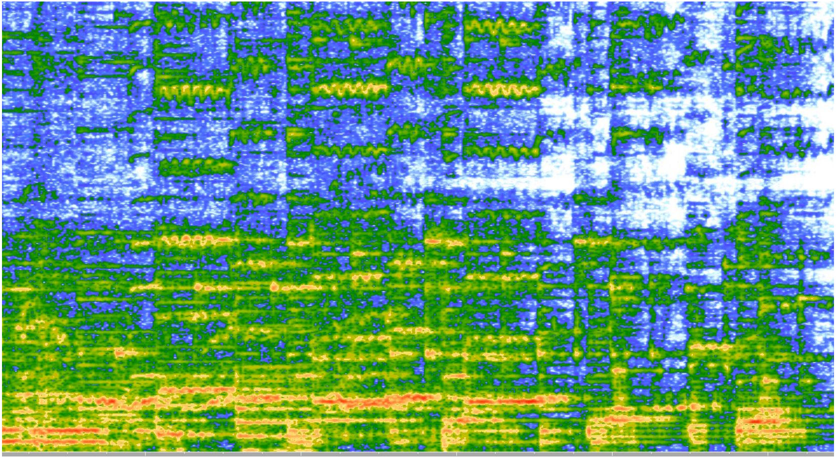
$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT

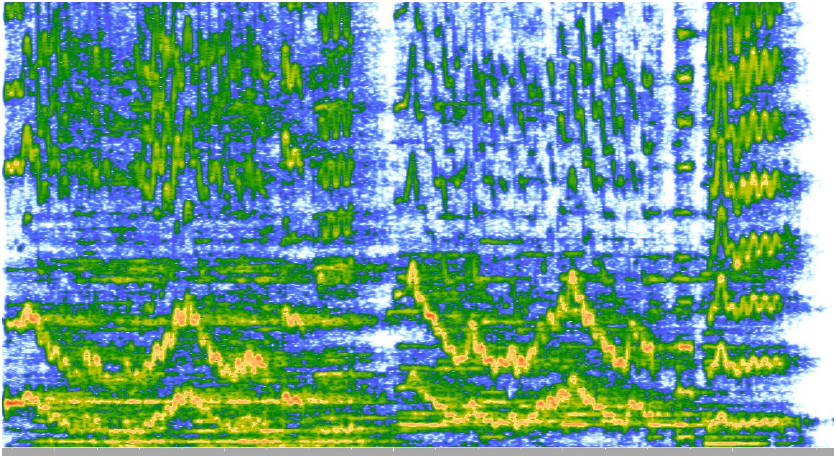
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello

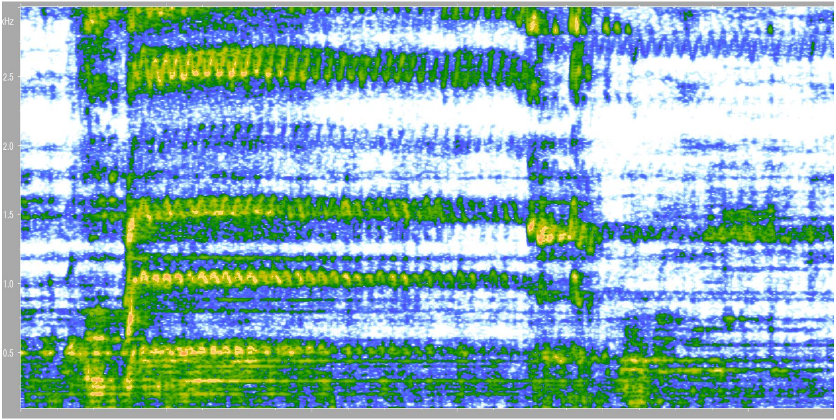


A Musical STFT: Maria Callas



A Musical STFT: Tenor: VINCERA!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

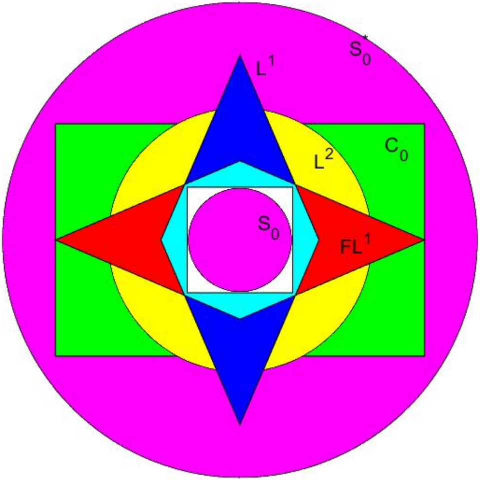
Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

Various Function Spaces



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

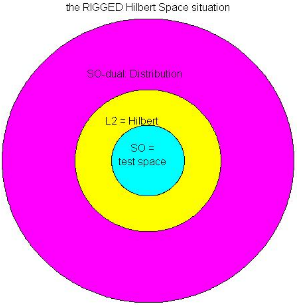
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}'_1)$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}'_2)$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}'_1$ and $\boldsymbol{B}'_2$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{2}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

What are the positive facts I

Compared to *wavelet theory* where the "mother wavelet" has to satisfy a certain *admissibility* condition, it looks as if the situation was much better for STFT: In fact, one has for any *Gabor atom* $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|g\|_{\boldsymbol{L}^2} = 1$ the isometric property for V_g, i.e.

What are the positive facts II

Theorem

For normalized Gabor atoms the STFT $f \mapsto V_g f$ *is an isometric embedding of* $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ *into* $(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2)$. *This implies the continuous reconstruction formula (using the adjoint operator):*

$$f = V_g^*(V_g f) = \int_{\mathbb{R}^{2d}} V_g f(\lambda)\pi(\lambda)g,$$

to be understood in the weak sense!
But it is also a non-expanding mapping from $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ *into* $(\boldsymbol{C}_0(\mathbb{R}^{2d}), \|\cdot\|_\infty)$. *Hence the range space* $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$ *is a reproducing kernel Hilbert space.*

What are the positive facts III

Question: But if we restrict the STFT to some lattice, say $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \lhd \mathbb{R} \times \widehat{\mathbb{R}}$, can we assume that the (analysis or sampling) mapping

$$f \mapsto V_g f|_\Lambda = (V_g f(\lambda))_{\lambda \in \Lambda} \tag{3}$$

which obviously maps $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $(c_0(\Lambda), \|\cdot\|_\infty)$, is also a bounded mapping into $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$? The answer is simply negative.

NOT every $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ generated a Bessel family of the form $(\pi(\lambda)g)_{\lambda \in \Lambda}$.

Equivalently, due to an adjointness relation, the *synthesis mapping*

$$(c_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g \tag{4}$$

What are the positive facts IV

is *not bounded* for an arbitrary $g \in \boldsymbol{L}^2(\mathbb{R}^d)$.
There are two ways out:

- Either one restricts that Gabor atom (both for analysis and synthesis) to belong to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Then both the analysis mapping $f \mapsto (V_g f(\lambda))_{\lambda\in\Lambda}$ and the synthesis mapping (4) are bounded mappings between the Hilbert spaces $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$. In other words, the family $(g_\lambda)_{\lambda\in\Lambda}$ is Bessel family (for any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$!); Furthermore, the combined mapping:

$$f \mapsto S_{g,\Lambda} := \sum_{\lambda\in\Lambda} \langle f, g_\lambda \rangle g_\lambda \tag{5}$$

is a bounded operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$.

What are the positive facts V

- Alternatively, one can restrict the domain of the analysis mapping to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and thus observe that the analysis mapping is bounded from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$. On the other hand (by another adjointness argument) the synthesis mapping maps $\ell^2(\Lambda)$ back into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (in a bounded way). Thus one can still describe the Gabor (pre)frame operator $S_{g,\Lambda}$ as a bounded linear operator from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $\left(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}\right)$.

Either way it is not difficult to verify that $S_{g,\Lambda}$ commutes with any $\pi(\lambda), \lambda \in \Lambda$.

Such considerations imply also that the mapping from $g \to S_{g,\Lambda} \in \mathcal{L}(\boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d))$ is continuous from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into the space of bounded linear operators on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. In particular (due to the continuity of inversion of

What are the positive facts VI

operators in this Banach algebra of operators) similar windows g in $\boldsymbol{S}_0(\mathbb{R}^d)$ have similar dual windows (for fixed Λ).
The dependence of $S_{g,\Lambda}$ on the lattice (parameters) is *not continuous* in the operator norm sense, but in the sense of the strong operator topology, i.e.

$$\Lambda_n \to \Lambda_0 \quad \Rightarrow \quad S_{g,\Lambda_n}(f) \to S_{g,\Lambda_0}(f), \forall f \in \boldsymbol{S}_0.$$

Hence it is not obvious (but a valid) statement that one has according to a joint paper with N. Kaiblinger

$$\widetilde{g}_n = S^{-1}_{g,\Lambda_n}(g) \quad \to \quad S^{-1}_{g,\Lambda_0}(g) = \widetilde{g}_0.$$

see: H. G. Feichtinger and N. Kaiblinger. *Varying the time-frequency lattice of Gabor frames.*
Trans. Amer. Math. Soc., *356(5):2001–2023, 2004.*

Articles concerning $S_0(\mathbb{R}^d)$

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

M. S. Jakobsen.
On a (no longer) New Segal Algebra: a review of the Feichtinger algebra.
J. Fourier Anal. Appl., 24(6):1579–1660, 2018.

H. G. Feichtinger and M. S. Jakobsen.
Distribution theory by Riemann integrals.
Mathematical Modelling, Optimization, Analytic and Numerical Solutions, pages 33–76, 2020.

H. G. Feichtinger and M. S. Jakobsen.
The inner kernel theorem for a certain Segal algebra.
2018.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

Poisson's Formula

One of the key results in Fourier Analysis is Poisson's formula, usually presented in the form

Theorem

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{n\in\mathbb{Z}} \widehat{f}(n), \quad \forall f \in \boldsymbol{S}_0 \tag{6}$$

Using the standard conventions to write $\sqcup\!\sqcup = \sum_{k\in\mathbb{Z}^d} \delta_k$ and the definition of the extended Fourier transform on $\boldsymbol{S}_0'(\mathbb{R}^d)$, via $\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in \boldsymbol{S}_0$, this is equivalent to the statement

$$\mathcal{F}(\sqcup\!\sqcup) = \sqcup\!\sqcup. \tag{7}$$

Using dilations one obtains corresponding formulas for Dirac combs over general lattices $\Lambda \lhd \mathbb{R}^d$.

ShannonLinAlg1

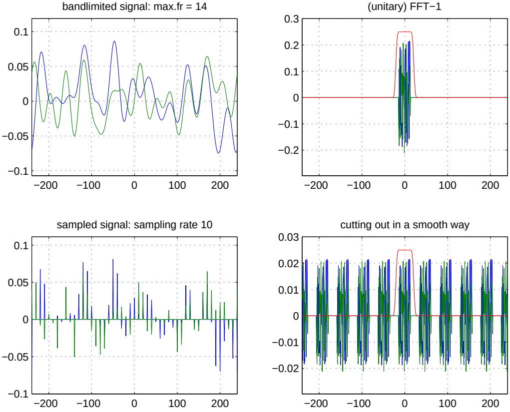


Abbildung: If there is a bit of oversampling, i.e. the s Dirac comb used for sampling is a bit more fine than the minimal requirement (Nyquist criterion) then one has more freedom in the choice of $\widehat{g}$.

Shannlocal1

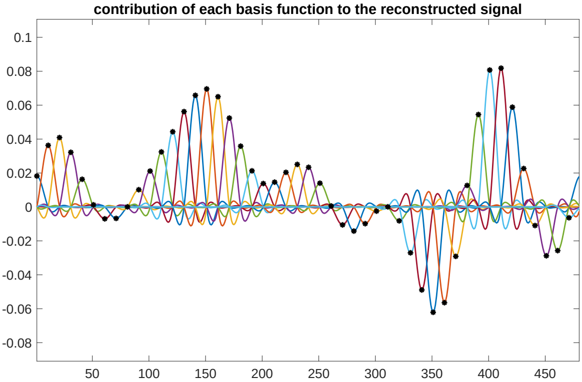


Abbildung: shannondem2C.pdf:
Showing the individual contributions of the well-localized version for the reconstruction of the **real part** of the signal.

ShannonSOfram

Theorem

Given a compact set $\Omega \subset \widehat{\mathbb{R}^d}$, and some lattice $\Lambda \lhd \mathbb{R}^d$ with the property that the $\Lambda^\perp$-translates of Ω are pairwise disjoint, then one recover any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}) \subset \Omega$ can be recovered from the Λ-samples of f by the series expansion

$$f(t) = \sum_{\lambda \in \Lambda} f(\lambda) g(t - \lambda), \tag{8}$$

with convergence in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, in particular absolutely for every $t \in \mathbb{R}^d$ and uniformly (in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$), for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\widehat{g}(\omega) \equiv 1$ for $\omega \in \Omega$ and $\operatorname{supp}(\widehat{g}) \cap \lambda^\perp + \Omega$ for any $\lambda^\perp \in \Lambda^\perp$, $\lambda^\perp \neq 0$.

Shann2dim

The general situation is described by the following picture:

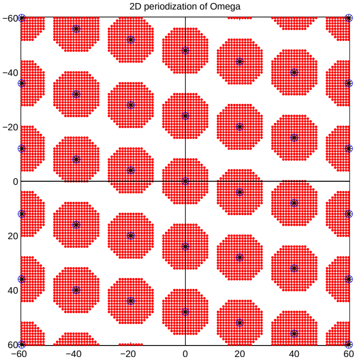


Abbildung: periodOmega2a.pdf: disjoint subsets of the fundamental domain along some lattice Λ.

The DFT/FFT as a special case

The extended Fourier transform $\boldsymbol{S}_0'(\mathbb{R}^d) \to \boldsymbol{S}_0'(\mathbb{R}^d)$ can also be applied to *periodic and discrete* signals (engineering terminology), which coincides with the DFT/FFT:

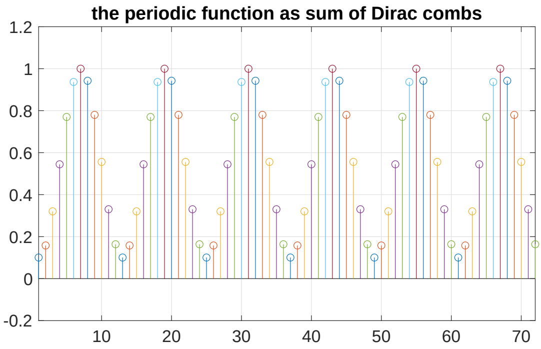


Abbildung: A periodic, discrete measure as a sum of Dirac combs.

Regularization of mild distributions

Note that $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)(\mathbb{R}^d)$ is the space of pointwise multipliers of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, and hence the corollary implies

$$(\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d), \tag{9}$$

and the same relationship, on the Fourier transform side reads:

$$(\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d). \tag{10}$$

In this way one can show that $\boldsymbol{S}_0(\mathbb{R}^d)$ is w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$, which means, that the spectrogram of $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ can be pointwise (uniformly over compact sets) be approximated by corresponding spectrograms of approximating test functions.

Approximation by discrete, periodic measures

The following theorem implies that a function $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be approximated recovered from regular samples:

Theorem

Assume that $\Psi = (T_k\psi)_{k\in\mathbb{Z}^d}$ *defines a BUPU in* $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ *and write* $\mathrm{D}_\rho\Psi$ *for the family* $\mathrm{D}_\rho(T_k\psi) = (T_{\alpha k}\mathrm{D}_\rho\Delta)_{k\in\mathbb{Z}^d}$*, with* $\alpha = 1/\rho \to 0$*. Then* $|\mathrm{D}_\rho\Psi| \leq r\alpha \to 0$ *for* $\alpha \to 0$*, and*

$$\|f - \alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\,\mathrm{St}_\alpha\,\psi\|_{\boldsymbol{S}_0} \to 0, \quad \text{for } \alpha \to 0, \forall f \in \boldsymbol{S}_0. \tag{11}$$

discrete approximation of FT I

Lemma

Given $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *we have* $\nu := f \cdot \sqcup\!\sqcup = \sum_{k\in\mathbb{Z}} h(k)\delta_k \in \boldsymbol{M}_d(\mathbb{R})$. *It has a Fourier transform is a* $\mathbb{Z}$*-periodic function, which can be described as* $\sqcup\!\sqcup * \widehat{f}$*:*

$$\mathcal{F}(f \cdot \sqcup\!\sqcup) = \widehat{f} * \sqcup\!\sqcup, \quad f \in \boldsymbol{S}_0. \tag{12}$$

In a similar way the periodized version of f is a periodic function whose Fourier coefficients are just the samples of $\widehat{f}$:

$$\mathcal{F}(f * ⨿⨿) = \widehat{f} \cdot ⨿⨿, \quad f \in \boldsymbol{S}_0. \tag{13}$$

For positive values a, b, with $a = N \cdot b$ and $c = 1/b, d = 1/a$ we have, up to the constant $C = C_{a,b,c,d}$, for every $f \in \boldsymbol{S}_0$:

$$\mathcal{F}[⨿⨿_a * (⨿⨿_b \cdot f)] = C \cdot [⨿⨿_d \cdot (⨿⨿_c * f)] = C \cdot [⨿⨿_c * (⨿⨿_d \cdot \widehat{f})]. \tag{14}$$

But for sufficiently large values of a and sufficiently small $b = a/N$ one can recover f approximately from these versions. In this way one can demonstrate that $\widehat{f}$ can be approximately computed via FFTs.

Kernel Theorems I

The so-called Kernel Theorem for $\boldsymbol{S}_0$-spaces allows to establish a number of further unitary BGTr-isomorphism. It involves certain types of operators, their (integral) kernels, but also their representation as *pseudo-differential* operators, via the Weyl or Kohn-Nirenberg symbol, or (important for applications in mobile communication) their spreading distribution.
These situations allow to make use of the general principles. In order to understand the transformation one can start from the core spaces, where the analogy with the finite dimensional case is valid in a very natural sense, e.g. if one has $K(x,y) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, then red $K(x,y) = T(\delta_y)(x)$, in analogy to the matrices

$$a_{n,k} = [T(\mathbf{e}_k)]_n.$$

The Hilbert space case of the well-known characterization

Kernel Theorems II

Theorem

There is a unitary BGTr isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and* $\left(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')\right)(\mathbb{R}^d)$, *which is a unitary mapping between* $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ *and* $\left(\mathcal{HS}, \|\cdot\|_{\mathcal{HS}}\right)$, *with*

$$\langle T_1, T_2\rangle_{\mathcal{HS}} = \mathrm{trace}(T_1 \circ T_2^*).$$

Alternative unitary BGTr describe operators via *Kohn-Nirenberg* symbol resp. their spreading representation, such as $T \in \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} H(\lambda)\pi(\lambda), \quad H \in \boldsymbol{S}_0(\mathbb{R}^{2d}).$$

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
Gabor Book 1998.

Wilson Bases

For the case $G = \mathbb{R}^d$ one can derive the kernel theorem also from the description of operators mapping ℓ^1 to ℓ^∞ or vice versa (in a w^*-to-norm continuous way).
The key is the fact, that local Fourier basis, but in particular the so-called Wilson bases are suitable for modulation spaces. In our situation we can formulat the following

Theorem

Any ON Wilson basis (obtained by a smart pairwise recombination of the elements of a tight Gabor frame of redundancy 2) establishes a unitary BGTr between $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ *and* $(\ell^1, \ell^2, \ell^\infty)$.

H. G. Feichtinger, K. Gröchenig, and D. F. Walnut.
Wilson bases and modulation spaces.
Math. Nachr., 155:7–17, 1992.

Thank you and further links

Thank for your attention!

... and please think about application oriented analysis!

A number of talks by the speaker are found at

www.nuhag.eu/talks : access via visitor//nuhagtalks

Details on the course: see www.nuhag.eu/ETH20
Direct question to: hans.feichtinger@univie.ac.at