NuHAG Numerical Harmonic Analysis Group

Fourier Standard Spaces

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Fourier Standard Spaces I

Title: Fourier Standard Spaces
The aim of this talk is to present a family of Banach spaces of functions or (mild) distributions as the natural setting for many questions of classical and modern Fourier analysis (i.e. for Gabor and time-frequency analysis).
These spaces are - by definition - double Banach modules, namely with respect to convolution by L^1 and pointwise multiplication under FL^1. There is a smallest element in this family, namely the Segal algebra S_0 and a biggest one, the dual space S_0^*, the space of mild distributions. Together with the Hilbert space L^2 they form the Banach Gelfand Triple (S_0, L^2, S_0^*). It can be characterized within the tempered distributions by the membership of the Short-time Fourier Transform (STFT) of its members, which belong to L^1, L^2 and L^∞ respectively.

Fourier Standard Spaces II

The talk is going to explain in which sense many results from the literature, concerning e.g. L^p-spaces, Wiener amalgams, modulation spaces, or Fourier multipliers, can be viewed as special cases of results concerning Fourier Standard Spaces. We will reformulate a few early results by the author (e.g. on compactness or concerning double modules) in this framework.
The methods employed can be extended to translation and modulation invariant Banach spaces of (tempered) distributions or even ultra-distributions over LCA groups, but we avoid this level of generality in order to minimize the level of technicalities (mostly the use of weight functions can be ignored in the proposed setting).

Overview I

The purpose of this talk is to describe the scenario of what I am calling *Fourier Standard Spaces*.
This family of spaces appears to be the most natural domain for *Abelian Harmonic Analysis*, i.e. the study of signals defined over LCA (locally compact Abelian groups). In fact, we it allows to take up the perspective of Andre Weil (Fourier Analysis can and should be done naturally over LCA groups), or abstract Gabor Analysis (as a branch of TF-Analysis).
On the other hand it is of great value for the discussion of concepts of great practical value, e.g. concerning the transition from the discrete setting accessible to computations, to the continuous case. While engineers distinguish between discrete and continuous, between periodic and non-periodic signals mathematicians tend invoke Lebesgues spaces (requiring decay at infinity) or complicated topological vector spaces.

The various settings I

In order to give you an idea about the purpose of this talk (aside from technicalities) let me compare the setting with studies in the humanities.

If we want to understand the behaviour of mankind we can do this at various levels:

1. One can look at individuals (psychiatry, psycho-analysis);
2. We can look at families (Familienaufstellung);
3. We can study societies (sociology);
4. One can study the development of societies (anthropology);

Function Spaces come in Families

It was the rise of *Interpolation Theory* (for Banach spaces) which put a focus on *parameterized families of function spaces*.

Obviously the classical results (Hausdorff-Young and Marcinkiewicz) concerning the Fourier transform on $\boldsymbol{L}^p$-spaces was initiating the idea of deriving results for a *scale of space* (e.g. $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for $1 \leq p \leq \infty$) by establishing it for two limiting cases (e.g. $p = 1$: Riemann-Lebesgue and $p = 2$: Plancherel) first and do the rest by interpolation.

The so-called *complex interpolation method* (Calderon) is particularly useful when it comes to *bilinear mappings* and *weighted spaces*.

Other Families of Function Spaces

The list of *families of function spaces* is endless, and it is certainly wise to follow the inspiration and track of some of the early masters of this area, namely G. Köthe (Banach spaces of measurable functions), J. Peetre (with his "New Thoughts on Besov Spaces") and H. Triebel, who developed a *Theory of Functions Spaces*, dealing with the now classical Besov-Triebel-Lizorkin spaces, which are all Banach spaces of *tempered distributions*.
Obviously the family of *Besov spaces* $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$ has been a model for the family of *Modulation Spaces* .
All of these families are closed under interpolation and under duality (if test functions are dense). Thus obviously it is not enough to stay in the world of (smooth,or continuous) *functions*, but one has to allow distributions.
One may consider traces, but not tensor products of Fourier transforms.

How to deal with the Fourier Transform I

If one looks up a standard text-book on Fourier Analysis one finds very few *function spaces* which are Fourier invariant. Usually one only has $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (Plancherel's Theorem), the *Schwartz space* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and its dual, the *Tempered Distributions*.

Occasionally one will find that *Shubin classes* are Hilbert spaces which can be characterized by Hermite functions, but they also can be viewed as modulation spaces $\boldsymbol{M}^p_{v_s}(\mathbb{R}^d)$, which can be characterized via the membership of the STFT $V_g f$ of a function $f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ in some weighted $\boldsymbol{L}^p$-space, with radial symmetric weight $v_s(t,y) = (1+t^2+y^2)^{s/2}$.
But, as a family these spaces are of special interest, as the family is closed under duality and interpolation *and* the Fourier Transf..
Their intersection $(s > 0)$ is $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, their union $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$.

Double Modules I

Already early on (in my work on Segal algebras and later for the characterization of compact subsets in function spaces or discretization of convolution operators) function spaces with a double module structure, namely *pointwise multiplication* and *convolution* turned out to be a very useful setting. One could, for example start with Banach spaces which of *tempered distributions* which are *translation and modulation invariant*, and with growth estimates on the translation resp. modulation operators by a polynomial of the form $\langle t \rangle^s = (1 + t^2)^{s/2}$ resp. $\langle y \rangle^r$, for some $s, r \geq 0$.

This setting appeared in the last years in papers by S. Pilipovic and his coauthors (Dimovski, Prangovski, Vindas).

The Herz Algebras I

There are several reasons why the combination of convolution with pointwise multiplication was a striking feature of the work of Figa-Talamanca, Gaudry and Herz, which resulted in the observation, that the convolutors of $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ (the space of all bounded linear operators on $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ which commute with all translations) can be characterized as the dual space of

$$\boldsymbol{A}_p(\mathbb{R}^d) : \{h = \sum_{k=1}^{\infty} f_n * g_n, \text{ with } \sum_{k=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty\}$$

and the corresponding quotient norm (with $1/p + 1/q = 1$, $1 \leq p < \infty$).

The Herz Algebras II

This non-trivial characterization of the space of convolutors implies that the convolutors are so-called *quasi-measures*, but viewed as tempered distributions their FT (the tansfer function of the operator) was also identified as a quasi-measure.
Most surprisingly, C. Herz was able to show that is a Banach algebra with respect to *pointwise multiplication*.
Another early appearance of double module was in the study of the (weak) factorization problem for Segal algebras, which could also be solved under the additional assumption that the Segal algebra (an ideal in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$) is also a pointwise module over $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$!
But unfortunately the space of quasi-measures is too big to allow any reasonable extension of the Fourier transform up to this level. In fact, any continuous function defines a quasi-measure.

FOURIER STANDARD SPACES I

The term *Fourier Standard Spaces* has been coined ca. 4 years ago and meanwhile been discussed during serval talks, emphasizing different aspects of this family of spaces.
While the natural level of generality is that of TMIBs of ultra-distributions over LCA groups (which requires the use of suitable Wiener amalgam and general modulation spaces) one can restrict the discussion to two proto-typical settings, avoiding in this way the setting of abstact LCA groups as well as the more involved discussion of weight functions (be they sub-multiplicative *Beurling weights*, or polynomial weights on $\mathbb{R}^d$, or even stronger (sub-exponential) weights satisfying the Beurling-Domar non-quasi-analyticity condition $\sum_{k=1}^{\infty} w(nx)/n^2 < \infty$, or simply moderateness conditions (used to create translation invariant weighted spaces).

FOURIER STANDARD SPACES II

1. The Banach Gelfand Tripel $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$, which consists just of the *Segal algebra* $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right) = (\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\mathbf{W}(\mathcal{F}\mathbf{L}^1,\ell^1)})$, the Hilbert space $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space, $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$, als named the space of *mild distributions*;
2. OR: The family of all the Banach spaces which are sandwiched between $\mathbf{S}_0(\mathbb{R}^d)$ and $\mathbf{S}_0'(\mathbb{R}^d)$ and have a double module structure, namely over $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ with respect to convolution and with respect to pointwise multiplication over $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$.

FOURIER STANDARD SPACES III

The connection between the two settings is the *observation* that $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is contained in each of the (non-trivial) spaces which have the double module structure, while all the spaces, even if they are a priori only subspaces of $\mathcal{S}'(\mathbb{R}^d)$, are in fact continously embedded into $\boldsymbol{S}_0'(\mathbb{R}^d)$.
Thus we are assuming overall

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$$

and

$$\boldsymbol{L}^1(\mathbb{R}^d) \star \boldsymbol{B} \subset \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subset \boldsymbol{B},$$

combined with the corresponding norm estimates.

A long list of properties I

Due to the nice properties of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual (e.g. both of them are Fourier invariant), one has number of good properties of this family, e.g.

- closed under interpolation and duality;
- closed under the (fractional) Fourier transform;
- closed under tensor products;
- allowing the formation of Wiener amalgams $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$;
- all spaces of pointwise multipliers are included;
- hence also Fourier multipliers;
- forming modulation spaces with $V_g f \in \boldsymbol{B}(\mathbb{R}^{2d})$.
- taking kernels of operators between two such spaces!
- closed under the "Herz-construction" (convol.tens.prod.).

A long list of properties II

Obviously the ordinary/classical modulation spaces $\left(\boldsymbol{M}^{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^{p,q}}\right)$ belong to this family, as they are just the inverse Fourier transforms of the *Wiener Amalgam Spaces* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$.
But not every (general) modulation spaces is just a Wiener amalgam space on the Fourier transform side.

The interesting part - generally speaking - is the question, which properties of the *ingredients* go into the newly constructed space, or under which conditions.

Typical cases I

An important class of function spaces is the family of *solid BF-spaces*, i.e. spaces of locally integrable functions, with the property that $|f(x)| \leq |g(x)|$ a.e. for some $g \in \boldsymbol{B}$ implies that $f \in \boldsymbol{B}$ and $\|(\|_{\boldsymbol{B}} f) \leq \|(\|_{\boldsymbol{B}} g)$.
This family is closed under duality (if the norm is absolutely continues, and thus $\boldsymbol{C}_c(\mathbb{R}^d)$ is dense, and the dual space is just $\boldsymbol{B}^\alpha$, the Köthe dual.
If $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is translation invariant, then the weighted space

$$\boldsymbol{B}_w := \{f \mid fw \in \boldsymbol{B}\}$$

is translation invariant if and only if w is a *moderate* weight function (e.g. submultiplicative, or some - negative - power of such a function, such as $\langle x \rangle^s$, *forsomes* $\in \mathbb{R}$.).

But obviously one does not have invariance for the FT.

Wiener Amalgam Spaces I

Among the most important families of function spaces one may mention the *Wiener Amalgam Spaces*, which for me - is the most important tool, as it allows to describe local and global behaviour of functions in a much more flexible way than just the spaces $\boldsymbol{L}^p(\mathbb{R}^d)$.

If $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ has absolutely continuous norm, then $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$ has absolutely continuous norm for $q < \infty$, or one has translation invariance if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is translation invariant.

Details concerning the use of Wiener Amalgam spaces for classical Fourier Analysis or Time-Frequency Analysis, for the *irregular sampling problem* or *Spline-type spaces* have been given in various talks and papers.

Early Standard Spaces I

An important area where the double module structure arose was given in the papers on the characterization of compact sets in Banach spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. For Fourier Standard spaces we have:

Theorem

A bounded and closed subset M in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is relatively compact if for every $\varepsilon > 0$ there exist functions $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\|g\|_{\boldsymbol{L}^1} \leq 1$ and $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with $\|h\|_{\mathcal{F}\boldsymbol{L}^1} \leq 1$ such that for all $f \in M$

$$\|g \star f - f\|_{\boldsymbol{B}} \leq \varepsilon;$$

$$\|h \cdot f - f\|_{\boldsymbol{B}} \leq \varepsilon.$$

The MAIN DIAGRAM

[scale=1.1] (1) at (0,0) $\boldsymbol{B}^{\boldsymbol{AG}} = \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{\boldsymbol{GA}}$; (2) at (-2,-1) $\boldsymbol{B}^{A}$; (3) at (-2,-2) $\boldsymbol{B}_{\boldsymbol{G}}{}^{\boldsymbol{A}}$; (4) at (-2,-3) $\boldsymbol{B}^{\boldsymbol{A}}{}_{\boldsymbol{G}}$; (5) at (-2,-4) $\boldsymbol{B}_{G}$; (6) at (0,-5) $\boldsymbol{B}_{\boldsymbol{GA}} = \boldsymbol{B}_{0} = \boldsymbol{B}_{\boldsymbol{AG}}$; (7) at (2,-4) $\boldsymbol{B}_{A}$; (8) at (2,-3) $\boldsymbol{B}^{\boldsymbol{G}}{}_{\boldsymbol{A}}$; (9) at (2,-2) $\boldsymbol{B}_{\boldsymbol{A}}{}^{\boldsymbol{G}}$; (10) at (2,-1) $\boldsymbol{B}^{G}$;
[semithick,-] (1) – (2) node [midway,above] $+1$; [semithick,-] (2) – (3) node [midway,left] $+3$; [semithick,-] (3) – (4) node [midway,left] 5 ; [semithick,-] (4) – (5) node [midway,left] -2 ; [semithick,-] (5) – (6) node [midway,below] -4 ; [semithick,-] (6) – (7) node [midway,below] -3; [semithick,-] (7) – (8) node [midway,right] -1 ; [semithick,-] (8) – (9) node [midway,right] 6; [semithick,-] (9) – (10) node [midway,right] $+4$; [semithick,-] (10) – (1) node [midway,above] $+2$;

Abbildung: The MAIN DIAGRAM for double modules

Characterization via BAIs,I

For the rest take (e_α) a Dirac sequence and (h_β) plateau functions.

1. $$\lim_\alpha \|e_\alpha * f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{G}}.$$

2. $$\lim_\beta \|h_\beta \cdot f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{A}}.$$

3. $$\lim_{(\alpha,\beta)} \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\alpha \lim_\beta \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\alpha \lim_\beta \|h_\beta \cdot (e_\alpha * f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

$$\lim_\beta \lim_\alpha \|e_\alpha * (h_\beta \cdot f) - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_0.$$

Non-obvious Examples I

In a series of papers in the 70th K. McKennon has published papers concerning the space of *tempered elements in* $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ (also for more general groups), also showing that this space is a Banach algebra with respect to convolution and has the same multiplier space as $\boldsymbol{L}^p(\mathbb{R}^d)$ itself. Let us assume $1 < p < \infty$.

Definition

$$\boldsymbol{L}_p^t := \boldsymbol{L}^p(\mathbb{R}^d) \cap \mathcal{CV}_p(\mathbb{R}^d).$$

The key result is the following claim:

Theorem

$\mathcal{CV}(\boldsymbol{L}_p^t(\mathbb{R}^d)) = \mathcal{CV}(\boldsymbol{L}^p(\mathbb{R}^d))$, *with equivalence of norms.*

Non-obvious Examples II

Note that all the spaces involved are Fourier standard spaces. For example, $\boldsymbol{L}_p^t(\mathbb{R}^d)$ is just the intersection of two Fourier standard spaces, hence another Fourier standard space.
But when it comes to say that some $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ defines a convolution operator on $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ it is *tempting* (but also misleading) to assume that this should imply that

$$f \star g(x) = \int_{\mathbb{R}^d} g(x-y)f(y)dy \; \exists a.e., \quad \forall g \in \boldsymbol{L}^p(\mathbb{R}^d).$$

The correct interpretation is that the operator, which can be written as an integral operator for $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ extends to a bounded linear operator on all of $\boldsymbol{L}^p(\mathbb{R}^d)$!
For $1 \leq p \leq 2$ it is easy to check that integrals exists , but for $p > 2$ I am confident that a counter- example can be found (there are some ideas, but no explicit example written up).

Transformable measures I

In some early work on the Fourier transform of unbounded measures L. Argabright and J. Gil de Lamadrid have published papers on a space of so-called *transformable measures*. The idea is to say that there are measures which have a Fourier transform (somehow well defined, in a way which can be carried out for LCA groups, so avoiding the theory of tempered distributions), such that their Fourier transforms are also measures.
The classical case are the Haar measures of subgroups, which have (in the sense of $\boldsymbol{S}_0'(G)$) a Fourier transform, which turn out to be - up to normalization - just the Haar measure of the orthogonal group. The study of such measures has gained a lot of attention in the last years due to the connection with *quasi-crystals*, and the work of A. Olevskii and coauthors (e.g. Inv. Math.).

Transformable measures II

The classical case is Poisson's formula

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{n\in\mathbb{Z}} \widehat{f}(n), \quad f \in \boldsymbol{S}_0(\mathbb{R}),$$

can be reformulated simply to the claim that

$$\mathcal{F}(\sqcup\!\sqcup) = \sqcup\!\sqcup,$$

$$\text{for } \sqcup\!\sqcup(f) = \sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \delta_k(f) = [\sum_{k\in\mathbb{Z}} \delta_k](f), \quad f \in \boldsymbol{S}_0(\mathbb{R}).$$

The formal definition of *transformable measures* is a bit cumbersome and will not be presented here. An obvious subspace of measures (including the above example) is the space of all translation bounded measures $\boldsymbol{W}(\boldsymbol{M},\ell^\infty)(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d)$ which have a FT transform which is also in $\boldsymbol{W}(\boldsymbol{M},\ell^\infty)(\mathbb{R}^d)$.

Transformable measures III

It turns out that the formalities of the definition of *transformable measures* was based on ideas which lead to the definition of quasi-measures (going back to the work of G. Gaudry).
One can show, that a *Radon measure* $\mu \in \boldsymbol{S}_0'(\mathbb{R}^d)$ is transformable, of its FT belongs to $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d)$!

SUMMARY I

As a summary, one can say, that in the frame-work of Fourier Standard spaces (and within this family the Wiener amalgam spaces) many complicated ad-hoc constructions found in the literature can be avoided, or the situation can be analyzed in a more efficient way.
The benefit of this view-point is the fact that the tools arising from the theory of Banach modules and distribution theory (if one wants the theory of ultra-distributions) can be invoked.
Among others this shift of focus avoids the discussion concerning the existence of integrals, at least almost everywhere. Instead, one works with approximation methods or regularization operators, which is much more transparent.
Recall my comparison $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ with $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$!

Fourier Standard Spaces of Kernels I

As a final example of spaces (of distributions) which can (or should) be studied based on this context can be found via the so-called kernel theorem for $\boldsymbol{S}_0$.

Given any two Fourier Standard Spaces $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ (with $\boldsymbol{B}^1$ being minimal, i.e. containing $\mathcal{S}(\mathbb{R}^d)$ or $\boldsymbol{S}_0(\mathbb{R}^d)$ as a dense subspace), also the distributional kernels of all the bounded linear operators from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ form a Fourier Standard Space.

The same is true for the set of all spreading functions of such operators, or the set of all Kohn-Nirenberg symbols of such operators.

The generic question is then: What is the shape of the main diagram for such spaces?

Generalizations I

As already mentioned in the introductory part of this talk most of these questions can be formulated in the context of TMIBs, i.e. translation and modulation invariant Banach space of (ultra-)distibutions over $\mathbb{R}^d$ or even over LCA groups.
One should also mention the connection to the theory of *Heisenberg modules*, because the double action of TF-shifts can of course be combined elegantly to the action of the Heisenberg group (via the Schrödinger representation).
In the current talk we have tried to restrict the attention to the basic ideas, avoiding technicalities as much as possible.

Approximation by translates I

The setting of Fourier Standard Spaces can also be used to explain the basic idea of an approach to the approximation of functions by translates of a bounded approximate unit generated by dilation of an element $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, valid for all the *minimal Fourier Standard spaces*, as realized in two recent preprints of joint work with Anupam Gumber.

We do not have time to discuss details, but just mention, that the key ingredients are just

- The use of approximate units (for convolution and multiplication);
- The approximation of convolution integrals by finite sums of shifts.

Usefulness for Teaching I

There is another, maybe more important aspects of the approach to function spaces (for time-frequency analysis, but also for classical Fourier analysis, including the theory of Almost Periodic Functions etc.):
The setting of Fourier Standard Spaces just requires *basic functional analysis* and no deep theory of topological vector spaces, but still allows to discuss important aspects relevant for applications, e.g. the fact that pointwise multiplication corresponds to convolution on the Fourier transform side, thus sampling along a lattice corresponds to periodization on the FT side, and so on.

Further information, reading material

Thanks for your attention

The NuHAG webpage offers a large amount of further information, including talks and MATLAB code:

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www.nuhag.eu
www.nuhag.eu/bibtex (all papers)
www.nuhag.eu/talks (all talks, access: visitor/nuhagtalks)

www.nuhag.eu/skripten (lecture notes,open)
www.nuhag.eu/ETH20 (course notes, YouTube)
www.nuhag.eu/matlab (MATLAB code)
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