NuHAG Numerical Harmonic Analysis Group

Invariant function spaces as double modules illustrated via Fourier Standard Spaces

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The Double Module Diagram

The key object of this talks is a certain diagram which can be associated to most of the usual function spaces arising in Fourier analysis over LCA groups.
For the sake of convenience we restrict our attention here to the choice $G = \mathbb{R}^d$ and the realm of *tempered distributions*, as a widely known setup. The general results are valid for Banach spaces of *ultra-distributions* over LCA groups.
For each such Banach space we will assign a collection of subspaces, all "with the same norm", and the diagram will express how they are inter-connected by inclusions (as closed subspaces, from low to high in the diagram).

The MAIN DIAGRAM

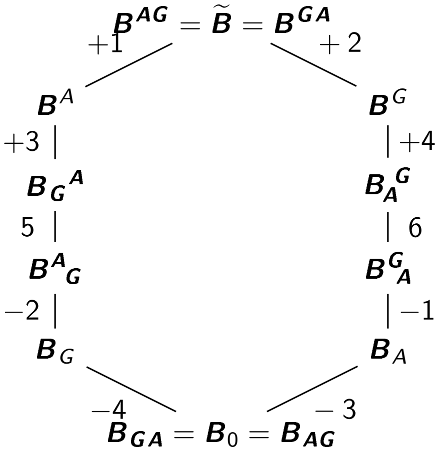

Figure: The MAIN DIAGRAM for double modules

I will restrict my attention here (for convenience) to the commutative case, and the to LCA group $G = \mathbb{R}^d$, but many of the claims made below are valid for much more general situations. Recall that a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is nothing else but a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ combined with a distinguished Banach algebra of operators acting on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. The trivial case is the algebra of scalar multiples of the $\mathrm{Id}_{\boldsymbol{B}}$, which corresponds to the ordinary concept of Banach spaces. We also could view $\mathbb{C}^n$ as a Banach module over $\mathcal{M}_{n,n}(\mathbb{C})$, and we would be back to linear algebra and matrix analysis.
For us Banach algebras which do not have a unit, but which have a *bounded approximate unit* will be the most interesting ones. The prototype is $(\boldsymbol{c}_0, \|\cdot\|_\infty)$ (the space of null-sequences), an ideal inside of $(\ell^\infty, \|\cdot\|_\infty)$, or in the continuous case $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ inside of $(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$. Note that the bigger space is in each case just the space of pointwise multipliers on the smaller one (a

closed ideal), resp. the space of all bounded linear operators on the small one, which commutes with pointwise multiplication (and hence has to be a pointwise multiplier, as a matter of fact)!
In the context of harmonic analysis we have of course $(\boldsymbol{L}^1(G), \|\cdot\|_1)$, which is a proper ideal in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ as long as G is not discrete. Let us therefore consider mostly again $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ as the typical case to learn from. Again, we obtain $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ characterizes the space of translation invariant operators (again as convolution operators) on $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$, according to *Wendel's Theorem* (see the book of R. Larsen on Multipliers).
Any space $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, with $1 \leq p < \infty$, or $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ (as a replacement for $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$), or $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ (the Fourier algebra), or Segal algebras such as $\boldsymbol{L}^1 \cap \boldsymbol{L}^p(\mathbb{R}^d)$ of $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ and of course $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ are so-called homogeneous Banach spaces (in the sense of Y. Katznelson's book of 1968).

Let us shortly recall how one proves (usually!) that $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is a Banach module over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ with respect to convolution, i.e. mostly the norm estimate:

$$\|g \star f\|_p \leq \|g\|_1 \|f\|_p, \quad g \in \boldsymbol{L}^1, f \in \boldsymbol{L}^p. \tag{1}$$

On makes sure that the convolution product exists pointwise almost everywhere, and that - by a duality argument, involving the fact that one has explicit knowledge about the dual space, which is $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$, with $1/p + 1/q = 1$, and that $\boldsymbol{C}_c(\mathbb{R}^d)$ is dense (for $q < \infty$) allows to derive the estimate (1).
In contrast, I have pointed out that my alternative approach to $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}) = \left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$ allows to make use of discretization of measures and (by viewing $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as a closed subspace of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}))$ to show, that we have

$$\lim_{|\Psi| \to 0} \|D_\Psi \mu * f - \mu * f\|_\infty = 0, \quad \forall y \in \mathbb{R}^d. \tag{2}$$

Here we have to choose $|\Psi|$ small enough (i.e. a "fine BUPU") and determine the coefficients of the discrete measures δ_{x_i}, so that for $\mu = \mu_g$ we have coefficients

$$c_i = \int_{\mathbb{R}^d} \psi_i(y) g(y) dy, \quad i \in I,$$

hence

$$D_\Psi \mu \star f = \sum_{i \in I} c_i T_{x_i} f, \text{ with } \|D_\Psi \mu \star f\|_p \leq \sum_{i \in I} |c_i| \|f\|_p \leq \|g\|_1 \|f\|_p.$$

But in fact we have more do not only have convergence in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, but rather norm-convergence in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for $f \in \boldsymbol{L}^p(\mathbb{R}^d)$, because translation is isometric and strongly continuous in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for $1 \leq p < \infty$.

Recall that a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called an *essential* Banach module if the linear span of $\boldsymbol{A} \cdot \boldsymbol{B}$ coincides with $\boldsymbol{B}$. If $\boldsymbol{A}$ has bounded approximate units this is equivalent (due to the Cohen-Hewitt Theorem) to the fact that $\boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{B}$, and also to the claim that one has for any bd. appr. unit $(e_\alpha)_{\alpha \in I}$ one has

$$\lim_{\alpha \to \infty} e_\alpha \cdot f = f, \quad \forall f \in \boldsymbol{B}.$$

Since translation is continuous for a homogeneous Banach spaces (by assumption) one has the expected action of Dirac sequences, and hence they are *essential* $\boldsymbol{L}^1$-modules with respect to convolution.

1. $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a Banach module over $\left(M_b(\mathbb{R}^d), \|\cdot\|_{M_b}\right)$ and hence over $\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)$ with respect to convolution. In fact, it is an essential BM.
2. $\left(M_b(\mathbb{R}^d), \|\cdot\|_{M_b}\right)$ is an essential module over $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ with respect to pointwise multiplication, hence also $\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)$.
3. $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is a essential Banach module over $\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)$ with respect to convolution.
4. $\left(\mathcal{F}L^p(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}L^p}\right)$ is an essential Banach module over $\left(\mathcal{F}L^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}L^1}\right)$ with respect to pointwise multiplication over.

Similar properties hold true for Wiener amalgam spaces, modulation spaces, or the space of Fourier multipliers for $\boldsymbol{L}^p(\mathbb{R}^d)$, which is the space of pointwise multipliers of $\left(\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^p}\right)$, or its (inverse) Fourier version, the set of all convolution kernels within $\boldsymbol{S}_0'(\mathbb{R}^d)$ which define bounded convolution operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ (endowed with the norm $\|\cdot\|_p$).

These results rely on the representation of $\mathbb{R}^d$ by translation operators $T_x, x \in \mathbb{R}^d$, while the pointwise action of $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ represents the outcome of the abstract approach by choosing $\rho(s) = M_s, s \in \mathbb{R}^d$, the *modulation operators*.

We leave it to the reader to replace $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ by a Beurling algebra $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$, in case of non-isometric action of translation operators, e.g. on $\left(\boldsymbol{L}^p_w(\mathbb{R}^d), \|\cdot\|_{p,w}\right)$, and similarly to replace the Fourier algebra $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ by a *Fourier-Beurling* algebra $\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d)$ in the sequel, but there are no important changes!

Next we talk about spaces having two (i.e. two, only *"somewhat"*) compatible module structures. They are NOT so-called *bi-modules* but still, for the most important cases, of the convolution algebra action by $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and the pointwise multiplicative structure using $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$, their *infinitesimal generators* are the time- resp. frequency shifts, which satisfy the canonical commutation relation!

Let us recall the situation: Assume that we start from a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is given, which has bounded approximate units. The we can form two objects, namely the essential part $\boldsymbol{B}_{\boldsymbol{A}} := \boldsymbol{A} \cdot \boldsymbol{B}$ and the $\boldsymbol{A}$-completion, which is given as

$$\boldsymbol{B}^{\boldsymbol{A}} := \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{B}) := \{T : \boldsymbol{A} \to \boldsymbol{B}, T(ab) = aT(b), \ a \in \boldsymbol{A}, b \in \boldsymbol{B}\}.$$

The statements coming now are all described in the following paper of 1983:

[brfe83] W. Braun and H. G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

Together we have the following chain of closed subspaces (usually even isometric embeddings!):

$$\boldsymbol{B_A} \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{B^A}. \tag{3}$$

These operations are like the kernel and the hull of a nice set, say the unit ball in $\mathbb{R}^n$. Accordingly we have simple rules such as

$$\boldsymbol{B^A}_{\boldsymbol{A}} = \boldsymbol{B_A}, \quad \boldsymbol{B_A}^{\boldsymbol{A}} = \boldsymbol{B^A}.,$$

and also the almost trivial (expected) results

$$(\boldsymbol{B_A})_{\boldsymbol{A}} = \boldsymbol{B_A}, \quad (\boldsymbol{B^A})^{\boldsymbol{A}} = \boldsymbol{B^A}.$$

For $\boldsymbol{A} = \boldsymbol{c}_0$ this reduces to

$$\boldsymbol{c}_{0\,\boldsymbol{c}_0} = \boldsymbol{c}_0 \hookrightarrow \ell^\infty$$

or for the Banach algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ we have

$$\boldsymbol{L}^1(\mathbb{R}^d)_{\boldsymbol{L}^1(\mathbb{R}^d)} = \boldsymbol{L}^1(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}_b(\mathbb{R}^d).$$

The situation is more interesting if we consider $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ as a Banach module over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$. Then we have

$$\boldsymbol{L}^\infty(\mathbb{R}^d)_{\boldsymbol{L}^1} = \boldsymbol{C}_{ub}(\mathbb{R}^d) \subset \boldsymbol{L}^\infty(\mathbb{R}^d) = \boldsymbol{L}^\infty(\mathbb{R}^d)^{\boldsymbol{L}^1}.$$

So $\boldsymbol{L}^\infty(\mathbb{R}^d)$ is not essential, but it is complete (which is in fact typical for a dual space).
But since we have for e.g. so-called Fourier Standard Spaces (!there are a lot of them) two module operations we have to choose two symbols.

We decided to use the symbol $\boldsymbol{A}$ for $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$ with *pointwise multiplication* and to write $\boldsymbol{G}$ for the action of the group, namely the *integrated group action*, of $\boldsymbol{L}^1(G)$ via convolution.
Thus instead of $\boldsymbol{B}_{\boldsymbol{L}^1}$ we will write $\boldsymbol{B}_{\boldsymbol{G}}$, and for $\boldsymbol{B}^{\boldsymbol{L}^1} := \mathcal{H}_{\boldsymbol{G}}(\boldsymbol{L}^1, \boldsymbol{B})$ we write simply $\boldsymbol{B}^{\boldsymbol{G}}$.
Due to the iteration principles (the same symbol is applied in any order, but only the last version "survives") we only have to look at the mixed terms, i.e. symbols using $\boldsymbol{A}$ and $\boldsymbol{G}$ alternating (in an upper or lower position).
The overall finding can be expressed by suitable diagrams!

One has to read this diagram by the following rules:
Given any Fourier Standard Space (or more generally a Translation and Modulation Invariant Banach space of (ultra-) distributions) one can form a collection of closed subspaces, based on the two (non-commuting) module structures (which can be unified to *one* such structure making use of the *Schrödinger representation* of the *reduced Heisenberg group*, but this is outside of the scope of this talk.
We rather concentrate on the two module structures developed in [1]. Similar ideas appear in [3] (tempered distributions) and [2]. Abstract version of the results presented have been developed in connection with the compactness criterion in the spirit of Kolmogorov and Riesz in [4].

Banach Algebras and Banach Modules

The presentation will be on *Banach modules* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with respect to different *Banach algebras* $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.
In other words, we call a given Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ a (left/right) Banach module over $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with respect to some action denoted by "$\bullet$" if there is a bilinear operation $(a, b) \mapsto a \bullet b$ which corresponds (essentially) to an embedding of $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ into the operator algebra$\mathcal{L}(\boldsymbol{B})$ of bounded, linear operators. In particular we request

$$a \bullet (a' \bullet b) = (a \cdot a') \bullet b, \quad a, a' \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

Such a Banach module is called *essential* if the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ coincides with all of $\boldsymbol{B}$.

Pointwise Multiplication and Convolution

For us the two prototypical (!commutative) Banach algebras are

1. $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ with respect to pointwise multiplication;
2. $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ with respect to convolution.

In order to make the situation more symmetric with respect to the Fourier transform one can also consider, instead of $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$, the *Fourier algebra* $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$, with respect to the norm

$$\|\widehat{f}\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)} = \|f\|_{\boldsymbol{L}^1(\mathbb{R}^d)}, \; f \in \boldsymbol{L}^1(\mathbb{R}^d).$$

For both $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ and $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ the *Gelfand space* is just $\mathbb{R}^d$ with the usual topology. Moreover, in both cases we do not have a unit element, but a bounded approximate unit (i.e. a bounded family $(e_\alpha)_{\alpha\in I}$ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with

$$\lim_{\alpha\to 0} \|e_\alpha \cdot a - a\|_{\boldsymbol{A}} = 0, \quad \forall a \in \boldsymbol{A}.$$

Pointwise Multiplication and Convolution II

Typical examples of the family of Banach spaces we want to study are the spaces $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, with $1 \leq p \leq \infty$. **They are BOTH Banach modules over $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ with respect to convolution and over $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ under pointwise multiplication.**
Moreover, the essential part $\boldsymbol{B}_\rceil = \boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^\infty(\mathbb{R}^d) = \boldsymbol{C}_{ub}(\mathbb{R}^d)$, the space of uniformly continuous, bounded functions (closed subalgebra in $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$!). The essential part of $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ with respect to the pointwise action of $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ is the space of functions "vanishing at infinity". The intersection of these two subspaces is just $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$!
It is also obvious that they are essential Banach modules if and only if $p < \infty$! On the other hand, these space are dual Banach spaces if and only of $1 < p \leq \infty$.

Pointwise Multiplication and Convolution III

Taking the essential part of such a Banach algebra (closed linear span if $\boldsymbol{A} \bullet \boldsymbol{B}$ inside $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$) is in fact the same as just taking the set $\boldsymbol{A} \bullet \boldsymbol{B}$, is a consequence of the *Cohen-Hewitt factorization Theorem* for Banach modules over Banach algebras with BAIs!
We will write $\boldsymbol{B}_{\boldsymbol{A}}$ for this essential part in the sequel, and observe that of course $(\boldsymbol{B}_{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}$, and also we will write (for simplicity of symbols) $\boldsymbol{B}_{\boldsymbol{G}}$ instead of $\boldsymbol{B}_{\boldsymbol{L}^1(G)}$.

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^p(\mathbb{R}^d) \subseteq \boldsymbol{L}^p(\mathbb{R}^d),$$

the convolution inequality together with the norm inequality

$$\|g * f\|_{\boldsymbol{L}^p} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{L}^p}, g \in \boldsymbol{A} = \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B} = \boldsymbol{L}^p(\mathbb{R}^d)$$

results from the fact that translation is isometric on $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ and continuous, in the sense of

$$\lim_{x \to 0} \|T_x f - f\|_{\boldsymbol{L}^p(\mathbb{R}^d)} = 0, \quad \forall f \in \boldsymbol{L}^p(\mathbb{R}^d),$$

The Banach module completion

While taking the "essential part" with respect to a given Banach algebra is an operation comparable to taking the *interior* of a nice set (say a ball), there is a complementary operation, the so-called module completion: $\boldsymbol{B} \mapsto \boldsymbol{B}^{\boldsymbol{A}}$.
Let us start with examples, and recall the concept of a *Banach module homomorphism*, which we denote by $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$.

Definition

A bounded, linear mapping between two Banach $\boldsymbol{A}$-modules $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ is called a *Banach module homomorphism* if

$$T(a \bullet_1 b) = a \bullet_2 T(a), \quad a \in \boldsymbol{A}, b \in \boldsymbol{B}^1.$$

Pointwise multipliers

It is not hard to found out that the $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ module homomorphism of $\boldsymbol{B}^1 = \boldsymbol{C}_0(\mathbb{R}^d) = \boldsymbol{B}^2$ are just pointwise multipliers, thus

$$\boldsymbol{H}_{\boldsymbol{C}_0}(\boldsymbol{C}_0(\mathbb{R}^d), \boldsymbol{C}_0(\mathbb{R}^d)) = \boldsymbol{C}_b(\mathbb{R}^d),$$

in fact in the sense of an isometric isomorphism (the operator norm of $M_h : f \to h \cdot f$ coincides withe $\|h\|_\infty$ for any $h \in \left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$.
In a similar way we can enlarge $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ to $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, by observing that

$$\boldsymbol{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1(\mathbb{R}^d), \boldsymbol{L}^1(\mathbb{R}^d)) \approx (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$$

according to *Wendel's Theorem*.

Module Completion II

The abstract definition of $\boldsymbol{B^A}$ is via

Definition

$$\boldsymbol{B^A} := \boldsymbol{H_A}(\boldsymbol{A}, \boldsymbol{B}).$$

It is easy to check that one has

$$(\boldsymbol{B^A})^{\boldsymbol{A}} = \boldsymbol{B^A}, (\boldsymbol{B^A})_{\boldsymbol{A}} = \boldsymbol{B_A}, (\boldsymbol{B_A})^{\boldsymbol{A}} = \boldsymbol{B^A}, (\boldsymbol{B_A})_{\boldsymbol{A}} = \boldsymbol{B_A}.$$

Note that we can replace any Banach algebra that we have by a smaller Banach algebra, as long as it is dense and still has bounded approximate units of its own. So we could replace $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ by *Beurling algebras* $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$, and we will substitute $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ by the *Fourier algebra* $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$! We could use *Beurling-Fourier algebras* $(\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d)})$!

Non-Commutativity

So far we have two commutative Banach algebras (without units, but with BAIs, both embedded into some larger Banach algebra with unit, namely $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and $\left(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}\right)$ resp.).
BUT THESE TWO MODULE ACTIONS do NOT commute!

$$h \cdot (g * f) \neq g * (h \cdot f), \quad g \in \boldsymbol{L}^1(\mathbb{R}^d), h \in \boldsymbol{C}_0(\mathbb{R}^d)!$$

But since convolution results from translation and multiplication by elements of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ results from translation and modulation

$$[T_z f](x) = f(x - z), \quad x, z \in \mathbb{R}^d,$$

$$[M_s f](x) = e^{2\pi i \langle s, x \rangle} f(x), \quad x, s \in \mathbb{R}^d,$$

and translation and modulation operators commute up to *phase factors* there is some hope that the two module structures show some compatibility.

Iterated constructions

For the rest of this section we will consider the very concrete situation of Banach spaces of tempered distributions with a double module structure, which I like to call **Fourier Standard Spaces**. What is important to know is the following: In the situation describe above each of the new spaces allows to apply one of the discussed operations again.

Hence for these space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ not only the symbols

$$\boldsymbol{B}_{\boldsymbol{A}}, \boldsymbol{B}_{\boldsymbol{G}}, \boldsymbol{B}_{\boldsymbol{AG}}, \boldsymbol{B}_{\boldsymbol{GA}}, \boldsymbol{B}^{\boldsymbol{A}}, \boldsymbol{B}^{\mathbf{G}}, \boldsymbol{B}^{\boldsymbol{AG}}, \boldsymbol{B}^{\boldsymbol{GA}}$$

are well defined, but one can think, in addition to the mixed symbols

$$\boldsymbol{B}_{\boldsymbol{A}}^{\boldsymbol{G}}, \boldsymbol{B}_{\boldsymbol{G}}{}^{\boldsymbol{A}}, \boldsymbol{B}^{\boldsymbol{G}}{}_{\boldsymbol{A}}, \boldsymbol{B}^{\boldsymbol{A}}{}_{\boldsymbol{G}}$$

and longer chains, such as

$$(((\boldsymbol{B}_{\boldsymbol{A}})^{\boldsymbol{G}})_{\boldsymbol{A}})_{\boldsymbol{G}}.$$

The basic rules

1. Any higher order chain of symbols can be replaced by a chain of order two, which is given by the appearance of the last two symbols of type $\boldsymbol{A}$ and $\boldsymbol{G}$ respectively;
2. $\boldsymbol{B}^{\boldsymbol{AG}} = \boldsymbol{B}^{\boldsymbol{GA}} = \widetilde{\boldsymbol{B}}$, relative completion of $\boldsymbol{B}$;
3. $\boldsymbol{B}_{\boldsymbol{AG}} = \boldsymbol{B}_{\boldsymbol{GA}} = \boldsymbol{B}_0$, the closure of $\mathcal{S}(\mathbb{R}^d)$ in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
4. The main diagram shows a chain of inclusions, labelled by the numbers $1, 2, 3, 4, 5, 6$. Equality appears at $+n$ if and only if appears at $-n$, $n \in \{1, 2, 3, 4\}$.
5. Minimal spaces show equality at $3, 4$;
6. Dual spaces are maximal and show equality at $1, 2$;

Fourier Standard Spaces, the Idea

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, sandwiched between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions, i.e. with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \tag{4}$$

is called a **Fourier Standard Space** on $\mathbb{R}^d$ (or a **FouSS**) if it has a double module structure over $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution and over the (Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d))$ with respect to pointwise multiplication.

Essentially we require that in addition to (4) one has:

$$\boldsymbol{L}^1 * \boldsymbol{B} \subseteq \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1 \cdot \boldsymbol{B} \subseteq \boldsymbol{B}. \tag{5}$$

Minimal Fourier Standard Spaces

Typically the situation arises in the following context:
Assume in addition to the sandwiching property, that $\mathcal{S}(\mathbb{R}^d)$ is a *dense subspace* of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and that translation is isometric on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ as well as modulation, i.e.

$$\|M_y T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall f \in \boldsymbol{B}, x, y \in \mathbb{R}^d. \tag{6}$$

Then $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a (minimal) Fourier standard space. The dual space of such a minimal standard space is also a FouSS (in fact a maximal one).
Methods from time-frequency analysis allow to show that there is a smallest and a largest member in this family, namely the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (also known as modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$), and its dual, $\boldsymbol{S}_0'(\mathbb{R}^d)$ (or $\boldsymbol{M}^{\infty,\infty}(\mathbb{R}^d)$).

Constructions within the FSS Family

1. Taking **Fourier transforms**;
2. Conditional dual spaces, i.e. the **dual space** of the closure of $S_0(G)$ within $(B, \|\cdot\|_B)$;
3. With two spaces B^1, B^2: take **intersection or sum**
4. forming **amalgam spaces** $W(B, \ell^q)$; e.g. $W(\mathcal{F}L^1, \ell^1)$;
5. defining pointwise or convolution **multipliers**;
6. using complex (or real) **interpolation methods**, so that we get the spaces $M^{p,p} = W(\mathcal{F}L^p, \ell^p)$ (all Fourier invariant);
7. any **metaplectic** image of such a space, e.g. the **fractional Fourier transform**.

Further properties

Theorem

1. *If FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a minimal FouSS, then the dual space belongs to the family and is maximal;*
2. *Conversely, any maximal space is a dual space, with the predual $((\boldsymbol{B}_0)^*)_0$.*

Remark: We all know that $\boldsymbol{B} = \left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a dual space. The predual can be obtained as follows: $\boldsymbol{B}_0 = \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, the dual space is $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. The closure of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is just $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$!

Corollary

A FouSS is reflexive if and only if both $\boldsymbol{B}$ and $\boldsymbol{B}^*$ are minimal.

Minimal Spaces Diagram

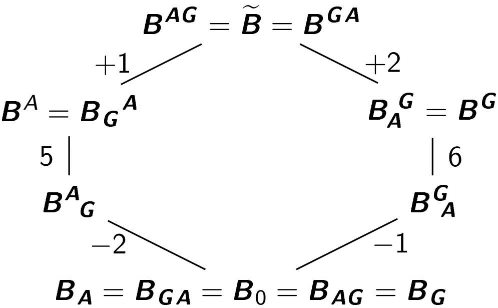

Figure: MINIMAL Spaces Diagram

Maximal Spaces Diagram

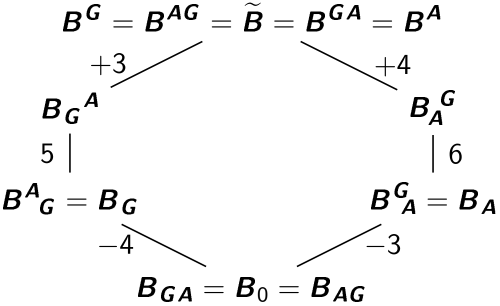

Figure: MAXIMAL Spaces Diagram

sup-norm cases

All the spaces in one of these diagrams have "the same norm" and form FouSS (in particular Banach).

Starting with $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (minimal!) one obtains actually 6 different spaces, with $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ at the top.

On the other hand one could start from any of the spaces in that diagram, e.g. $\boldsymbol{B} = \left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and obtain the same collection of spaces, but with a different list of non-collapsing connections (because $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a maximal space)! For the case $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (minimal) the diagram consists only of 2 spaces, namely $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, with e.g. $\boldsymbol{L}^1(\mathbb{R}^d) = (\boldsymbol{M}_b)_{\boldsymbol{G}}$ (continuous shift!).

Characterization via BAIs,I

For the rest take (e_α) a Dirac sequence and (h_β) plateau functions.

1. $$\lim_\alpha \|e_\alpha * f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{G}}.$$

2. $$\lim_\beta \|h_\beta \cdot f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{A}}.$$

3. $$\lim_{(\alpha,\beta)} \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\alpha \lim_\beta \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\alpha \lim_\beta \|h_\beta \cdot (e_\alpha * f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\beta \lim_\alpha \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

Characterization via BAIs, II

1. $$\sup_\alpha \sup_\beta \|h_\beta \cdot (e_\alpha * f)\|_{\boldsymbol{B}} < \infty \quad \Leftrightarrow \quad f \in \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{\boldsymbol{AG}} = \boldsymbol{B}^{\boldsymbol{GA}}.$$

   $$\sup_\alpha \sup_\beta \|e_\alpha * (h_\beta \cdot f)\|_{\boldsymbol{B}} < \infty \quad \Leftrightarrow \quad f \in \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{AG} = \boldsymbol{B}^{GA}.$$

2. $$\forall \beta \ \lim_\alpha \|e_\alpha * (h_\beta \cdot f) - h_\beta \cdot f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}};$$

3. $$\forall \alpha \ \lim_\beta \|h_\beta \cdot (e_\alpha * f) - e_\alpha * f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}^{\boldsymbol{A}}}_{\boldsymbol{G}}.$$

4. $$\forall \alpha \quad \lim_\beta \|e_\alpha * (h_\beta \cdot f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}_{\boldsymbol{G}}}^{\boldsymbol{A}}.$$

5. $$\forall \beta \quad \lim_\alpha \|h_\beta \cdot (e_\alpha * f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{G}}.$$

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19.08.2020 THANKS for your ATTENTION!

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