NuHAG Numerical Harmonic Analysis Group

Gabor Analysis: State of the Art and Computational Issues

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Summary

Since Gabor Analysis can be realized over any LCA group, including the finite Abelian groups, we can structure this talk as follows:

1. First Gabor Analysis is described in the case $G = \mathbb{U}_n$, the cyclic group of order $n \in \mathbb{N}$; this setting can be dealt with in a linear algebra (e.g. MATLAB) setting;
2. Then we discuss the shortly the functional analytic background; we have frames, function spaces, etc.;
3. The structural properties of Gabor families will us then allow to understand the situation better and design efficient algorithms;
4. As time permits the connection between Gabor Analysis over $\mathbb{R}^d$ and its finite-dimensional approximation will be shortly discussed.

The original idea

The original suggestion of D. Gabor from 1946 was to expand "every function" in a double series with optimally concentrated (from the phase space point of view) building blocks, so that one could interpret the (unique?) coefficients as the energy found within the signal at a given time and frequency.
From a modern point of view one would say, that he was *expecting that the double indexed family of TF-shifted Gaussians* (the Gabor family proposed by him) was a Riesz basis (because it is obviously non-orthogonal) for the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.

Different application areas

While the concrete suggestion did not work out well from a technical point of view, the variety of applications confirms Gabor's intuition:

1. expansions of signals into Gabor series;
2. based on this e.g. sparse approximation;
3. realization of Gabor multipliers
4. study of slowly varying channels (Gabor matrix representations) (for mobile communication, patent);
5. study of all kinds of pseudo-differential operators;
6. Schrödinger equation and so on (PDEs).

Moving between Linear Algebra and Functional Analysis

There are different aspects of the MATLAB experiments and demos given in this course (and beyond). They serve and served several different purposes:

1. demonstration, motivation (a posteriori)
2. simulation, providing intuition (a priori): exploration, e.g. concerning Gabor multipliers
3. what about *quantitative aspects*: Can you deduce from the finite, discrete case what will happen in the continuous limit?

The last step requires various arguments from functional analysis (typically w^*-convergence). As we will try show for all the hard questions the setting of the so-called Banach Gelfand-Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ will provide a satisfactory answer.

Various Paths through Gabor-Land

There are many "GUIDED TOURS" through the landscape of Gabor Analysis, e.g.

1. Wiener Amalgam Spaces for Gabor Analysis;
2. Applications of Gabor Analysis to *underspread operators*;
3. Modulation Spaces and Gabor Expansions;
4. Banach Frames for Modulation Spaces;
5. Functional Analytic Methods for Time-Frequency Analysis;
6. The Banach Gelfand Triple$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and Fourier Analysis;
7. Efficient methods for computational Gabor Analysis;
8. Modulation Spaces and Coorbit Theory;
9. Fourier Standard Spaces: $(\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d))$.

The Actual Plan

We will pick out the following two aspects:

1. First we recall some linear algebra concepts;
2. Using MATLAB we then illustrate these concepts;
3. Moving to the continuous setting we point out the relevance of the so-called Banach Gelfand triple;
4. Discuss a number of topics typically for Gabor Analysis.

H. G. Feichtinger and F. Luef.
Wiener amalgam spaces for the Fundamental Identity of Gabor Analysis.
Collect. Math., 57(Extra Volume (2006)):233–253, 2006.

H. G. Feichtinger, F. Luef, and T. Werther.
A guided tour from linear algebra to the foundations of Gabor analysis.
In *Gabor and Wavelet Frames*, volume 10 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 1–49. World Sci. Publ., Hackensack, 2007.

E. Cordero, H. G. Feichtinger, and F. Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

H. G. Feichtinger, W. Kozek, and F. Luef.
Gabor Analysis over finite Abelian groups.
Appl. Comput. Harmon. Anal., 26(2):230–248, 2009.

H. G. Feichtinger.
Gabor expansions of signals: computational aspects and open questions.
In *Landscapes of Time-Frequency Analysis*, volume ATFA17, pages 173–206. Birkhäuser/Springer, 2019.

The linear algebra setting I

First we will discuss Gabor families in the finite-dimensional setting. We are going to expand vectors in $\mathbb{C}^n$ into a (double-indexed) series of building blocks of a specific form.
It is important that the DFT/FFT as well as Gabor analysis is based on the interpretation of finite length (complex-valued) vectors as function on $\boldsymbol{U}_n$, the (multiplicative) group of unit-roots of order n. As it is natural to label vectors $(x_k)_{k=1}^n \in \mathbb{C}^n$ with indices from $\{1, ..., n\}$ we have to fix the convention that the elements of $\boldsymbol{U}_n$ are coming as $\omega^k, k = 0, 1, ..., n-1$, with $\omega = exp(2\pi i/n)$, the primitive unit root of order n. In other words, for us $\boldsymbol{U}_n$ is the collection of unit roots or order n, starting with $1 = \omega^0$, taken in the *mathematical positive sense*.

The linear algebra setting II

This implies immediately that $\mathbf{x} \in \mathbb{C}^n$ viewed as a function on $\boldsymbol{U}_n$, is *even* if and only if $x_k = x_{n+1-k}$, for $k = 1, ..., n$, because the inverse element of $u \in \boldsymbol{U}_n$ is just $\overline{(u)}$, and the coordinates of x have to be equal at conjugate (multiplicative inverse) positions, so we will have $x_2 = x_n$, $x_3 = x_{n-1}$ and so on (and not just naive flip-invariance!

The questions

1. What is a Gabor family $\mathbb{C}^n = \ell^2(\mathbb{Z}_n)$?
2. When is a Gabor family spanning the signal space?
3. How can we compute (minimal norm) coefficients?
4. Efficient code for dual Gabor atom? $(f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda)$
5. When is a Gabor family linear independent?
6. What are the properties of decent Gabor families (condition number, TF-concentration, etc.)
7. How can we build Gabor multipliers $\sum_{\lambda \in \Lambda} m_\lambda P_\lambda$?
8. How can we approximate Gabor multipliers?
9. Uniqueness of representation, condition numbers.

H. G. Feichtinger.
Spline-type spaces in Gabor analysis.
In D. X. Zhou, editor, *Wavelet Analysis: Twenty Years Developments Proceedings of the International Conference of Computational Harmonic Analysis, Hong Kong, China, June 4–8, 2001*, volume 1 of *Ser. Anal.*, pages 100–122. World Sci.Pub., River Edge, NJ, 2002.

H. G. Feichtinger and K. Nowak.
A first survey of Gabor multipliers.
In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.

H. G. Feichtinger, M. Hampejs, and G. Kracher.
Approximation of matrices by Gabor multipliers.
IEEE Signal Proc. Letters, 11(11):883– 886, November 2004.

M. Dörfler and B. Torresani.
Representation of operators in the time-frequency domain and generalized Gabor multipliers.
J. Fourier Anal. Appl., 16(2):261–293, 2010.

K. Gröchenig.
Representation and approximation of pseudodifferential operators by sums of Gabor multipliers.
Appl. Anal., 90(3-4):385–401, 2011.

Some notes on my ETH course

In the course on **Mathematical Methods for Signal Processing** (as in earlier courses, e.g. at TU Muenich) I have tried to start from Linear Algebra, use MATLAB as an illustration and come up with a theory of *mild distributions*, i.e. the space of all the "signals" which have a bounded spectrogram (e.g. sound signals, which may not have pointwise values, but they can be measured in a linear way).

The goal was to establish the basics of Fourier and Gabor Analysis *without relying on Lebesgue integration theory or the Schwartz Theory of Tempered Distributions*.

MATERIAL and LINKS (and MATLAB code) is available from `www.nuhag.eu/ETH20`, and on YouTube.

Several Publications in this direction I

H. G. Feichtinger.
Ingredients for Applied Fourier Analysis.
In *Sharda Conference Feb. 2018*, pages 1–22. Taylor and Francis, 2020.

H. G. Feichtinger and M. S. Jakobsen.
Distribution theory by Riemann integrals.
Mathematical Modelling, Optimization, Analytic and Numerical Solutions, pages 33–76, 2020.

H. G. Feichtinger.
A sequential approach to mild distributions.
Axioms, 9(1):1–25, 2020.

H. G. Feichtinger.
Classical Fourier Analysis via mild distributions.
MESA, Non-linear Studies, 26(4):783–804, 2019.

H. G. Feichtinger and M. S. Jakobsen.
The inner kernel theorem for a certain Segal algebra.
2018, under final revision.

A musical histogram (Callas!)

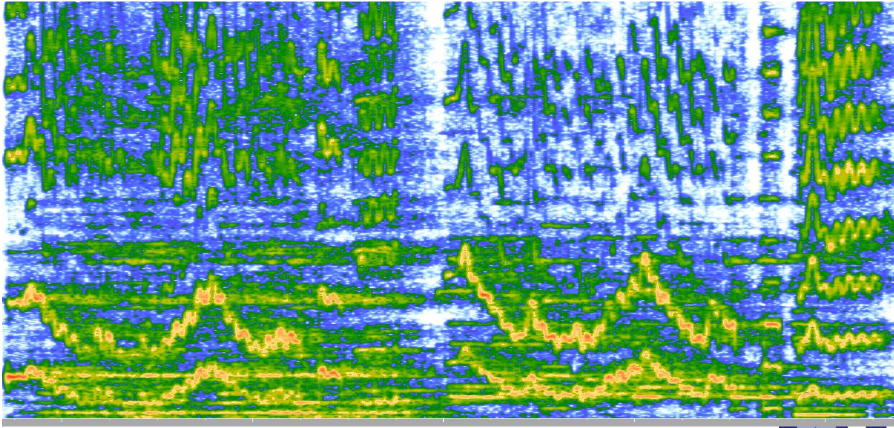
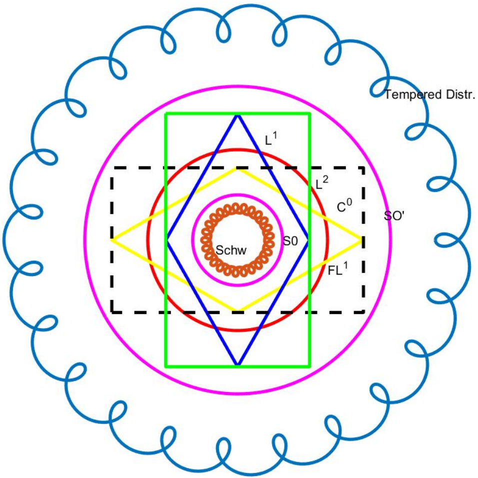


Abbildung: Spectrogramm of a wild area by M. Callas

Summarizing the situation: test functions & distributions



The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{1}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

A few relevant references

K. Gröchenig: Foundations of Time-Frequency Analysis, Birkhäuser, 2001.
H.G. Feichtinger and T. Strohmer: Gabor Analysis, Birkhäuser, 1998.
H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, Birkhäuser, 2003.
G. Folland: Harmonic Analysis in Phase Space. Princeton University Press, 1989.
I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.

Some further books in the field are in preparation, e.g. on modulation spaces and pseudo-differential operators.
See also www.nuhag.eu/talks (!ask for access code!)

Thanks!

THANK you for your attention

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