NuHAG Numerical Harmonic Analysis Group

Approximation by translates in invariant Banach spaces of distributions and the bounded approximation property

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The Topic of this Talk

This talk is presenting joint work with **Anupam Gumber** PostDoc at the Indian Institute of Science, 560012 Bangalore).

It is a summary of 3 papers on the following subject:

Given an invariant Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of functions (or typically tempered distributions), when can one assure that the closed linear span of the set of translates of a single function, *or:* the set of all translates of all the dilated version of one suitable function coincides with the whole Banach space.
fegu20:[1], **fegu21:[2]**, **fegu21-1:[3]**

Key bibliographic References I

H. G. Feichtinger and A. Gumber.
Completeness of shifted dilates in invariant Banach spaces of tempered distributions.
Proc. Amer. Math. Soc., 12 pages, accepted, 2021.

H. G. Feichtinger and A. Gumber.
Completeness of sets of shifts in invariant Banach spaces of tempered distributions using Tauberian conditions.
submitted, page 10, 2021.

H. G. Feichtinger and A. Gumber and N. Teofanov.
Completeness of sets of translates in invariant Banach spaces of ultra-distributions over locally compact Abelian groups.
in preparation, 2021.

Key bibliographic References II

V. Katsnelson.
On the completeness of Gaussians in a Hilbert functional space.
Complex Anal. Oper. Theory, pages 1–22, Nov 2017.

W. Braun and H. G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

H. G. Feichtinger.
Compactness in translation invariant Banach spaces of distributions and compact multipliers.
J. Math. Anal. Appl., 102:289–327, 1984.

P. Dimovski, S. Pilipovic, and J. Vindas.
New distribution spaces associated to translation-invariant Banach spaces.
Monatsh. Math., 177(4):495–515, 2015.

P. Dimovski, S. Pilipovic, B. Prangoski, and J. Vindas.
Translation–modulation invariant Banach spaces of ultradistributions.
J. Fourier Anal. Appl., 25(3):819–841, 2019.

The starting point

The paper by V. Katsnelson was stating that for certain *Hilbert spaces* $\mathcal{H}$ of functions, continuously embedded into $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ satisfying a few additional (!invariance) conditions has the property that the set of shifted and dilated Gaussian's g span a dense subspace of $\mathcal{H}$, with $g(t) = exp(-\pi t^2)$, t $\in \mathbb{R}$.
This raises a number of questions: What are the relevant conditions for such a statement? Is the Hilbert space structure important? Can one have similar statements for Banach spaces of functions or distributions even if they are not inside of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (to jump right away to the multi-dimensional setting)? And recalling Tauberian Theorems (non-vanishing of the Fourier transform of $g = \widehat{g}$!), is it really necessary to use all the dilates of g (the answer will be no!).

ABSTRACT I

In this talk we shall present a few recent results concerning the approximation of functions in Banach spaces of distributions (or just measurable functions) by finite linear combinations of *translations of a given functions*.
The key feature of the approach is the use of integrated group actions (realized as convolution with elements from a Beurling algebra or and pointwise multiplication with respect to some Fourier-Beurling algebra), the use of bounded approximate units, and discretization of such convolution products.
These results have been obtained in joint work with Anupam Gumber (the first paper is about to appear in Proc. AMS soon). In a second paper (preprint available on ARXIV) we show how these results can be improved: using a Tauberian condition a single function (without dilations) can be used.

ABSTRACT II

As a by-product (making use of the characterization of compact sets in those Banach spaces, which are double Banach modules) it can be demonstrated that all those (separable) Banach spaces of tempered distributions satisfy the bounded approximation property. Such a statement had been shown already in a widely unknown paper by the author (jointly with W.Braun), published in 1985, using fairly abstract methods involving twisted convolution.

[brfe85] H. G. Feichtinger and W. Braun.
Banach spaces of distributions with double module structure and twisted convolution.
In *Proc. Alfred Haar Memorial Conf.*, volume 49 of *Coll. J. Bolyai Soc.*, pages 225 – 246. North Holland Publ. Comp., 1985.

The setting

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a **MINTSTA**, i.e. a *minimal tempered standard space* if one has:

1.
$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d); \tag{1}$$

2. $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (minimality);
3. $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is translation invariant, and for $n_1 \in \mathbb{N}, C_1 > 0$
$$\|T_x f\|_{\boldsymbol{B}} \leq C_1 \langle x \rangle^{n_1} \|f\|_{\boldsymbol{B}} \quad \forall x \in \mathbb{R}^d, f \in \boldsymbol{B}; \tag{2}$$
4. $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is modulation invariant, and for $n_2 \in \mathbb{N}, C_2 > 0$
$$\|M_y f\|_{\boldsymbol{B}} \leq C_2 \langle y \rangle^{n_2} \|f\|_{\boldsymbol{B}} \quad \forall y \in \mathbb{R}^d, f \in \boldsymbol{B}. \tag{3}$$

The long list of Function Spaces

There is a long list of function spaces to which the above principle applies:

- Weighted $\boldsymbol{L}^p$-spaces;
- Sobolev spaces, Triebel-Lizorkin and Besov spaces;
- Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$;
- modulation spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$;
- suitable coorbit spaces, e.g. $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$;

You may find further spaces in the books on *function spaces*, e.g. by H. Triebel or more recently S. Samko (and coauthors) on **Integral Operators in Non-standard Function Spaces.**

Lemma

(i) For any MINTSTA $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ also its Fourier version

$$\mathcal{F}\boldsymbol{B} = \{\widehat{f} \mid f \in \boldsymbol{B}\}$$

is a MINTSTA with respect to the norm

$$\|\widehat{f}\|_{\mathcal{F}\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{4}$$

(ii) Given two MINTSTAs $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, also their intersection is a MINTSTA, with the norm

$$\|f\|_{\boldsymbol{B}^1\cap\boldsymbol{B}^2} := \|f\|_{\boldsymbol{B}^1} + \|f\|_{\boldsymbol{B}^2}, \quad f \in \boldsymbol{B}^1 \cap \boldsymbol{B}^2.$$

A similar statement is valid for $\boldsymbol{B}^1 + \boldsymbol{B}^2$.

Shubin Classes

Lemma

For $m_1(x) = \langle x \rangle^s = m_2(x)$, $s \in \mathbb{R}$ the corresponding spaces $\boldsymbol{L}^2_{m_1} \cap \mathcal{F}\boldsymbol{L}^2_{m_2}$ are Fourier invariant, as the intersection of a Sobolev space with the corresponding weighted $\boldsymbol{L}^2$-space.

They can also be identified with the so-called Shubin classes $(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s})$, characterized (for any $s \in \mathbb{R}$!) as the Banach spaces of all tempered distributions with a short-time Fourier transform in $\boldsymbol{L}^2_{v_s}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, i.e.

$$\|f\|_{\boldsymbol{Q}_s(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)|(1 + |\lambda|^2)^s d\lambda \right)^{1/2} < \infty. \quad (5)$$

Double Module Structure

Any TMIB $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module over* $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$, and in fact over $\left(\boldsymbol{M}^1_w(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1_w(\mathbb{R}^d)}\right)$. Hence we have

$$\|\mu * f\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}^1_w} \|f\|_{\boldsymbol{B}}, \quad \forall \mu \in \boldsymbol{M}^1_w(\mathbb{R}^d), f \in \boldsymbol{B}. \tag{6}$$

Let us not forget to mention the associative law:

$$(\mu_1 * \mu_2) * f = \mu_1 * (\mu_2 * f), \quad \mu_1, \mu_2 \in \boldsymbol{M}^1_w(\mathbb{R}^d), f \in \boldsymbol{B}. \tag{7}$$

This result concerning "integrated group representations" can be considered a folklore result, see for example Bourbaki, Integration, Chap.8, integrated version for working for general group representations on a Banach space. Similar results are given in [5] and [7], for example.

Discretization of Convolution

Theorem

Any minimal TMIB Banach space of tempered distributions $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is an essential Banach module over some Beurling algebra* $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$*. Moreover, one has for any* $g \in \boldsymbol{B}$ *and* $k \in \boldsymbol{L}^1_w(\mathbb{R}^d)$*:*

$$\|g * k - g * \mathrm{D}_\Psi k\|_{\boldsymbol{B}} \to 0 \quad \text{for } |\Psi| \to 0. \tag{8}$$

By way of symmetry one has a pointwise module structure under the Fourier Beurling algebra, now with the weight function controlling the modulation invariance of the given TMIB

Here we use BUPU's, i.e. FINE (uniform) partitions of unity, which can be obtained from a given smooth partitionof unity, say $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}^d)$, with $\sum_{k\in\mathbb{Z}^d} T_i\varphi(x) = 1$.
Such partitions of unity can be for example the family of cubic B-splines or smooth variants. Also a periodized Gauss function divided by the sum.
For any such BUPU we define a bounded, w^*-w^*- continuous and non-expansive discretization operator

$$\mathrm{D}_\Psi : \mu \mapsto \sum_{i\in I} \mu(\psi_i)\delta_{x_i}.$$

The details of this construction are e.g. in the course notes of my recent ETH20 course: www.nuhag.eu/ETH20.

General considerations of Convolution

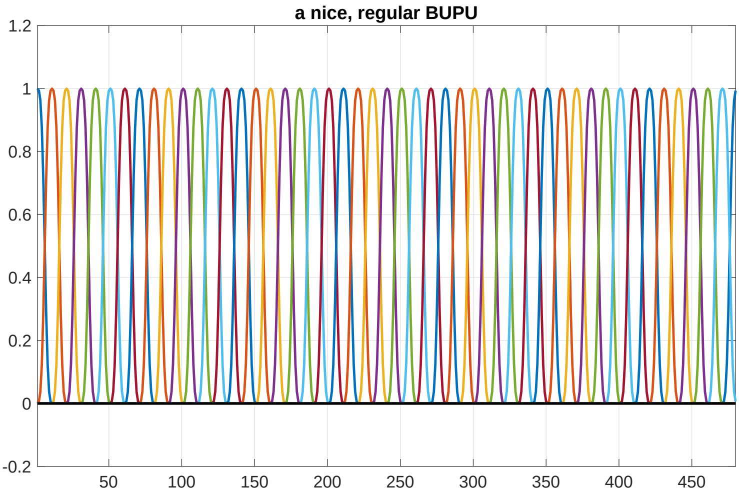
While the standard approach to convolution starts with *measure theoretical* methods (Bochner integrals, weak definitions etc.) we have in mind that one should introduce convolution as the natural extension of discrete convolutions, based on the obvious rule

$$\delta_x \star \delta_y := \delta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

Details of such an approach have been given in the ETH20 course (2020/21) resp. in the paper [fe17] H. G. Feichtinger.

A novel mathematical approach to the theory of translation invariant linear systems.
In I. Pesenson, Q. Le Gia, A. Mayeli, H. Mhaskar, and D. Zhou, editors, *Recent Applications of Harmonic Analysis to Function Spaces, Differential Equations, and Data Science.*, Applied and Numerical Harmonic Analysis., pages 483–516. Birkhäuser, Cham, 2017.

Illustrating BUPUs



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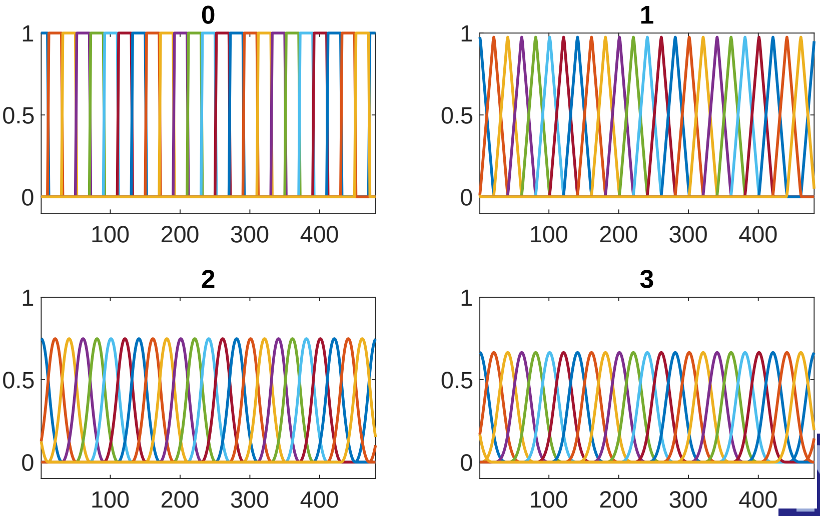
Different regular BUPUs



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Convolution without Measure Theory

This are the key points in alternative approach:

1. The translation invariant operators (systems) on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ are exactly represented by the continuous linear functionals on this Banach space, i.e. $T(f) = \mu \star f$, with

$$\mu \star f(x) = \mu(T_x f^{\checkmark}), \quad x \in \mathbb{R}^d.$$

2. This correspondence between operators is isometric;
3. Thus composition of operators can be transferred to $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}) = \left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$!
4. $T_x f = \delta_x \star f, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d)$
5. The action of general a general measure $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ can be approximated by its discretized version (i.e. $D_\Psi \mu \star f$ is a Cauchy-net in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$!).

Connections to Histograms

Recall that the coefficients are obtained by integrating k over suitable members of the BUPU $\Psi = (\psi_i)_{i \in I}$, via

$$c_i = k(\psi_i) = \int_{\mathbb{R}^d} \psi_i(x) k(x) dx.$$

For the case of the most simple BUPU (using rectangular functions, resp. using a uniform decomposition as it is used for Riemann integrals) the information stored in $D_\Psi \mu$ corresponds to the information that can be read of from a *histogram*!
In fact in this case we have (on $\mathbb{R}$)

$$c_k = \int_{\alpha k}^{\alpha(k+1)} d\mu,$$

which is just the hight of the k-th bin in the histogram.

Some Functional Analytic Comments

Just as Riemannian sums represent a w^*-convergent net of discrete (bounded) measures approximating the function

$$f \mapsto \int_a^b f(x)dx, \quad f \in \boldsymbol{C}([a,b]),$$

known as Riemann integral we have the situation, that

Lemma

For any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d) = \boldsymbol{C}_0'(\mathbb{R}^d)$ the family $(D_\Psi \mu)_{|\Psi| \to 0}$ is a bounded, uniformly tight family in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, which is w^*-convergent to μ, i.e. for any given $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ on has:

$$D_\Psi \mu(f) \to \mu(f), \quad |\Psi| \to 0.$$

The Main Result I

Theorem

Given a minimal tempered standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ on $\mathbb{R}^d$, and any $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = \widehat{g}(0) \neq 0$, the set

$$S(g) := \{ T_x \mathrm{St}_\rho g \,|\, x \in \mathbb{R}^d, \rho \in (0,1] \}$$

is total in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. the finite linear combinations are dense.

Here we use

$$g_\rho(x) := \mathrm{St}_\rho g(x) = \rho^{-d} g(r/\rho), \quad \rho > 0,$$

and

$$T_x g(y) = g(y - x), \quad x, y \in \mathbb{R}^d.$$

The claim requires to find, for any given $f \in \boldsymbol{B}$ and $\varepsilon > 0$, some finite linear combination φ of elements from $S(g)$ such that

$$\|f - \varphi\|_{\boldsymbol{B}} < \varepsilon. \tag{9}$$

We will verify something slightly stronger: Given $f, \varepsilon > 0$ and finitely many MINSTA norms there exists $\rho_0 < 1$ such that for any (fixed) $\rho \in (0, \rho_0]$ one can find a finite set $(x_i)_{i \in F}$ and coefficients $(c_i)_{i \in F}$ such that $\varphi = \sum_{i \in F} c_i T_{x_i} g_\rho$ satisfies (9).

In fact, we will find finite rank operators with the range in the described finite dimensional spaces, and with the additional property, that the ε-approximation is uniform over a given compact set. In other words, be show that these spaces have the bounded approximation property. In fact, the family of operators has to be uniformly bounded!

This approximation will be achieved in four steps: By the density of $\mathcal{S}(\mathbb{R}^d)$ in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and the density of compactly supported functions in $\mathcal{S}(\mathbb{R}^d)$ (in the Schwartz topology), we can find some $k \in \mathcal{D}(\mathbb{R}^d) \subset \boldsymbol{B} \cap \boldsymbol{L}^1_w$ with

$$\|f - k\|_{\boldsymbol{B}} < \varepsilon/4. \tag{10}$$

In the next step we we choose ρ_0 such that for any $\rho \in (0, \rho_0]$ one has

$$\|g_\rho * k - k\|_{\boldsymbol{B}} < \varepsilon/4. \tag{11}$$

Let us fix one such parameter ρ for the rest. The final step is the discretization of the convolution $g_\rho * k$, by replacing k by some finite, discrete measure in $\boldsymbol{M}^1_w(\mathbb{R}^d)$, by applying Theorem 4 with $g = g_\rho$ and $k \in \mathcal{D}(\mathbb{R}^d)$. By choosing $\delta_0 > 0$ properly we can guarantee that $|\Psi| \leq \delta_0$ implies

$$\|k * g_\rho - (\mathrm{D}_\Psi k) * g_\rho\|_{\boldsymbol{B}} < \varepsilon/4. \tag{12}$$

Note that

$$\varphi = (\mathrm{D}_\Psi k) * g_\rho = \sum_{i \in I} c_i \delta_{x_i} * g_\rho = \sum_{i \in F} c_i T_{x_i} g_\rho \qquad (13)$$

has the required form, because
$F = \{i \in I \mid \mathrm{supp}(k) \cap \mathrm{supp}(\psi_i) \neq \emptyset\}$ is a finite set, due to the compactness of $\mathrm{supp}(k)$. It depends only on $\mathrm{supp}(k)$ and Ψ.
Combining the estimates (10), (11) and (12), we have

$$\|f - \varphi\|_{\boldsymbol{B}} \leq \|f - k\|_{\boldsymbol{B}} + \|k - g_\rho * k\|_{\boldsymbol{B}} + \|k * g_\rho - \varphi\|_{\boldsymbol{B}} \leq \frac{3\varepsilon}{4},$$

i.e., we have obtained the desired estimate:

$$\|f - \sum_{i \in F} c_i T_{x_i} g_\rho\|_{\boldsymbol{B}} = \|f - \varphi\|_{\boldsymbol{B}} < \varepsilon, \qquad (14)$$

and the proof is complete.

Extension to compact sets

In order to extend the approximation to a compact set we have to recall a compactness criterion published by the author in 1984.

[fe84] H. G. Feichtinger.
Compactness in translation invariant Banach spaces of distributions and compact multipliers.
J. Math. Anal. Appl., 102:289–327, 1984.

It is a (quite general) version of the Riesz-Kolomogorov Theorem for $\left(\boldsymbol{L}^p(\mathbb{R}), \|\cdot\|_p\right)$, using the concepts of *tightness* and *equicontinuity*.

Compactness Criterion for MINTSTAs

Theorem

A bounded and closed subset of a MINTSTA $M \subset \boldsymbol{B}$ is compact in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only if for each $\varepsilon > 0$ there exist two compactly supported functions g and h such that

1. $\|g * f - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M$;
2. $\|h \cdot f - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M$;

One may think of g as a member of a *Dirac sequence* (of course it could also be a compressed Gaussian as well) and of h is a kind of *plateau-like* function. Here one may think of a dilated version of a Gauss function. In this sense the two properties are exchanging their roles under the Fourier transform.

The long list of Function Spaces

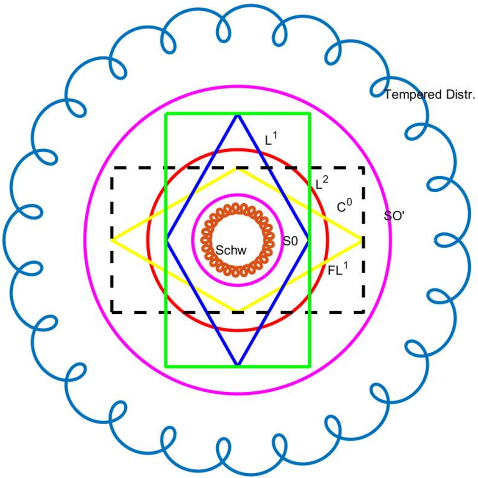
There is a long list of function spaces to which the above principle applies:

- Weighted $\boldsymbol{L}^p$-spaces;
- Sobolev spaces, Triebel-Lizorkin and Besov spaces;
- Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$;
- modulation spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$;
- suitable coorbit spaces, e.g. $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$;

In conclusion this paper provides extensions of the main result of V. Katsnelson (from 2019) in the following directions:

1. The Gauss function can be replaced by any Schwartz function with non-zero integral;
2. The results are valid for $\mathbb{R}^d$, for any $d \geq 1$;
3. We abolish the assumption that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Hilbert space, as well as the rather restrictive property that it should be contained in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$;
4. Our main result applies to an abundance of function spaces for which such completeness statements can be shown; we just list particular examples;
5. As a benefit we establish a connection to the so-called Shubin classes $\boldsymbol{Q}_s(\mathbb{R}^d)$ and show that the completeness statement is valid for for the full range of $s \in \mathbb{R}$.

A Zoo of Banach Spaces for Fourier Analysis



The Metric Approximation Property

The result above can even be strengthened in the following fashion. It is not only true that for any given $f \in \boldsymbol{B}$ and $\varepsilon > 0$ one can find a compression parameter ρ and a finite linear combination h of translates of $\mathrm{St}_\rho(g)$ which approximates f, but even more is true:

Lemma

Given any compact subset of $M \subset \boldsymbol{B}$ and $\varepsilon > 0$ one has the following: There exists $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$ there is a finite subset $F = F(g, \rho, M) \subset \mathbb{R}^d$ such that any $f \in M$ has some ε-approximation by a suitable linear combination of these translates of g_ρ with parameters only from F.

In fact, we can show that any minimal TMIB satisfies the metric approximation property, which (by definition) means: given any compact subset $M \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $\varepsilon > 0$ there exists a finite rank operator $T = T(M, \varepsilon > 0)$ such that

$$\|f - T(f)\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M. \tag{15}$$

And even this statement can be made stronger, more or less by the same argument. While the above statement allows to choose this linear operator in dependence of the space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ used, resp. the norm we can achieve such a family of operators universally: There exists a sequence of bounded linear operators $T_n : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ of of finite rank with the property that

- T_n is uniformly bounded on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$; and such that for any compact set $M \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$
-
$$\lim_{n\to\infty} [\sup_{f \in M} \{\|T_n f - f\|_{\boldsymbol{B}}\}] = 0, \tag{16}$$

Iterated limits

Lemma

Assume that $(T_\alpha)_{\alpha \in I}$ *and* $(S_\beta)_{\beta \in J}$ *are two* **bounded** *(!) nets of operators in* $\mathcal{L}(\boldsymbol{V})$*, strongly convergent* T_0*, and* S_0 *resp. i.e.*

$$T_o(\mathbf{v}) = \lim_\alpha T_\alpha(\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{V} \quad \textit{and} \quad S_0(\mathbf{w}) = \lim_\beta S_\beta(\mathbf{w}) \quad \forall \mathbf{w} \in \boldsymbol{V}.$$

Then the net $(T_\alpha \circ S_\beta)_{(\alpha,\beta)}$ *is strongly convergent to* $T_0 \circ S_0$*:*

$$T_o[S_0(\mathbf{v})] = [T_0 \circ S_0](\mathbf{v}) = \lim_{\alpha,\beta} [T_\alpha \circ S_\beta](\mathbf{v}). \tag{17}$$

In detail: Given $\mathbf{v} \in \boldsymbol{V}, \varepsilon > 0$ *there exists a pair of indices* $(\alpha_0, \beta_0) \in I \times J$ *such that for* $\alpha \succeq \alpha_0$ *in* I *and* $\beta \succeq \beta_0$ *in* J

$$\| T_0(S_0(\mathbf{v})) - T_\alpha(S_\beta(\mathbf{v}))\| \leq \varepsilon. \tag{18}$$

Note that this allows a change in the order in which limits are taken (not the order of operators);

$$T_0 \circ S_0 = \lim_\alpha \lim_\beta T_\alpha \circ S_\beta = \lim_\beta \lim_\alpha T_\alpha \circ S_\beta. \tag{19}$$

Observe that of course the composed operators also form a bounded net and thus it is a routine task to prove that the convergence stated above will be obtained

uniformly over compact subsets $M \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

We will make use of this fact by making use of approximate units in the Banach algebras $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$ and the Fourier-Beurling algebra $(\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d)})$ respectively.

Approximate units by dilation

In the first case let us assume (for convenience) that the weight function w is of polynomial type, e.g. of the form $w(x) = \langle x \rangle^2 = (1 + |x|^2)^{s/2}$, hence radial symmetric and increasing with $|x|$.
Here we can create from any Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{g}(0) = \int_{\mathbb{R}^d} g(x)dx = 1$ an approximate unit for $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$ by applying the usual ($\boldsymbol{L}^1$-normalized) dilation operator St_ρ:

$$\mathrm{St}_\rho(g)(x) := \rho^{-d} g(x/\rho), \quad x \in \mathbb{R}^d, \rho \to 0.$$

Correspondingly the dilation operator, given as

$$\mathrm{D}_\rho h(x) = h(\rho x), \quad \rho \to 0,$$

produces bounded approximate units for ptw. multiplication in the Fourier-Beurling algebra $(\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d)})$ if $h(0) = 1$.

Consequence: relevant special case

For our purpose, i.e. for ensuring that the choice of k in the proof of the main theorem is the consequence of the application of as suitable *regularization operator* it make sense the choose $h \in \mathcal{D}(\mathbb{R}^d)$ (i.e. with compact support) and look at the operators

$$R_\rho : f \mapsto \mathrm{D}_\rho h \cdot (\mathrm{St}_\rho g * f).$$

Then clearly these operators are uniformly bounded on any TMIB and we have for $\rho \to 0$, uniformly over the compact set M:

$$\lim_{\rho \to 0} \|R_\rho(f) - f\|_{\boldsymbol{B}} < 0,$$

But $R_\rho(M)$ is even compact in $\boldsymbol{B}$, since

$$k := R_\rho(f) \in \mathcal{S}(\mathbb{R}^d) \cdot \left(\mathcal{S}(\mathbb{R}^d) * \mathcal{S}'(\mathbb{R}^d)\right) \subset \mathcal{S}(\mathbb{R}^d) \subset \boldsymbol{B}.$$

The Metric Approximation Property

Observing that also the estimates (11) and (12) are valid uniformly for compact sets of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (now applied to $R_\rho(M)$, for ρ chosen appropriately) we come to the conclusion that on has (g is now the function in the original statement):

$$\|f - \mathrm{D}_\Psi(R_\rho f) * g_\rho\|_{\boldsymbol{B}} < \varepsilon, \quad \forall f \in M. \tag{20}$$

Since $\mathrm{supp}(\mathrm{D}_\rho(h)) = \mathrm{supp}(h)/\rho$ is compact it is clear that the discrete measure $\mathrm{D}_\Psi(k)$ has finite support, and thus the approximation operator can be written as the finite rank operator

$$f \mapsto \sum_{i \in F} c_i T_{y_i} g_\rho,$$

i.e. the range is in the finite dimensional span of the family $T_{y_i} \mathrm{St}_\rho(g)$, for some finite set F.

Statement added in proof: 18.03.

The claim that spaces with a double module structure satisfy the bounded approximation property has already been observed in **brfe85:[3]**, Corollary 10 (see reference given below, downloadable from `www.nuhag.eu/bibtex`, request access code).

[brfe85] H. G. Feichtinger and W. Braun.
Banach spaces of distributions with double module structure and twisted convolution.
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The Tauberian Approach

Another direction of improvement of the results of Katsnelson (under similar assumptions concerning the function spaces involved) can be based on the use of Tauberian Theorems.
It is well known, that for any Beurling algebra $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$ the following fact is well known (see the book of H. Reiter):
If $\widehat{g}(s) \neq 0$ for any $s \in \mathbb{R}^d$, then it is not necessary to use all the dilations of g. It is enough to use g "as it is".
What is needed is the approximation of $f \in \boldsymbol{B}$ by band-limited functions in $\boldsymbol{L}^1_w(\mathbb{R}^d) \cap \boldsymbol{B}$, because in this case one can start from the factorization.

For such a band-limited $h \in \boldsymbol{B} \cap \boldsymbol{L}^1_w(\mathbb{R}^d) \subset \boldsymbol{L}^1_w(\mathbb{R}^d)$ with $\mathrm{spec}(h) := \mathrm{supp}(\hat{h}) = Q$ (some compact set) we can find, according to Wiener's inversion theorem for Beurling algebras (see p.13 of [1]) some $g_1 \in \boldsymbol{L}^1_w(\mathbb{R}^d)$ with $\widehat{g_1}(y) = 1/\widehat{g}(y)$ for all $y \in Q$. As a consequence we can write

$$h = (g * g_1) * h = g * (g_1 * h) \tag{21}$$

where $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \subset \boldsymbol{L}^1_w(\mathbb{R}^d)$ and

$$g_1 * h \in \boldsymbol{L}^1_{v_s}(\mathbb{R}^d) * \boldsymbol{B}_{1,w}(\mathbb{R}^d) \subset \boldsymbol{B}_{1,w}(\mathbb{R}^d) \subset \boldsymbol{B}.$$

Outlook

The results presented can be extended in various directions:

1. The natural setting for the problem is with weight functions satisfying the so-called *Beurling-Domar condition*, these include, beyond polynomial weights the so-called *sub-exponential weights*.
2. The natural setting for the application of Wiener's Theorem are Beurling algebras over LCA groups, so we are not restricted to $G = \mathbb{R}^d$.

Corresponding Tauberian Theorems for LCA groups are discussed in the book of H. Reiter (**rest00:[1]**), the details are to be given in **fegu21-1:[3]**.

Typical Arguments

In the next step we approximate $g_1 * h \in \boldsymbol{B}_{1,w}(\mathbb{R}^d) \subset \left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$ by some compactly supported function $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, i.e. we can have

$$\|g_1 * h - k\|_{\boldsymbol{L}^1_w} < \varepsilon/4\|g\|_{\boldsymbol{B}}. \tag{22}$$

As a consequence we obtain, recalling the identity (21)

$$\|h - k * g\|_{\boldsymbol{B}} = \|(g_1 * h - k) * g\|_{\boldsymbol{B}} \leq \|g_1 * h - k\|_{\boldsymbol{L}^1_w} \|g\|_{\boldsymbol{B}} < \varepsilon/4.$$

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THANK you for your attention!
ALL THE BEST to Stefan!
Happy birthday!

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