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# 1 Novi Sad Talk I

1. We start with the pointwise Banach algebra $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, of continuous, complex-valued (hence bounded) functions vanishing at infinity, endowed with the sup-norm (written as $\|f\|_\infty$ for $f \in \boldsymbol{C}_b(\mathbb{R}^d)$). It contains $\boldsymbol{C}_c(\mathbb{R}^d)$ as a dense subspace, in fact it coincides with the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$!

2. By (our, justified) definition the bounded linear functionals are called bounded measures, and we use the symbol $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ for $\left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$.

3. Using the simple fact that $\|\delta_x\|_{\boldsymbol{M}_b} = 1$ one observes that the closed linear span of the *Dirac functionals* $\delta_x(f) := f(x)$, for $f \in \boldsymbol{C}_0(\mathbb{R}^d)$, are the discrete [bounded] measures, can be characterized as absolutely convergent series of the form $\nu = \sum_{k=1}^\infty c_k \delta_{x_k}$, with $\sum_{k=1}^\infty |c_k| < \infty$. We use the symbol $\boldsymbol{M}_d(\mathbb{R}^d)$ for this closed subspace of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.

Note that translations acts in a natural way on e.g. $\boldsymbol{C}_b(\mathbb{R}^d)$, given by $T_x(f)(z) = T(z - x)$. The fact, these translations form a commutative group of isometric operators, i.e. satisfy

$$\|T_x f\|_\infty = \|f\|_\infty, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d), x \in \mathbb{R}^d, \tag{1}$$ C0trans02

An object of great interest is the subalgebra of all bounded linear operators which commute with the shift operators! We use the symbol $\mathcal{H}_G(\boldsymbol{C}_0(G))$.

chartinv00 **Theorem 1.** *There is an (natural) isometric identification between translation invariant, linear systems on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (i.e. bounded linear mappings on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ commuting with translations, and the space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.*

*In fact, every such operator $T \in$ is a convolution operator by a uniquely determined bounded measure $\mu$, we write $C_\mu(f)$ or (later) $\mu * f$, for $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.*

The "non-trivial" part is of course to show that $C_\mu(f)$ is not only bounded and (in fact uniformly) continuous, but also still tending to zero at infinity.

The space $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ has another important feature: It has *bounded approximate units*. One can think of these as collections of (almost) plateau functions $(p_\alpha)$, which are uniformly bounded (without loss with $\|p_\alpha\|_\infty \leq 1, \forall \alpha$, such that

$$\lim_{\alpha\to\infty} \|p_\alpha \cdot f - f\|_\infty = 0, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{2}$$ BAPPR01

Over the Euclidean space such BAU's can be obtained by dilating any $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ (e.g. in $\boldsymbol{C}_c(\mathbb{R}^d)$ or $\boldsymbol{C}_0(\mathbb{R}^d)$) with $h(0) = 1$, and putting

$$\mathrm{D}_\rho(h)(z) := h(\rho z), \quad \text{for} \quad \rho \to 0.$$

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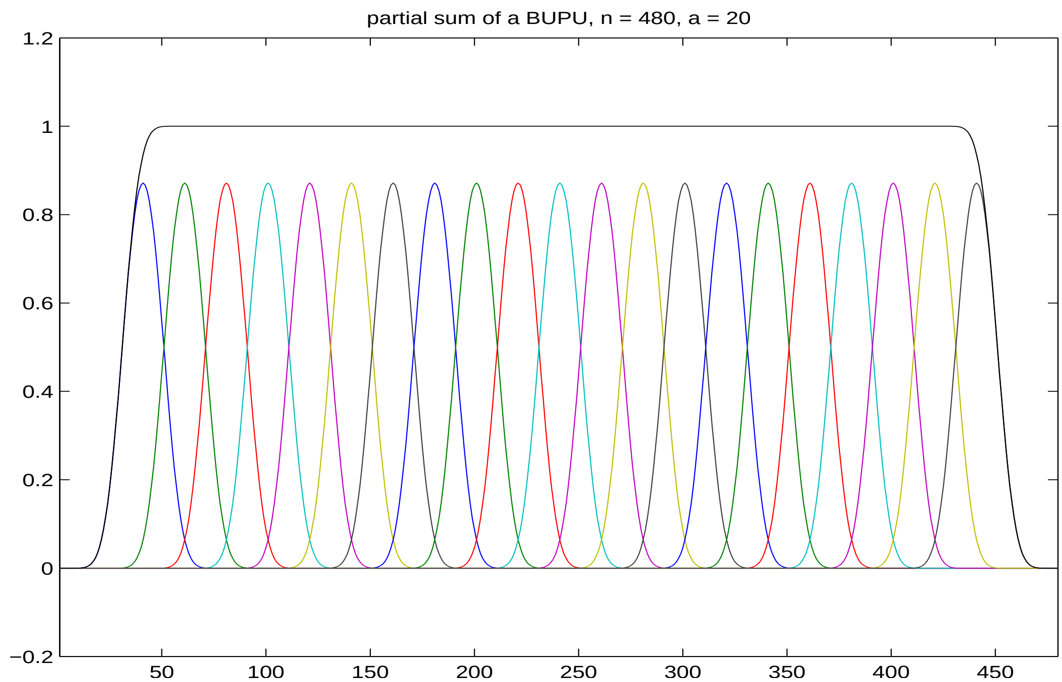


## 2 Operations on the dual space

The fact that $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a pointwise algebra, invariant under translations, allows to define the corresponding *adjoint* operations on the dual space, i.e. we can define

mumult01 **Definition 1.**

$$(\mu \bullet h)(f) := \mu(h \cdot f), \quad h \in \boldsymbol{C}_b(\mathbb{R}^d), f \in \boldsymbol{C}_0(\mathbb{R}^d), \mu \in \boldsymbol{M}_b(\mathbb{R}^d). \tag{3}$$ mumult02

Of course we will just write "$\cdot$" later on instead of "$\bullet$". It is a good exercise to check out what the effect of the adjoint of $T_x$ is on $\boldsymbol{M}_b(\mathbb{R}^d)$.

As we will see in a moment we have the non obvious fact that

$$\lim_{\alpha \to \infty} \|p_\alpha \bullet \mu - \mu\|_{\boldsymbol{M}_b} = 0. \tag{4}$$ muBAU01

From a measure theoretic view-points this can be derived from the $\sigma$-additivity of bounded measures.

In fact, this will be a consequence of an even stronger property using BUPUs (*Bounded Partitions of Unity*).

We need only a simple version of this term for $\mathscr{G} = \mathbb{R}^d$:

DefBUPU01 **Definition 2.** A (ideally non-negative) function $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$ defines a (regular) BUPU (bounded partition of unity) $(\varphi_\lambda)_{\lambda \in \Lambda}$ if one has for some lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}^d$

$$\sum_{\lambda \in \Lambda} \varphi(z - \lambda) := \sum_{\lambda \in \Lambda} \varphi_\lambda(z) = 1, \quad \forall z \in \mathbb{R}^d. \tag{5}$$ regBUPU03

For any such BUPU the family of finite partial sums

$$p_F = \sum_{\lambda \in F} \varphi_\lambda \tag{6}$$ BUPUPS1

elxBUPU1.eps

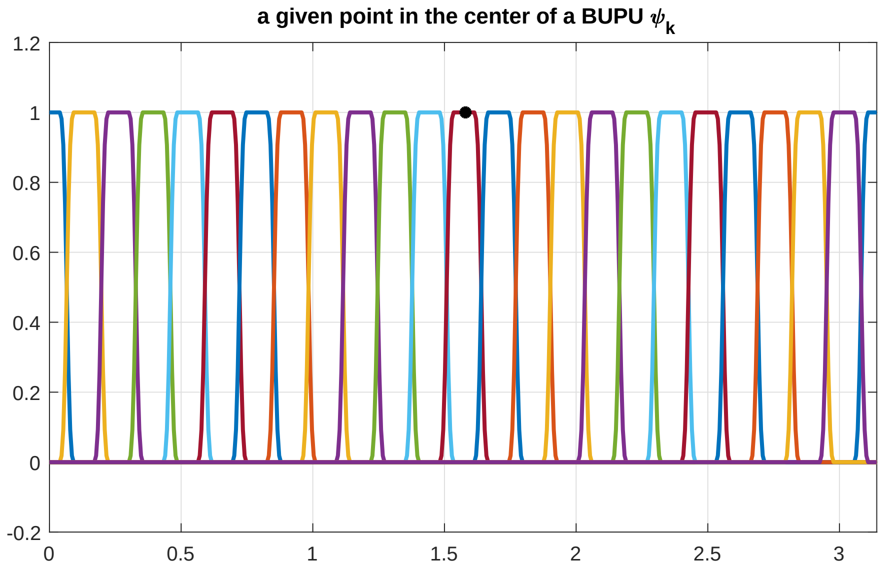


Abbildung 1: delxBUPU1.eps

where $F$ runs through the family of finite subsets of $\Lambda$, forms a BAU!

Again, for the Euclidean case, it is enough to take BUPUs which are obtained from B-splines (so you can think of a collection of triangular functions, or cubic B-splines on the line, and tensor products of such BUPUs for $\mathbb{R}^d$, $d \geq 2$. Note also, that the dilation operators $\mathrm{D}_\rho$, for $\rho \to \infty$ allows to generate “arbitrary fine” BUPUs, i.e. collections as above with $\operatorname{supp}(\varphi) \subseteq B_\delta(0)$, for any given $\delta > 0$.

Consequently we can form the family $(\mu_\lambda)_{\lambda\in\Lambda}$, by putting $\mu_\lambda := \mu \cdot \varphi_\lambda$ and observe that $\mu = \sum_{\lambda\in\Lambda} \mu_\lambda$ (in the $w^*$-sense), but even more is true:

muabsconv04 **Lemma 1.**

- $\sum_{\lambda\in\Lambda} \|\mu_\lambda\|_{\boldsymbol{M}_b} = \|\mu\|_{\boldsymbol{M}_b}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$;
- $\sum_{\lambda\in\Lambda} |\mu(\varphi_\lambda)| \leq \|\mu\|_{\boldsymbol{M}_b}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

This implies *absolute* hence *unconditional* convergence of $\sum_{\lambda\in\Lambda} \mu_\lambda$ in $\boldsymbol{M}_b(\mathbb{R}^d)$.

One can and should consider the subspace of $\boldsymbol{C}_0(\mathbb{R}^d)$ which consists of absolutely convergent decompositions $(f\varphi_\lambda)$. In fact, one can (easily) show that the following definition does NOT depend on the particular choice of the BUPU): $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$

WC0liRddef2 **Definition 3.** $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d) := \{f \in \boldsymbol{C}_0(\mathbb{R}^d) \mid \|f\|_{\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^1)} := \sum_{\lambda\in\Lambda} \|f \cdot \varphi_\lambda\|_\infty < \infty\}$.

Of course the norm on this (proper) subspace of $\boldsymbol{C}_0(\mathbb{R}^d)$, which consists of continuous, absolutely Riemann integrable functions on $\mathbb{R}^d$, are equivalent for different BUPUs.

**Lemma 2.**

- $\mathrm{Sp}_\Psi$ *is a nonexpansive operator*
- *For* $|\Psi| \to 0$ *one has* $\|\mathrm{Sp}_\Psi f - f\|_\infty = 0, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d)$.
- *The operator* $\mathrm{D}_\Psi$ *is the adjoint of the operator* $\mathrm{Sp}_\Psi$.
- *Consequently*

$$w^*\text{-}\lim{}_{|\Psi|\to 0} D_\Psi \mu = \mu.$$

In other words, we have for any $f \in \boldsymbol{C}_0(\mathbb{R}^d)$:

$$\lim_{|\Psi|\to 0} D_\Psi \mu(f) = \lim_{|\Psi|\to 0} \mu(\mathrm{Sp}_\Psi f) = \mu(f).$$

**We have to emphasize that this is a** *constructive* **way of approximating any bounded measure by a (!tight) and bounded net of bounded, discrete measures.** On closer inspection one can see that this is reminiscent of the definition of the Riemann integral (as a limit of Riemannian sums, with respect to trivial BUPUS, using step functions!).

Recall the notion of a FLIP operator:

$$\check{f}(z) = f^{\checkmark}(z) = f(-z).$$

mustarf1 **Definition 4.** Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the *convolution operator* $C_\mu$ by:

$$C_\mu(f)(z) := \mu(T_z f^{\checkmark}). \tag{7}$$ Cmudef1

The *reverse mapping* $R$ recovers the measure $\mu = \mu_T$ from the system $T$ via

$$\mu(f) := T(f^{\checkmark})(0). \tag{8}$$ muTdef1

ization-TLIS **Theorem 2.** *[Characterization of LTISs on $\boldsymbol{C}_0(\mathbb{R}^d)$]*
*There is a natural isometric isomorphism between the Banach space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, endowed with the operator norm, and $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, the dual of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, by means of the following pair of mappings:*

1. *Given a bounded measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the operator $C_\mu$ (to be called convolution operator with convolution kernel $\mu$ later on) via:*

$$C_\mu f(x) = \mu(T_x f^{\checkmark}). \tag{9}$$ conv-measdef

2. *Conversely we define $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ the linear functional $\mu = \mu_T$ by*

$$\mu_T(f) = [T f^{\checkmark}](0). \tag{10}$$ impuls-def

*The claim is that both of these mappings: $C : \mu \mapsto C_\mu$ and the mapping $T \mapsto \mu_T$ are linear, non-expansive, and inverse to each other. Consequently they establish an isometric isomorphism between the two Banach spaces with*

$$\|\mu_T\|_{\boldsymbol{M}} = \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \quad \text{and} \quad \|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|\mu\|_{\boldsymbol{M}}. \tag{11}$$ LTIS-isomet

THIS IS A RATHER IMPORTANT THEOREM, ALLOWING TO DEFINE CONVOLUTION, AND ALSO TO DERIVE THE CONVOLUTION THEOREM, NOT JUST IN AN $\boldsymbol{L}^1$-CONTEXT, BUT FOR GENERAL MEASURES!

The previous characterization allows to introduce in a natural way a Banach algebra structure on $\boldsymbol{M}(\mathbb{R}^d)$. In fact, given $\mu_1$ and $\mu_2$ the translation invariant system $C_{\mu_1} \circ C_{\mu_2}$ is represented by a bounded measure $\mu$. In other words, we can define a new (so-called) *convolution product* $\mu = \mu_1 * \mu_2$ of the two bounded measures such that the relation (completely characterizing the measure $\mu_1 * \mu_2$)

$$C_{\mu_1 * \mu_2} = C_{\mu_1} \circ C_{\mu_2} \tag{12}$$ CONVO1

Of course this observation can be turned into a formal definition of $\mu_1 \star \mu_2$!

It is immediately clear from this definition that $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is a **Banach algebra with respect to convolution**! Associativity is given for free, but *commutativity of the new convolution is not so obvious* (and will follow only later, clearly as a consequence of the commutativity of the underlying group).

The translation operators themselves, i.e. $T_z$ are elements of $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, which correspond exactly to the Dirac measures $\delta_z$, $z \in \mathbb{R}^d$, due to the following simple consideration[1].

$$C_{\delta_x} f(z) = \delta_x(T_z f^{\checkmark}) = [T_z f^{\checkmark}](x) = f^{\checkmark}(x - z) = f(z - x) = [T_x f](z), \tag{13}$$ conv-deltax

and thus $C_{\delta_x} = T_x$, in particular $C_{\delta_0} = T_0 = Id$, resp.

$$f = \delta_0 * f \quad \text{and} \quad T_x f = \delta_x * f, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{14}$$ convDirac0

[1] We start writing $\mu * f$ for $C_\mu(f)$, the convolution of $\mu$ with $f$. This is justified by the fact that in the case of double interpretation, e.g. for $\mu = \mu_g$ for some $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ either as a pointwise defined integral or the action of a measure on test functions one can *easily* check that there is consistency among the different view-points. From now on we will not discuss this issue in all details and leave the verification of details to the reader.

In the *engineering literature* this (trivial) fact is offered as the *sifting property* of the Dirac delta, and written as

$$\int_{\mathbb{R}^d} f(y)\delta(x-y)dy = f(x).$$

DPsiconvappr **Proposition 1.** *For every* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *and* $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ *one has*

$$lim_{|\Psi|\to 0} \|D_\Psi \mu * f - \mu * f\|_\infty = 0. \quad (15)$$ convdiscr1

*Moreover the limit is uniform for (bounded and) equicontinuous sets* $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$*.*

$$\|\sum_{k=1}^{n} c_n \delta_{x_i}\|_{\boldsymbol{M}_b} = \sum_{k=1}^{n} |c_k|. \quad (16)$$ normdiscmeas0

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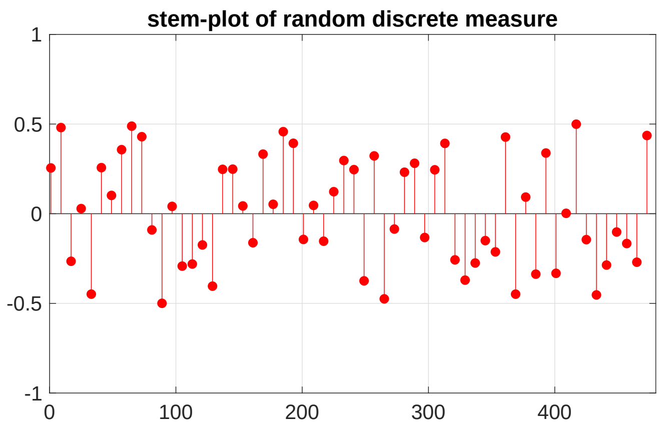


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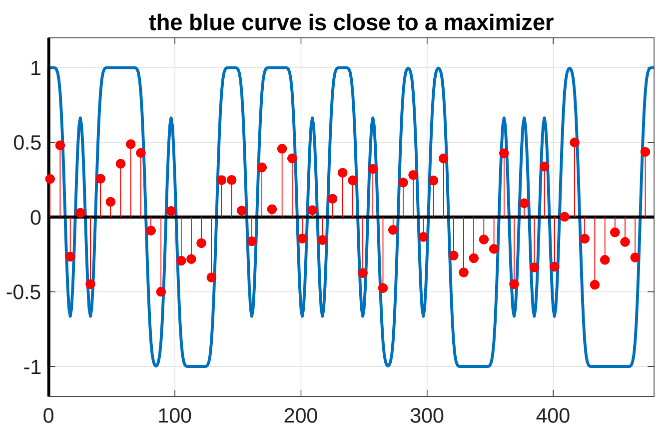


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It can also be shown that one has $w^*\text{-}\lim_{|\Psi|\to 0} D_\Psi \mu_1 \star D_\Psi \mu_2 = \mu_1 \star \mu_2$.

The above observation opens the path towards and action of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (as a Banach algebra with respect to convolution) on any homogeneous Banach space in the sense of Katznelson or any Segal algebra in the sense of Hans Reiter.

defhomBRKatz **Definition 5.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ embedded into the space of locally integrable functions with isometric and strongly continuous translations is called a *homogeneous Banach space* (in the sense of Y. Katznelson).[2] If, in addition, it is continuously embedded into $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and dense there, it is called a *Segal algebra*.

For our purpose it would be more convenient to define such spaces (slightly more general, in fact) as follows: Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into the dual of $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ and has these properties...
For any such space one can show that, given any $f \in \boldsymbol{B}$ and $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ one has: The family

$$D_\Psi \mu \star f := \sum_{i \in I} \mu(\psi_i) \delta_{x_i} \star f = \sum_{i \in I} \mu(\psi_i) T_{x_i} f$$

is relatively compact in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, hence it has a limit in such a Banach space which can be denoted by $\mu \star f$. The classical examples of this type are the spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for $1 \leq p < \infty$, or in fact any reflexive Banach function space (Orlicz spaces, Lorentz spaces etc.) in the sense of Luxemburg-Zaanen.

For us the following situation will be useful (unifying the properties of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$):

homFuncAlg **Definition 6.** A Banach algebra of continuous functions (with respect to pointwise multiplication) is called a *homogeneous Banach algebra* of functions if one has:

1. $\boldsymbol{A} \cap \boldsymbol{C}_c(\mathbb{R}^d) \hookrightarrow (\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) \hookrightarrow \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ as dense subspaces;

2. Translation is isometric on $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, i.e. $\|T_x h\|_{\boldsymbol{A}} = \|h\|_{\boldsymbol{A}}, h \in \boldsymbol{A}, x \in \mathbb{R}^d$;

3. Translation is continuous on $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, i.e.

$$\lim_{x \to 0} \|T_x h - h\|_{\boldsymbol{A}} = 0, \quad \forall h \in \boldsymbol{A}.$$

4. $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ has bounded approximate units;

5. There exists a regular BUPU $(\psi_k)_{k \in \mathbb{Z}^d}$ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, i.e. some $\psi \in \boldsymbol{A} \cap \boldsymbol{C}_c(\mathbb{R}^d)$ such that

$$\sum_{k \in \mathbb{Z}^d} \psi_k(x) = \sum_{k \in \mathbb{Z}^d} \psi(x-k) \equiv 1. \qquad (17)$$ BUPUinAa

As a consequence of 2. and 3. above one has $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{A} \subset \boldsymbol{A}$, we will need

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{A} \subseteq \boldsymbol{A}, \quad \text{with } \|g * f\|_{\boldsymbol{A}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{A}}, \quad g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{A}. \qquad (18)$$ LiConvMod1

Thus $\boldsymbol{M}_b(\mathbb{R}^d) * \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{C}_0(\mathbb{R}^d) \subseteq \boldsymbol{C}_0(\mathbb{R}^d)$, i.e. the action of convolution by bounded measures is well defined, but, does it also preserve the $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$-structure? The answer is provided by the following proposition.

[2] Such structures also appear in many papers on approximation theory.

MbconvFLi **Proposition 2.** *The Banach space* $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ *is a homogeneous Banach space, and any convolution operator* $C_\mu$ *with* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *leaves* $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ *invariant, i.e. we have*

$$\boldsymbol{M}_b(\mathbb{R}^d) * \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \quad \textit{hence } \boldsymbol{L}^1(\mathbb{R}^d) * \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subseteq \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \tag{19}$$ MbconvFLi1

*with the corresponding norm estimate*

$$\|\mu * h\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)} \leq \|\mu\|_{\boldsymbol{M}_b}\|h\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d), h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d). \tag{20}$$ MbconvFLiest

dardspacdef0 **Definition 7.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a Fourier Standard Space (earlier called a *(restricted) standard space* if

1. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (continuous embeddings);
2. $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$, with $\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\mathcal{F}\boldsymbol{L}^1}\|f\|_{\boldsymbol{B}}$ for $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;
3. $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$ with $\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1}\|f\|_{\boldsymbol{B}}$; for $g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$.

tandardonRd1 **Lemma 3.** *Assume that* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is a Banach space of locally integrable functions on* $\mathbb{R}^d$ *such that* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *is contained in* $\boldsymbol{B}$ *as a dense subspace and that* $\|M_\omega T_t f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ *for all* $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*. Then it is a Fourier Standard Space.*

The notion of a *Fourier Standard Space* offers great flexibility and allows a unified treatment of a huge collection of spaces which play a role in Fourier Analysis. It has been the topic of another talk, just recall that the notation is invariant under the (even !fractional) Fourier transform, it includes spaces of Fourier multipliers or convolution kernels, Wiener Amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^p, \boldsymbol{\ell}^q)$ as well as the (unweighted) Modulation Spaces $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$, or more abstract Generalized Modulation Spaces as they are discussed in a recent joint paper with Stevan Pilipovic and Bojan Prangovski (modulation spaces derived from tensor products of $\boldsymbol{L}^p$-spaces, for example).

One important point in the discussion if these spaces, that there is a smallest among them, namely $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d)$, also known as $\boldsymbol{S}_0(\mathbb{R}^d)$ or modulation space $\boldsymbol{M}^{1,1}_0(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d)$. It can also be seen as $\boldsymbol{\mathcal{C}o}(\boldsymbol{L}^1)$ with respect to the Schrödinger representation of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a homogeneous Banach-space with respect to some "abstract" group action $\rho$, i.e. we assume that $x \to \rho(x)f$ is continuous from $\mathscr{G}$ into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, isometric in the sense that $\|\rho(x)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ for all $x \in \mathscr{G}$ and $f \in \boldsymbol{B}$, and that $\rho(x_1x_2) = \rho(x_1)\rho(x_2)$. Of course we can define $\mu \bullet_\rho f$ for any discrete measure, hence for the family $D_\Psi\mu$ as the (unique) limit of this Cauchy net in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then one has

$$\|\mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}}\|f\|_{\boldsymbol{B}}, \quad \text{for all } f \in \boldsymbol{B}. \tag{21}$$ meas-onB1

In fact, $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ becomes a Banach module over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ in this way.

We are going to introduce $\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)$ resp. $\big(\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1), \|\cdot\|_{\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)}\big)$ in the following way.

The two prototypical examples (which we want to describe in a unified way) are given by $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) = \big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ or $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) = \big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$. The resulting space will be called Wiener algebra, denoted by $\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^1)(\mathbb{R}^d)$, and $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1,\boldsymbol{\ell}^1)(\mathbb{R}^d)$ for the second case, also known as *modulation space* $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$ or as **Segal algebra** $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, also called Feichtinger algebra in the literature.

WAlidef00 **Definition 8.** Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a homogeneous Banach algebra as defined in (6). Then we set

$$\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)(\mathbb{R}^d) := \left\{ f \in \boldsymbol{A} \,|\, \sum_{k\in\mathbb{Z}^d} \|f\psi_k\|_{\boldsymbol{A}} < \infty \right\}. \tag{22}$$ WAlidef

It is clear that the proposed expression actually defines a norm on $\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)(\mathbb{R}^d)$ and that the embedding of $\big(\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1), \|\cdot\|_{\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)}\big)$ into $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a continuous one. In the proof of the various statements we will make use of the following technical result (which also implies the equivalence of the norms introduced by different BUPUs).

As a preparation for the relevant estimates let us provide an alternative, so-called *atomic characterization* of $\boldsymbol{W}(\boldsymbol{A},\boldsymbol{L}^1)(\mathbb{R}^d)$.

WAliatomic **Proposition 3.** *For any fixed $R > 0$ the following characterization can be given: A function $h \in \boldsymbol{A}$ belongs to $\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)$ if and only if $h$ can be written as an absolutely convergent sum of so-called atoms: there exists a sequence $(h_k)_{k=1}^\infty$ with $\sum_{k=1}^\infty \|h_k\|_{\boldsymbol{A}} < \infty$ and $\mathrm{supp}(h_k) \subseteq B_R(x_k)$ for some sequence of points $(x_k)_{k=1}^\infty$. Any such representation of $h \in \boldsymbol{A}$ is called an admissible atomic representation of $h = \sum_{k=1}^\infty h_k$.*

MbWali **Proposition 4.** *There exists $C > 0$ (depending only on the BUPU etc.) such that:*

$$\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1) \subseteq \boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1), \quad \textit{with} \quad \|\mu * f\|_{\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)} \leq C\|\mu\|_{\boldsymbol{M}_b}\|f\|_{\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)}. \tag{23}$$ MbWAliest

Very similar methods can be applied, if the corresponding translation and modulation operators are not isometric anymore, but just bounded.

Clearly in this case Beurling weights come into the game, which can be assumed to be positive and continuous, *submultiplicative* functions on $\mathscr{G}$ or $\widehat{\mathscr{G}}$ respectively.

The minimal invariant space, satisfying

$$\|T_x f\|_{\boldsymbol{B}} \leq w_1(x)\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B} \tag{24}$$ transvarw1

and

$$\|M_\xi f\|_{\boldsymbol{B}} \leq w_2(\xi)\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B} \tag{25}$$ modvarw2

turns out to be $\boldsymbol{W}(\mathcal{F}(\boldsymbol{L}^1_{w_2}), \boldsymbol{L}^1_{w_1})(\mathbb{R}^d)$!

If the invariance is described by means of TF-shifts, i.e. via the assumption

$$\|\pi(x,\xi)f\|_{\boldsymbol{B}} = \|M_\xi T_x f\|_{\boldsymbol{B}} \leq \langle (x,\xi)\rangle^s = (1 + |x|^2 + |\xi|)^{s/2} \tag{26}$$ TFvarvs

it turns out that the modulation spaces $\boldsymbol{M}^1_{v_s}(\mathbb{R}^d) = \boldsymbol{\mathcal{C}o}(\boldsymbol{L}^1_{v_s})$ are the minimal spaces. They play an important role in the discussion of coorbit spaces (hence for general modulation spaces).

Using the BUPUs we can define two operators, one on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ and the other on the dual space, $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. We use the abstract notation $\Psi = (\psi_i)_{i\in I}$:

QuasiInt02 **Definition 9.** $\mathrm{Sp}_\Psi f := \sum_{i\in I} f(x_i)\psi_i, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d)$. QUASI-INTERPOLATION

histoPsi **Definition 10.** $D_\Psi \mu := \sum_{i\in I} \mu(\psi_i)\delta_{x_i}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. DISCRETIZATION

dztofive.pdf

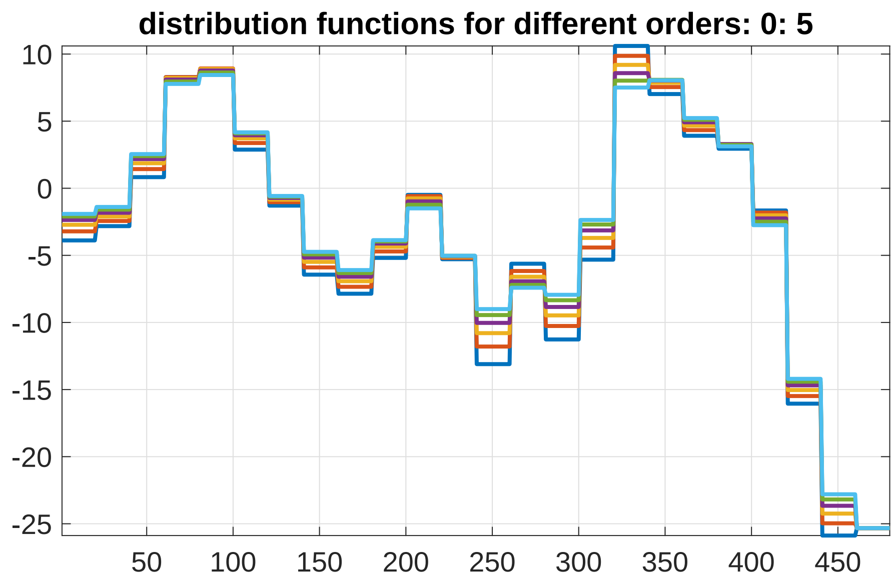


Abbildung 4: DPsimuordztofive.pdf:
Distribution function obtained by B-spline BUPUs of different order: the higher the order, the more smearing effect is obtained, because the supports of the B-splines are increasing with the order.

UUUUU

# 3 DPsi Demo

Psimu01b.pdf

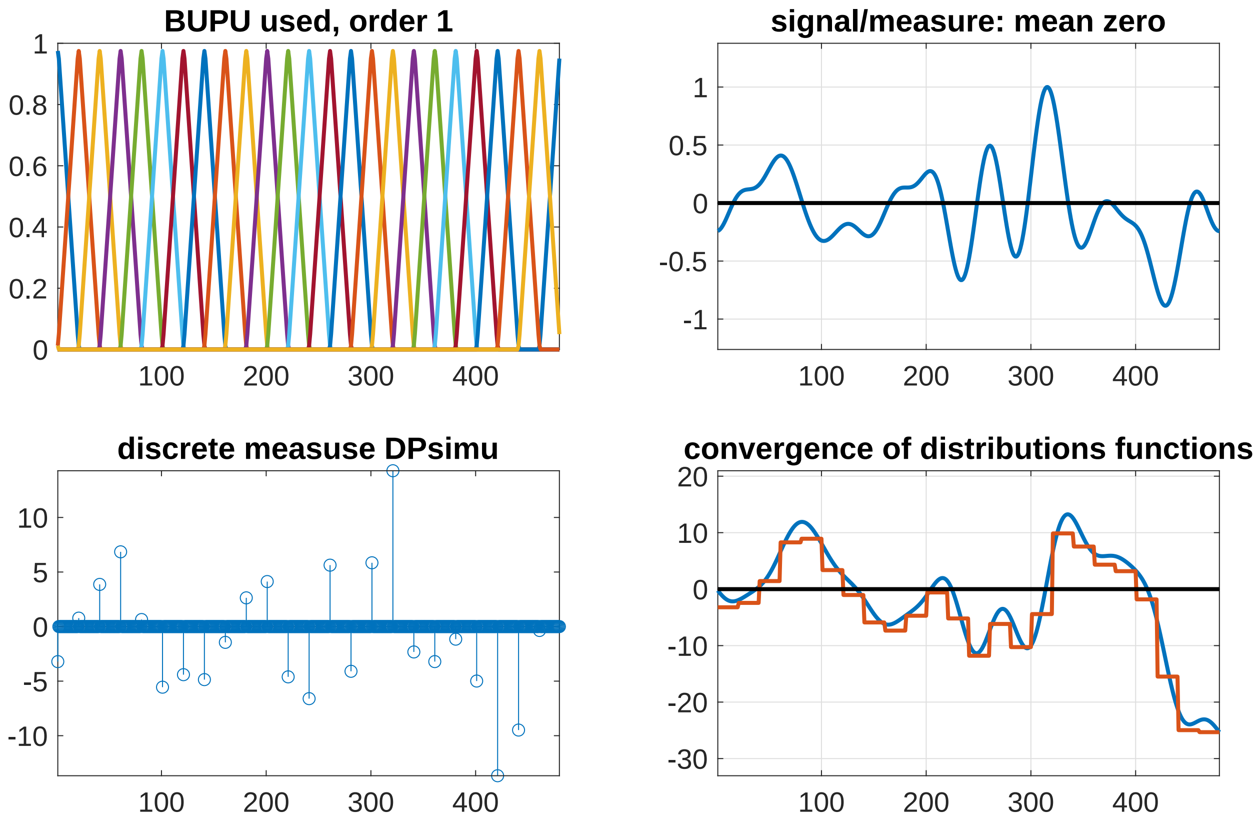


Abbildung 5: convDPsimu01b.pdf:
The (triangular) BUPU, the smooth signal/measure, the discretized measure illustrate by the STEM-command, and the corresponding distribution function, which has a jump of sice $c_k$ at the position of the corresponding (positive or negative) Dirac measure.

Using the BUPUs we can define two operators, one on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ and the other on the dual space, $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. We use the abstract notation $\Psi = (\psi_i)_{i\in I}$:

QuasiInt02 **Definition 11.** $\mathrm{Sp}_\Psi f := \sum_{i\in I} f(x_i)\psi_i, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d)$. QUASI-INTERPOLATION

histoPsi **Definition 12.** $D_\Psi \mu := \sum_{i\in I} \mu(\psi_i)\delta_{x_i}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. DISCRETIZATION vargap1c.pdf

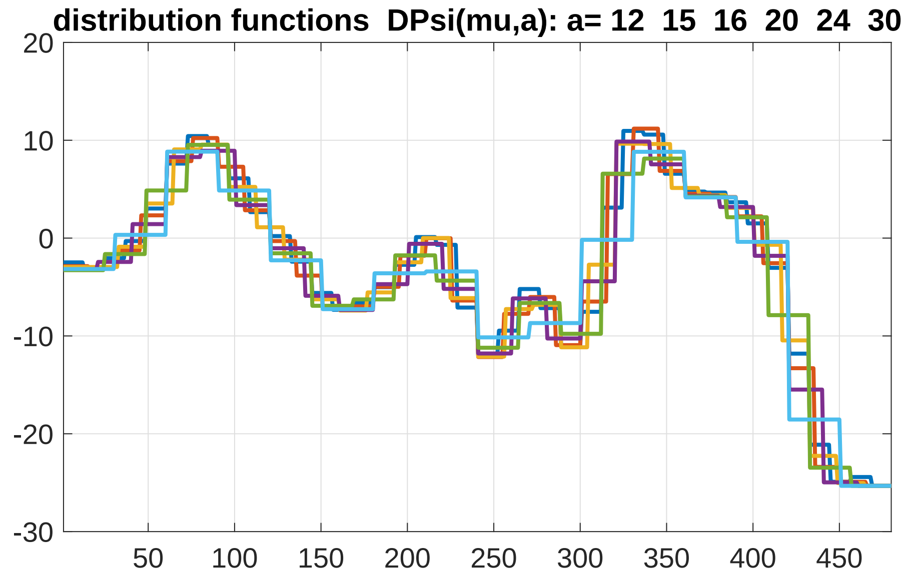


Abbildung 6: DPsimuvargap1c.pdf:
Distribution function of discrete measures, for finer and finer partitions of unity, so that the approximation of the resulting step-functions are better and better!

DiracWAli **Lemma 4.** *Assume that one has for any* $f \in \boldsymbol{A}$ *and* $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ *with* $\hat{g}(0) = 1$

$$\|\mathrm{St}_\rho g * f - f\|_{\boldsymbol{A}} \to 0 \text{ for } \rho \to 0, \tag{27}$$ DiracA1

*then the same relationship also holds for* $\boldsymbol{W}(\boldsymbol{A}, \boldsymbol{\ell}^1)(\mathbb{R}^d)$, *i.e.*

$$\lim_{\rho \to 0} \|\mathrm{St}_\rho g * f - f\|_{\boldsymbol{W}(\boldsymbol{A}, \boldsymbol{\ell}^1)} = 0. \tag{28}$$ StrhoWAli

*In particular, the band-limited elements are dense in* $\boldsymbol{W}(\boldsymbol{A}, \boldsymbol{\ell}^1)(\mathbb{R}^d)$.

DcombSOP **Lemma 5.** *For any* $\Lambda \lhd \mathbb{R}^d$ *the Dirac comb* $\delta_\Lambda := ⊔⊔_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ *belongs to* $\boldsymbol{W}(\mathbb{R}^d)^*$, *the dual of Wiener's algebra* $\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big)$, *hence the linear mapping*

$$f \mapsto \sum_{\lambda \in \Lambda} f(\lambda) \tag{29}$$ Dcombdef

*defines a mild distribution, i.e.* $⊔⊔_\Lambda \in \boldsymbol{S}_0'$.

# Literatur

dztofive.pdf

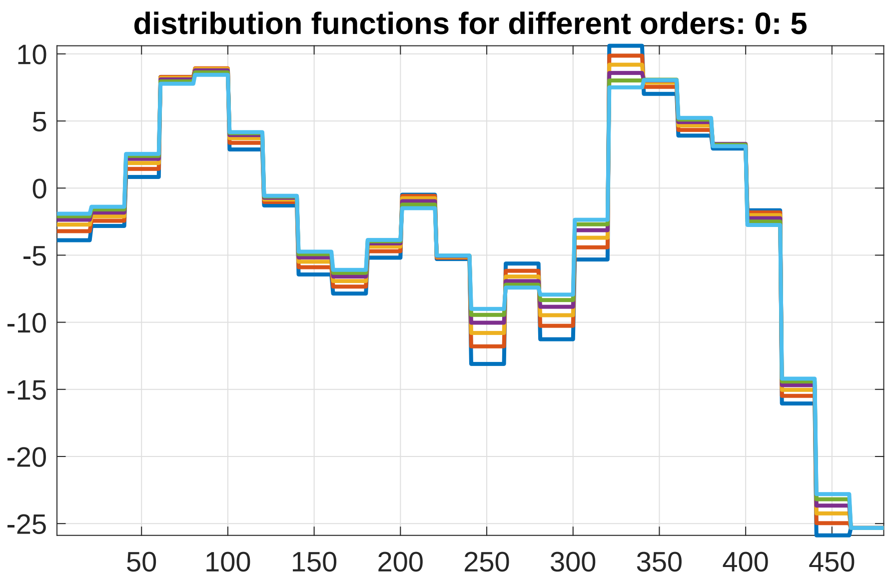


Abbildung 7: DPsimuordztofive.pdf:
Distribution function obtained by B-spline BUPUs of different order: the higher the order, the more smearing effect is obtained, because the supports of the B-splines are increasing with the order.