A Modern Approach to the FOURIER TRANSFORM

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Abstract Submitted I

Assuming the many of the participants of this conference may at some point also have to teach a course in Fourier Analysis I consider it eminently important that such a course should satisfy the needs of a modern society, with an increasing role for digital signal processing, or the simulation of linear systems using the computer. While Laurent Schwartz' theory of tempered distributions (almost as old as the speaker) is well established it is highly demanding in terms of mathematical background, and mostly relevant for the solution of PDEs.
This talks should point out that a theory of "*mild distributions*" can be based on quite simple ideas, and not much more than the Riemann integral.

Abstract Submitted II

The key to this modern (and *simple*!) approach to a theory of *generalized functions* provides a solid mathematical basis for objects such as Dirac measures or Dirac combs. Mild distributions have a Fourier transform, which is also a mild distributions, and they have a bounded spectrogram. Such spectrograms can nowadays be produced on real-time, and in this sense audio-signals are typical examples of objects which should undergo a "localized frequency analysis".

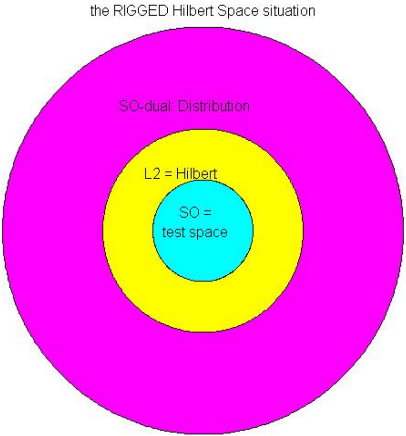
The analogy to the number systems, with the natural embedding $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ is used as a motivation and orientation for a sequential approach. Actual computations are done in the field of rational numbers $\mathbb{Q}$, and extended to $\mathbb{R}$ by taking limits (e.g. in order to define $\pi \cdot \sqrt{5}$). Finally a "trick" allows to extend the usual calculations to the even larger (and more flexible) field of complex numbers $\mathbb{C}$.

Abstract Submitted III

By its genesis the function spaces used, namely the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (also called *Feichtinger's algebra*), together with $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, the Hilbert space of square integrable functions, and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (of mild distributions) are very well suited to deal with questions of time-frequency analysis and discuss the stability of so-called Gabor expansions of signals. In the audio case this can be viewed as a kind of graphical score computed from the recorded sound signal.

A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



Diagnosis of the Current Situation I

Depending on the science a different perspective is taken:

- Mathematicians emphasize certain technical aspects of Fourier Theory (e.g. pointwise a.e. convergence of Fourier Series, existence of the Fourier integral, boundedness of the Hilbert transform on $\boldsymbol{L}^p(\mathbb{R})$, etc.);
- Engineers start from discrete signals, and describe linear, time-invariant systems as convolution operators (using the *impulse response*), which are diagonalized via the Fourier transform (*transfer function*);
- Physicist would consider the Fourier transform as a simple change of bases, from the "continuous basis of Dirac measures $(\delta_x)_{x\in\mathbb{R}}$ to the *pure frequencies* $\chi_s(t) = e^{2\pi i s t}, s \in \mathbb{R}$.

... but I have a dream ...

The ideal requests on the concept of CHA

The PLAN is to develop the idea of Conceptual Harmonic Analysis as a *unifying framework* which should serve a variety of purposes:

- From the point of view of mathematics (AHA) the approach should allow to deal with Fourier Analysis over LCA (locally compact Abelian) groups G;
- For engineers it should provide a unified framework for *discrete and continuous signals*, but also for periodic and non-periodic signals (all defined over $\mathbb{R}$ and $\mathbb{R}^d$, e.g.) or a correct proof of the Shannon Sampling Theorem;
- For physicists it should provide a justification for expressions such as

$$\int_{-\infty}^{\infty} e^{2\pi i s t} dt = \delta(t).$$

Towards a NEW INTRODUCTION to Fourier Analysis I

It SHOULD BE:

1. *mathematically correct* (FA);
2. *useful* for engineers, physicists;
3. suitable for course on TILS (transl. invar. systems);
4. provide support for the transition between the world of disrete/periodic and continuous signals;
5. not based on Lebesgue integration theory;
6. not making use of Schwartz's $\mathcal{S}'(\mathbb{R}^d)$;
7. should be possible for general LCA groups!
8. thus help to treat problems in Abstract HA (AHA);

The Tools used so far I

Most of the tools involved in the new approach are based on considerations arising in Time-Frequency Analysis and Gabor Analysis.
In this context are Banach frames, Gabor expansions of distributions, Wiener amalgam spaces, modulation spaces, and the Segal algebra $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ (Feichtinger's algebra), which is the basis of the Banach Gelfand Triples $(S_0, L^2, S_0')(G)$, Time-Frequency Analysis, *spreading functions*, *kernel theorem*, etc..
In any case it is clear that some (modest) form of Functional Analysis (FA) will be needed, because otherwise it is not possible to describe objects (described as signals, and modelled as function or generalized functions, i.e. *distributions*) as members of not-finite-dimensional linear spaces, but also various forms of convergence.

The Tools used so far II

- MATH: First one has to establish the existence of the Haar measure, then define $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ and convolution and the Fourier transform, mapping $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ into $\left(\boldsymbol{C}_0(\widehat{G}), \|\cdot\|_\infty\right)$ (Riemann-Lebesgue Lemma);
- FT as *integral transform*, and *convolution* understood pointwise (a.e.), all based on Lebesgue's integral;
- ENGINEERS: Start from the *sifting property* of the Dirac:

$$f = \int_{-\infty}^{\infty} f(x)\delta(x-t)dt,$$

  in order to derive that any TILS is a convolution operator!
- The FFT is the backbone of *digital signal processing*, is just a fast implementation of the DFT, with little immediate connection to the integral transform $\mathcal{F}$!

The Tools used so far III

After ca. 200 years of Fourier analysis we have a lot of tools:

1. We can do Fourier Analysis using $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, a Banach algebra with convolution and a well defined Fourier transform via Lebesgue integration;
2. We can do similar things over general LCA groups
3. We have fast algorithms, thanks to the FFT (basis for digital signal processing!);
4. Thanks to L. Schwartz we have the wonderful theory of tempered distributions, which is based on the properties of the *nuclear Frechet space* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ [Schwartz space].
5. We have a large variety of function space, in particular the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ which is required to prove Plancherel's Theorem and then the Hausdorff-Young inequality for the Fourier transform.

Some references I

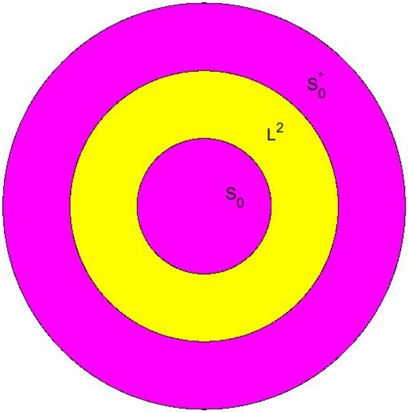
[rest00] H. Reiter and J. D. Stegeman.
Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.
Clarendon Press, Oxford, 2000.

[ca11] G. Cariolaro.
Unified Signal Theory.
Springer, London, 2011.

[fe20-1] H. G. Feichtinger.
A sequential approach to mild distributions.
Axioms, 9(1):1–25, 2020.

[we40] A. Weil.
L'integration dans les Groupes Topologiques et ses Applications.
Hermann and Cie, Paris, 1940.

The comparison with the number system



The three layers I

Rational Numbers $\mathbb{Q}$

Looking at the numbers systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ we easily diagnose that the field $\mathbb{Q}$ of rational numbers is the workhorse of numerical computations. It allows exact computations, and even the determination of the multiplicative inverse of $\frac{a}{b}$ can be done by every child, writing $\frac{b}{a}$.

Real number system $\mathbb{R}$

The incompleteness of $\mathbb{Q}$ combined with the fact that the equation $x^2 = 2$ does not have a solution in suggest to enlarge the rationals, by going over to the real numbers $\mathbb{R}$, where $\sqrt{2}$ is described as the "ideal limit" of approximate solutions (e.g. in the form of finite decimal expressions) to the above equation.

The three layers II

All the operations (like addition, multiplication, division) are then *extended* to the real number system, which is then a field of its own, containing (a COPY of) $\mathbb{Q}$ (periodic decimal expressions), as a dense subset. Moreover, $\mathbb{R}$ is now a *complete* space, i.e. any Cauchy sequence is convergent.

Complex Numbers

Finally one can extend the real number system to the complex field $\mathbb{C}$, by defining on the pairs $(x, y) \in \mathbb{R}^2$ (Gaussian plane), usually written as $z = x + iy$, addition and multiplication turning $\mathbb{C}$ into a field which allows to establish rules like Euler's law

$$e^{ix} = \cos(x) + i\sin(x), \quad x \in \mathbb{R}.$$

Going by ANALOGY I

We are going to discuss a chain of Banach spaces

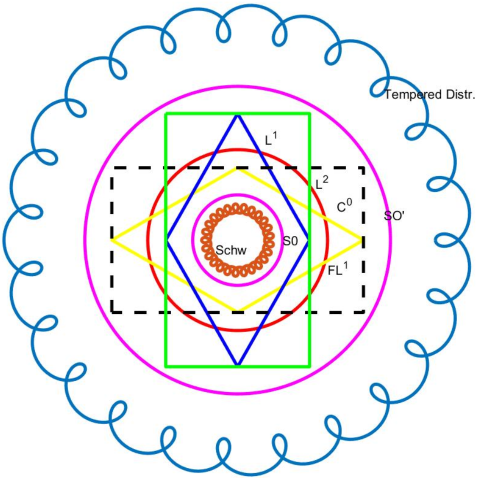
$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}) \qquad (1)$$

1. The Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ (Feichtinger's algebra), also known as modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$;
2. The Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ of square integrable (in the sense of Lebesgue, [classes of]) functions on $\mathbb{R}^d$;
3. The Banach space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ of mild distributions (also known as modulation spaces $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$.

We have $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^p(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$, for $1 \leq p \leq \infty$ and

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

A Zoo of Banach Spaces for Fourier Analysis



The Key-players for Time-Frequency Analysis (TFA)

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

–

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t),$$

with $x, \omega, t \in \mathbb{R}^d$, compatible with the Fourier transform

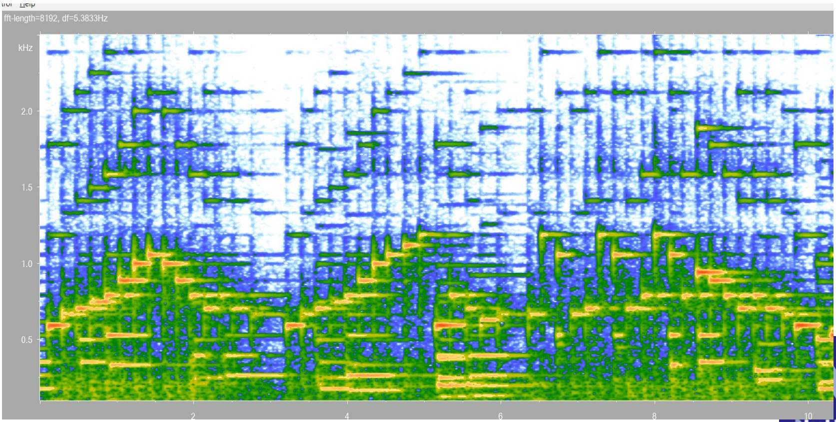
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

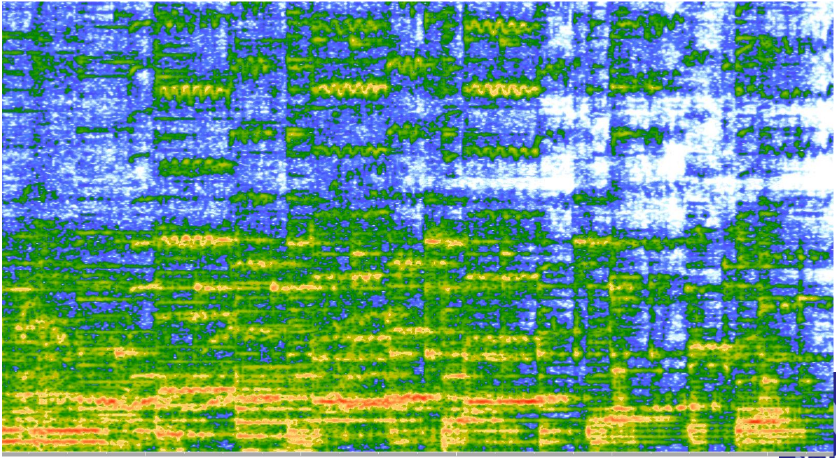
$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT

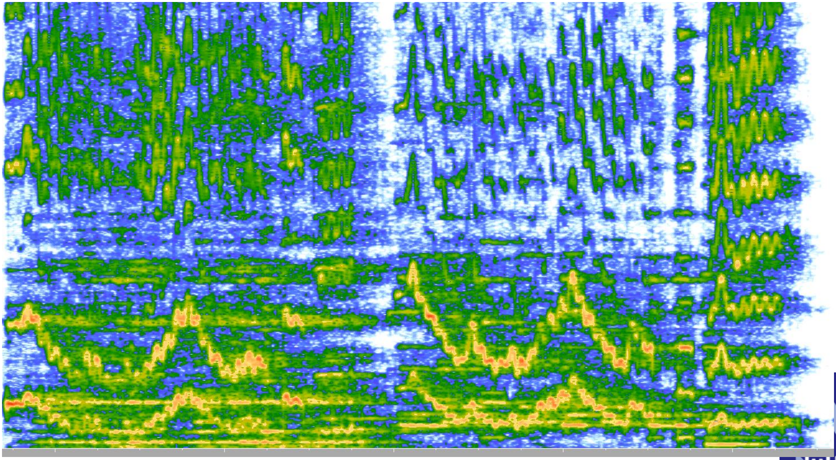
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

Different BUPUs using B-splines

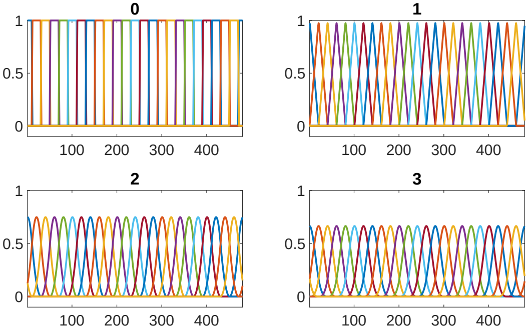


Abbildung: bupuitofourA.pdf
Different partitions of unity using B-splines of order $1, 2, 3, 4$ (degree $0, 1, 2, 3$).

The AXIOMATIC APPROACH

For a more axiomatic description (which allows us to use similar arguments) we assume:

1. $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a Banach algebra continuously and densely embedded into $(\boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_1 + \|\cdot\|_\infty)$;
2. $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is invariant under TF-shifts $\pi(\lambda)$ for $\lambda = (x, \xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and
$$\lim_{\lambda \to 0} \|\pi(\lambda)f - f\|_{\boldsymbol{A}} = 0, \quad \forall f \in \boldsymbol{A}.$$
3. $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is invariant under the Fourier transform;
4. For any $x \in \mathbb{R}^d$ and $\delta > 0$ $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ contains non-zero non-negative functions with $\operatorname{supp}(\varphi) \subset B_\delta(x)$.

Derived Properties

Assuming the assumption listed above (all satisfied by $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$) we have:

- There is a natural embedding of $(\mathbf{A}, \|\cdot\|_{\mathbf{A}})$ into its dual space (and in fact with the Hilbert space $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ as completion of $(\mathbf{C}_c(\mathbb{R}^d), \|\cdot\|_2)$ inside of $\mathbf{A}'$.
- The basic operators such as $\pi(\lambda)$ can be extended in a *unique* w^*-w^*--continuous fashion to all of $\mathbf{A}'$.
- Also the Fourier transform extends to a (unitary) automorphism of $\mathbf{A}'$, and in fact a Banach Gelfand Triple automorphism., based on the following simple definition:

$$\widehat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \mathbf{A}. \tag{2}$$

Atomic Characterization

The minimality implies (inspired by the atomic characterization of *real Hardy spaces*) the following result:

Theorem

Given any non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *one has*

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{ \sum_{k \geq 1} c_k M_{s_k} T_{x_k} g, \quad \textit{with} \sum_{k \geq 1} |c_k| < \infty \}.$$

This results also implies that the space is invariant under dilation, rotation and even fractional Fourier transforms.
A long list of sufficient conditions ensures that among others all classical summability kernels belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

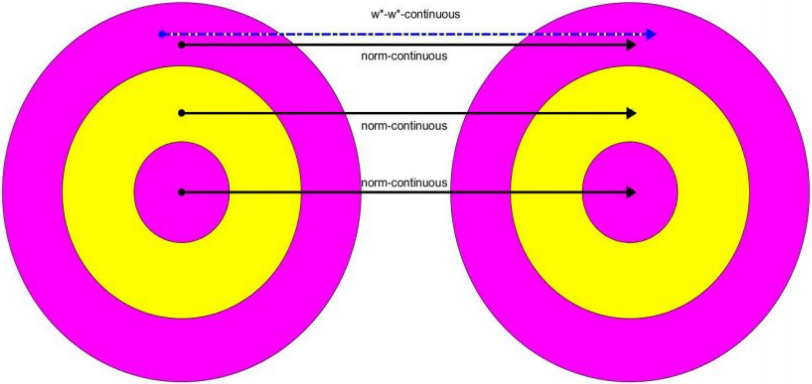
1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

The prototypical examples over the torus

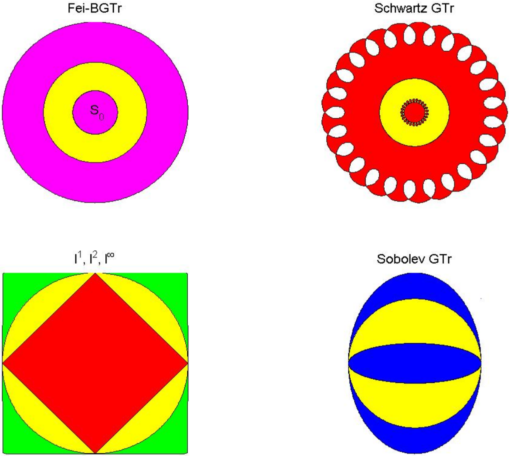
In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.

For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

Banach Gelfand Triple Morphism



Other Banach Gelfand Triples



The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{3}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Poisson Formula

Among others the so-called Dirac comb $⨿⨿_\Lambda$ defines a bounded linear functional on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, for any lattice $\Lambda = \mathbb{A} * \mathbb{Z}^d \lhd \mathbb{R}^d$ defines an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$ via

$$⨿⨿_\Lambda(f) := [\sum_{\lambda\in\Lambda} \delta_\lambda](f) = \sum_{\lambda\in\Lambda} f(\lambda).$$

The validity of Poisson's formula for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be formulated like this: For any $\Lambda \lhd \mathbb{R}^d$ there exists $C_\Lambda > 0$ such that

$$\sum_{\lambda\in\Lambda} f(\lambda) = C_\Lambda \sum_{\lambda^\perp\in\Lambda^\perp} \widehat{f}(\lambda^\perp) \quad (4)$$

for $\Lambda^\perp = (\boldsymbol{A}^t)^{-1} * \mathbb{Z}^d$, or equivalently, the validity (in the sense of $\boldsymbol{S}_0'(\mathbb{R}^d)$):

$$\mathcal{F}(⨿⨿_\Lambda) = C_\Lambda ⨿⨿_{\Lambda^\perp}. \quad (5)$$

Sampling corresponds to periodization on the FT side

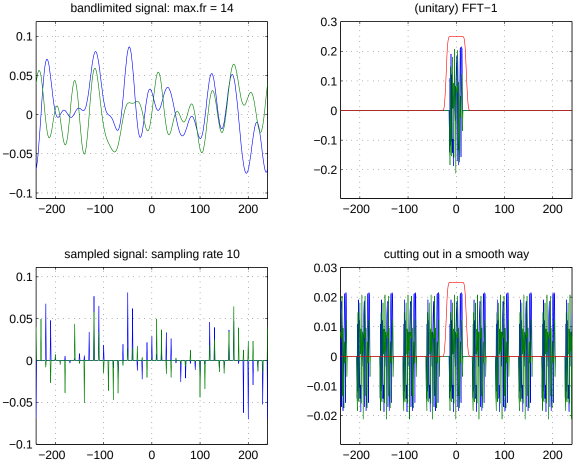


Abbildung: If there is a bit of oversampling, one can choose a better localized reconstruction atom (than SINC).

Distributional FT and FFT

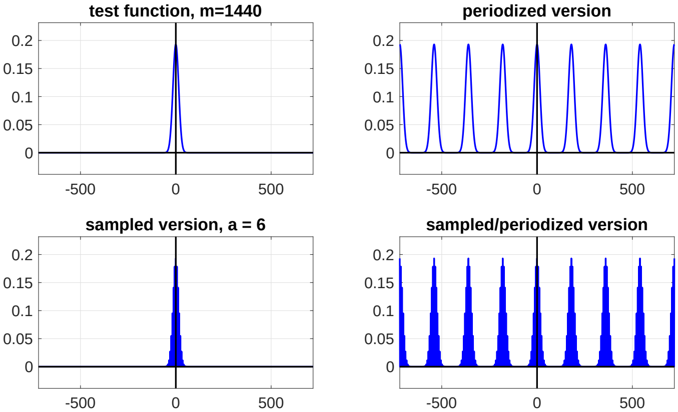


Abbildung: original(top left), periodize (top right),
or sample (left lower corner), or both (right lower corner).

The Spline Quasi-interpolation operators I

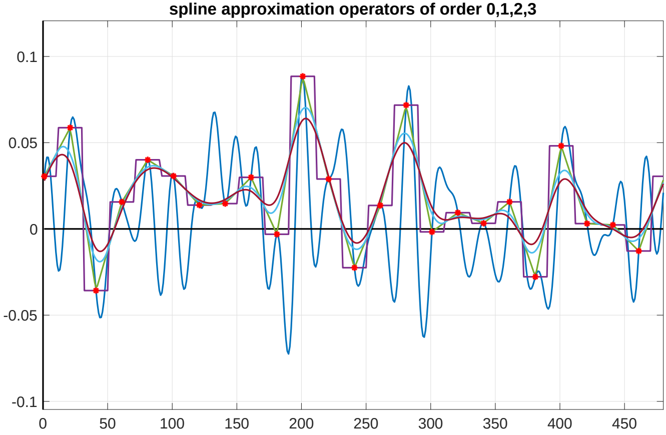


Abbildung: Approximation by spline functions of order 1, 2, 3, 4

Recovery from samples I

The information contained in the samples of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ is getting more and more. We know, that for any $f \in \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ one has norm convergence of $\mathrm{Sp}_\Psi f$ to f with respect to the sup-norm, as $|\Psi| \to 0$ (think of piecewise linear interpolation over $\mathbb{R}$).
Since the sequence of compressed triangular functions $\mathrm{St}_\rho \Delta$, $\rho \to 0$ forms a Dirac family one may expect that

$$\mathrm{St}_\rho \Delta * [\alpha^d \sum_{k \in \mathbb{Z}^d} f(\alpha k) \delta_{\alpha k}] \to \delta_0 * (\mathbf{1} \cdot f) \to f, \quad f \in \boldsymbol{S}_0. \tag{6}$$

We have to make two observations: First of all this is in fact an alternative description of piecewise linear interpolation, since $\mathrm{D}_{1\alpha} \Delta$ is just a triangular function with basis $[-\alpha, \alpha]$.

$$\mathrm{St}_\rho \Delta * [\alpha^d \sum_{k \in \mathbb{Z}^d} f(\alpha k) \delta_{\alpha k}] = \sum_{k \in \mathbb{Z}^d} f(\alpha k) T_{\alpha k} \mathrm{D}_\rho(\Delta). \tag{7}$$

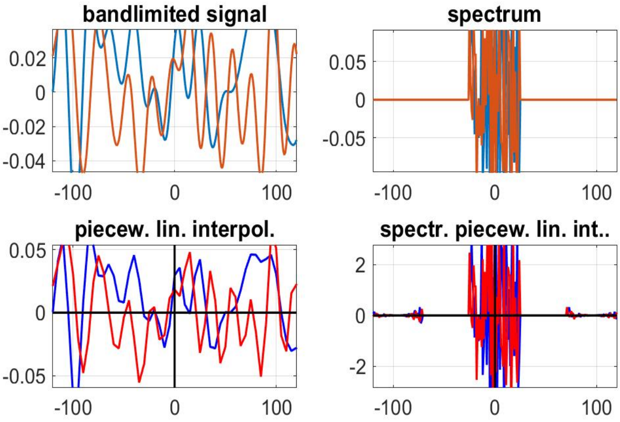


Abbildung: piecewise linear interpolation of a smooth, i.e. here band-limited signal (partial zoom in, to show the lack of smoothness).

Theorem

Assume that $\Psi = (T_k\psi)_{k\in\mathbb{Z}^d}$ *defines a BUPU in* $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$[a] *and write* $\mathrm{D}_\rho\Psi$ *for the family* $\mathrm{D}_\rho(T_k\psi) = (T_{\alpha k}\mathrm{D}_\rho\Delta)_{k\in\mathbb{Z}^d}$, *with* $\alpha = 1/\rho \to 0$. *Then* $|\mathrm{D}_\rho\Psi| \leq r\alpha \to 0$ *for* $\alpha \to 0$, *and*

$$\|f - \alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\, \mathrm{St}_\alpha\, \psi\|_{\mathbf{S}_0} \to 0, \qquad \text{for } \alpha \to 0, \forall f \in \mathbf{S}_0. \quad (8)$$

[a] As it is required for the definition of $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$, for some $\psi \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ with $\mathrm{supp}(\psi) \subset B_r(0)$.

This result was the cornerstone for the subsequent result published by Norbert Kaiblinger, concerning the approximate computation of the Fourier transform of a “nice function” $f \in \mathbf{S}_0(\mathbb{R}^d)$ via an FFT routine, applied to a sequence of samples of the original function! (a claim which is “obvious” to engineers).

TILS characterization

In this case the classical result is:

Theorem

Any $T \in \boldsymbol{H}_{\mathbb{R}^d}(\boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d))$ *can be described as a pointwise multiplier with some* $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$*:*

$$\widehat{Tf} = h \cdot \widehat{f}, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d)$$

(where the FT is taken in the sense of the Plancherel Theorem).

In the terminology of engineers: Any TILS (which is bounded on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$) can be described by the multiplication of the Fourier transform of the input function by some bounded TRANSFER function h.

The most general TILS

In order to describe bounded operators from $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ to some $(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q)$ one does not have such a simple description.

Theorem

The bounded linear operators T from $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ which commute with translations (or with convolutions) (i.e. the elements of $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{S}_0, \boldsymbol{S}_0')$) can be characterized as convolution operators by some uniquely determined $\sigma \in \boldsymbol{S}_0'$ (with $f^{\checkmark}(x) = f(-x)$) via

$$Tf(x) = \sigma(T_x f^{\checkmark}), \quad x \in \mathbb{R}^d, f \in \boldsymbol{S}_0,$$

and equivalence of norms (operator norm and $\|\sigma\|_{\boldsymbol{S}_0'}$).

Alternative Approach to Mild Distributions I

First we characterize $S_0'(\mathbb{R}^d)$ inside of $\mathcal{S}'(\mathbb{R}^d)$:

Theorem

*A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_0'(\mathbb{R}^d)$ (or equivalently: extends continuously from the dense subspace $\mathcal{S}(\mathbb{R}^d)$ of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ to all of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$) if and only if $V_g(\sigma)(\lambda) := \sigma(\pi(\lambda)g)$ is a bounded and continuous function on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and for any non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ the expression $\|V_g(\sigma)\|_\infty$ defines an equivalent norm on $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$.
Moreover, w^*-convergence of a sequence $\sigma_n \to \sigma_0 \in (S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is equivalent to the POINTWISE convergence of $V_g(\sigma_n)(\lambda)$ to $V_g(\sigma_0)(\lambda)$ (in fact, uniform convergence over compact subsets of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$!).*

Since clearly $(\mathbf{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$ is thus a subspace of $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ (and hence any *almost periodic function*, etc.) the above characterization allows to derive two possible approaches to the space $\mathbf{S}_0'(\mathbb{R}^d)$ of mild distributions via a *sequential approach*. The objects of the extended domain for the Short-Time Fourier Transform are (equivalence classes of) so-called *mild Cauchy sequences* (in short **ECmiCS**).

Definition

Fix any non-zero $g \in \mathbf{C}_c(\mathbb{R}^d)$ with $\hat{g} \in \mathbf{L}^1(\mathbb{R}^d)$. A sequence $(h_n)_{n\geq 1}$ in $\mathbf{C}_b(\mathbb{R}^d)$ is called a *mild Cauchy sequence* if the sequence $\|V_g(h_n)\|_\infty$ is bounded and for every pair $(t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we a Cauchy sequence: $(V_g(h_n)(s,t))_{n\geq 1}$.

The uniform continuity of the functions V_{g_n} implies that

$$\exists\, H(s,t) = \lim_{n\to\infty} V_g(h_n)(s,t) \tag{9}$$

defines a STFT for the "limit" $\|H\|_\infty \leq \sup_{n\geq 1} \|V_g(h_n)\|_\infty$.

Starting from these mild Cauchy sequences one can do the same thing as with the construction of $\mathbb{R}$ from $\mathbb{Q}$: One has to identify Cauchy sequences which "represent the same object", by determining **equivalence classes** of Cauchy sequences.
By identifying these equivalence classes with the limit of their STFTs we can thus claim that every such equivalence class has a bounded STFT and this determines a unique element in $\boldsymbol{S}_0'(\mathbb{R}^d)$.
IN SHORT: can be identified with with the space of all equivalence of mild Cauchy sequences! using

$$\boldsymbol{S}_0 \cdot (\boldsymbol{S}_0 * \boldsymbol{S}_0') \subset \boldsymbol{S}_0 \subset \boldsymbol{C}_b.$$

The KERNEL THEOREM for $\mathcal{S}(\mathbb{R}^d)$

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{10}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (10) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)
One of the corner stones for the kernel theorem is: One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert, showing a kind of uniqueness of the functor $G \mapsto \boldsymbol{S}_0(G)$) is the tensor-product factorization:

The KERNEL THEOREM for S_0 II

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \tag{11}$$

with equivalence of the corresponding norms.

A few more details

Given two functions f^1 and f^2 on $\mathbb{R}^d$ respectively, we set $f^1 \otimes f^2$

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1)f^2(x_2), \; x_i \in \mathbb{R}^d, i = 1, 2.$$

Given two Banach spaces $\boldsymbol{B}^1$ and $\boldsymbol{B}^2$ embedded into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, $\boldsymbol{B}^1\hat{\otimes}\boldsymbol{B}^2$ denotes their *projective tensor product*, i.e.

$$\left\{ f \mid f = \sum f_n^1 \otimes f_n^2, \; \sum \|f_n^1\|_{\boldsymbol{B}^1} \|f_n^2\|_{\boldsymbol{B}^2} < \infty \right\}; \tag{12}$$

The KERNEL THEOREM for S_0 III

Any such representation of f will be called an *admissible representation*, and of course this is not unique, because one can add certain terms and subtract part of it later on. There is also often no optimal or canonical representation (as we have it for finite, discrete measures). It is easy to show that (12) defines a Banach space of tempered distributions on $\mathbb{R}^{2d}$, in our case a Banach space of mild distributions or even with respect to the folowing (natural quotient) norm:

$$\|f\|_{\hat{\otimes}} := inf \left\{ \sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2}, \right\}, \tag{13}$$

where the infimum is taken over all *admissible representations*. One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert, is the tensor-product factorization:

The KERNEL THEOREM for S_0 IV

Lemma

$$\mathbf{S}_0(\mathbb{R}^k)\hat{\otimes}\mathbf{S}_0(\mathbb{R}^n) \cong \mathbf{S}_0(\mathbb{R}^{k+n}). \tag{14}$$

The KERNEL THEOREM for S_0 V

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ FORMALLY (!) as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the “outer shell” of the Gelfand triple isomorphism. The “middle = Hilbert” shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ *as scalar product on* $\mathcal{HS}$ *and the usual Hilbert space structure on* $\boldsymbol{L}^2(\mathbb{R}^{2d})$ *on the kernels.*
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ *if and only if the corresponding operator* K *maps* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *into* $\boldsymbol{S}_0(\mathbb{R}^d)$*, but not only in a bounded way, but also continuously from* w^*-*topology into the norm topology of* $\boldsymbol{S}_0(\mathbb{R}^d)$*.*

In analogy to the matrix case we have for $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ for the corresponding operator $T = T_K$:

$$K(x,y) = \delta_x(T(\delta_y)), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\mathbf{S}_0$ and $\mathbf{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$*, where the first set is understood as the* w^* *to norm continuous operators from* $\mathbf{S}_0'(\mathbb{R}^d)$ *to* $\mathbf{S}_0(\mathbb{R}^d)$*, the so-called regularizing operators.*

Spreading function and Kohn-Nirenberg symbol

1. For $\sigma \in S_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$$

The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}d\omega\right)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta,u)M_\eta T_{-u}f(x)\,du\,d\eta.$$

$\widehat{\sigma}$ is called the spreading function of T_σ.

Further details concerning Kohn-Nirenberg symbol

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

$$k(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x) = \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y-x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

The kernel $k(x,y)$ can be described as follows (imitating our MATLAB viewpoint: first reshape the matrix to side-diagonal form, then take a one-dimensional Fourier transform along the diagonals).

Spreading representation and commutation relations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x) f(x-t)\, dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu) f(x-t) e^{2\pi i x\cdot\nu}\, dt\, d\nu$
	$=$
	$\int\int \eta_H(t,\nu) M_\nu T_t f(x), dt\, d\nu\,,$

The description of operators through the spreading function and allows to understand a number of commutation relations.
If an operator is a limit (in the strong operator topology) of translation operators it is just a convolution operator with some $\tau \in \boldsymbol{S}_0'(\mathbb{R}^d)$, resp. its spreading representation is just an element

The symplectic Fourier transform

The *symplectic Fourier transform* connects the Kohn-Nirenberg symbol with the spreading function, i.e.

$$\mathcal{F}_s(\sigma(T)) = \eta(T) \quad \text{resp.} \quad \mathcal{F}_s(\eta(T)) = \sigma(T). \tag{15}$$

$$(\mathcal{F}_{symp} f)(k,l) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y) e^{-2\pi i (k\cdot y - l \cdot x)}; \quad f \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d). \tag{16}$$

It is completely characterized by its action on elementary tensors:

$$\mathcal{F}_{symp}(f \otimes \hat{g}) = g \otimes \hat{f}, \quad f,g \in \boldsymbol{S}_0(\mathbb{R}^d), \tag{17}$$

and extends from there in a unique way to a $w^* - w^*$ continuous mapping from $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ to $\boldsymbol{S}_0'(\mathbb{R}^{2d})$, also ${\mathcal{F}_s}^2 = Id$.

Best approximation by Gabor Multipliers

The Kohn-Nirenberg setting is also the best place, not only to compute a Gabor multiplier (given the ingredients, so ideally a pair (gt, Λ), with $gt \in \boldsymbol{S}_0(\mathbb{R}^d)$, such that the Gabor family

SOME Relevant References

[cofelu08] E. Cordero, H. G. Feichtinger, and F. Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lect. Notes Math.*, pages 1–33. Springer, Berlin, 2008.

[fe09] H. G. Feichtinger.
Banach Gelfand triples for applications in physics and engineering.
volume 1146 of *AIP Conf. Proc.*, pages 189–228. Amer. Inst. Phys., 2009.

[fe17] H. G. Feichtinger.
A novel math.approach to the theory of transl. invariant linear systems.
In I. Pesenson, et al.eds., *Recent Applications of Harmonic Analysis to Function Spaces, Differential Equations, and Data Science.*, Applied and Numerical Harmonic Analysis, 483–516. Birkhäuser, Cham, 2017.

[fe20-1] H. G. Feichtinger.
A sequential approach to mild distributions.
Axioms, 9(1):1–25, 2020.

[fe20-2] H. G. Feichtinger.
Ingredients for Applied Fourier Analysis.
In *Sharda Conference Feb. 2018*, pages 1–22. Taylor and Francis, 2020.

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