NuHAG Numerical Harmonic Analysis Group

Multiparameter Families of Function Spaces, Pitfalls and Chances

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at
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Lubos PICK 60th birthday
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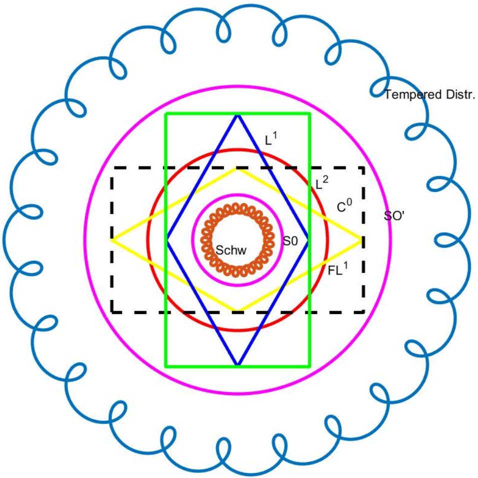
A Zoo of Banach Spaces for Fourier Analysis



Submitted Abstract I

Throughout my work in harmonic analysis and its applications families of function spaces have played a big role, sometimes as an technical tool, sometimes as an independent subject of interest. For me the design of families of function spaces (for me this is Banach spaces of distributions over locally compact Abelian groups, like the Euclidean space) is one of the key challenges. It is natural, that often progress is made by going from a few concrete cases to more general situations, always asking to which extent the existing function spaces are appropriate for a discussion of the continuity properties of linear operators arising in a certain context. Of course the classical Lebesgue spaces are the prototypical examples of such a "scale of spaces", and the masters of interpolation theory (Jaak Peetre and Hans Triebel) have shown us that the well-known Besov-Triebel-Lizorkin spaces form families with similar properties.

Submitted Abstract II

For me their work was the motivation to introduce Wiener amalgam spaces, modulation spaces and decomposition spaces. The short message of this short talk will be: If you extend or generalize a given result to a larger family, you have to do it in the right way. Just to give an example: Having a multi-parameter family of spaces (like weighed Lp-spaces with a family of weights) it is a big difference whether one claims that a given operator is bounded in any of these spaces of this family, or whether (hopefully constructively) a uniform bound can be established. I have seen various cases where the second claim can be obtained from the proofs, while the first one was stated in the results of a paper. Further examples are related to the irregular sampling problem or the theory of Banach frames.

The Topic of this Talk I

At an occasion like this one there is no need to motivate the audience to study function spaces. There exist huge collections of such spaces, and they are still getting more and more general.
A short inspection shows that they often appear in *scales* or *multiparameter families*, just think of (weighted) $\boldsymbol{L}^p$-spaces, or Sobolev spaces, or more generally (inhomogeneous) *Besov spaces* $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$ or the modulation spaces $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$ (partially modelled after Besov spaces).
Instead of providing new families here or derive the boundedness properties of some operators on such function spaces let me take the role of an "advocatus diaboli" with respect to such constructions.
A couple of relevant papers (see bibliography at the end):
fe83:[1], **fe83-4:[2]**, **fe15:[3]**

The Topic of this Talk II

FOR A WARM-UP let me mention the following opinions which I have read/heard over the years several times (paraphrasing the original statements):

- By enlarging the family of spaces we have some chance to improve the information about certain operators!
- The newly constructed spaces can be shown to be strictly larger than the well known spaces used so far!

Critical Comments

Although from a purely logical point if view such statements have some validity, it does not mean that the multi-parameter situation improves the available information.
Let me give a trivial example: Looking at Fourier series expansions the Hausdorff-Young Theorem tells us that

The Topic of this Talk III

Fourier coefficients
in $\ell^p(\mathbb{Z})$ give us functions in $\boldsymbol{L}^q(\mathbb{T})$.
Sometimes (!paraphrased) I get the impression that the following multi-parameter version appears to provide a more general result:

Lemma

If the three parameters p, r, s satisfy

$$1 \leq r \leq p \leq 2 \leq p' \leq s.$$

the Fourier synthesis operator maps $\ell^r(\mathbb{Z})$ boundedly into $\boldsymbol{L}^s(\mathbb{T})$

But I claim that this lemma is
less useful than the Hausdorff-Young Lemma!

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Sampling and Shannon's Sampling Theorem I

The core statement of this section is the Whittaker-Kotelnikov-Shannon Theorem, which states that an $\boldsymbol{L}^2(\mathbb{R})$-function whose Fourier transform is contained in the symmetric interval $I = [-1/2, 1/2]$ around zero (i.e. $\mathrm{supp}(\hat{f}) \subseteq I$) can be completely recovered from regular samples of the form $(f(\alpha n))_{n\in\mathbb{Z}}$ as long as $\alpha \leq 1$.
The reconstruction can be achieved using the so-called SINC-function, with $SINC(t) = sin(\pi t)/\pi t$, the *sinus cardinales*[1], which can be characterized as the inverse Fourier transform of the box-function $\mathbf{1}_I$, the indicator function of I.
It is convenient to apply the following notation:

$$\boldsymbol{B}_I := \{f \mid f \in \boldsymbol{L}^2(\mathbb{R}),\ \mathrm{supp}(\hat{f}) \subseteq I\}, \tag{1}$$

Sampling and Shannon's Sampling Theorem II

Due to the fact that $\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g$ it is clear that we have for every $f \in \boldsymbol{B}_I$:

$$f * \mathrm{SINC} = f. \tag{2}$$

In fact, by the convolution theorem this is a trivial statement, because it just tells us that pointwise multiplication of $\widehat{f}$ with the indicator function of the spectral support (box-car function) reproduces $\widehat{f}$.

[1] The word "cardinal" comes into the picture because of the *Lagrange type* interpolation property of the function *SINC*: $SINC(k) = \delta k, 0$.

Theorem

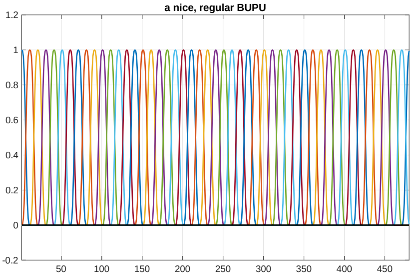
Assume that $\Omega \subset \mathbb{R}^d$ is a bounded subset with the property Ω is disjoint from all its (non-trivial) $\Lambda^\perp$-translates, i.e. $\lambda^\perp + \Omega \cap \Omega = \emptyset$ for all $\lambda^\perp \neq 0, \lambda^\perp \in \Lambda^\perp$. Then every Ω-band-limited function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ can be completely be recovered from the samples $(f(\lambda))_{\lambda\in\Lambda}$ by the Shannon Sampling expansion: $\exists C_\Lambda > 0$:

$$f(t) = C_\Lambda \sum_{\lambda\in\Lambda} f(\lambda) T_\lambda SINC_\Omega(t) = C_\Lambda \sum_{\lambda\in\Lambda} f(\lambda) SINC_\Omega(t-\lambda), \quad (3)$$

where $SINC_\Omega = IFFT(\mathbf{1}_\Omega)$. The convergence takes place in the $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$-sense, but also uniformly (even in $\boldsymbol{C}_0(\mathbb{R}^d)$).

Spline-type spaces

Let us consider the space of *cubic polynomials* in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, then it is known to be generated by the shifted versions of the cubic B-spline, which is the 4-fold convolution product of the box-car function and its shift, along (say) $\mathbb{Z}$.
They also form a BUPU,a Bounded Uniform Partitions of Unity and a Riesz basis for their closed linear span:



Orthogonal Projection

The key-properties of cubic splines in $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$ can be described as follows:

Lemma

The family $(T_n\phi)_{n\in\mathbb{Z}}$ forms a Riesz basis for the closed linear subspace $\boldsymbol{V}_\phi$ of $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$ constituted by the cubic splines in $\boldsymbol{L}^2(\mathbb{R})$. In fact, there is a biorthogonal family $(T_n\tilde{\phi})_{n\in\mathbb{Z}}$, which allows to describe the orthogonal projection onto $\boldsymbol{V}_\phi$ as

$$P_\phi(f) = \sum_{n\in\mathbb{Z}} f * \tilde{\phi}(n) T_n\phi.$$

There is also another function $\varphi \in \boldsymbol{V}_\phi$ such that $(T_n\varphi)_{n\in\mathbb{Z}}$ constitutes an ONB for $\boldsymbol{V}_\phi$.

BUT what happens on $\boldsymbol{L}^p(\mathbb{R})$? Is P_ϕ still continuous there?

LESSON 1

1. Given a family of function spaces one should try to expand the validity of a given formula (usually the Hilbert spaces setting, or the extreme cases) to a larger range of parameters, e.g. from $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R})$ to $\boldsymbol{L}^p(\mathbb{R})$.
2. Sometimes the range is open (Shannon: $1 < p < \infty$), sometimes the range can be a closed set (Spline Projections): $1 \leq p \leq \infty$.
3. While the constants blow up close to the critical boundary in the first case we have uniform bounds on the operator norms in the second case.

Improvements

1. We all know that Shannon's formula can be extended to a larger range of spaces, preserving the uniform boundedness of the synthesis mapping (from $\ell^p(\mathbb{Z})$ to $\left(\boldsymbol{L}^p(\mathbb{R}), \|\cdot\|_p\right)$) by replacing SINC $\notin \boldsymbol{L}^1(\mathbb{R})$ by better kernels $g \in \boldsymbol{L}^1(\mathbb{R})$ (via some smoothness of $\widehat{g}$).
2. In both cases it is in fact improving the claims very much if one makes use of the Wiener amalgam spaces, specifically the space .
3. These spaces are also suitable to formulate the so-called *irregular sampling problem*: can we reconstruct from irregular samples?

Added benefits of Wiener Amalgams

1. Using Wiener amalgam spaces it is possible to derive L^p-versions of results otherwise done via Plancherel's Theorem;
2. They also allow to show robustness with respect to the involved ingredients (i.e. jitter error, minimal norm interpolation in Sobolev spaces with variable degree of smoothness or variable interpolation lattice, etc.);
3. By separating local and globald properties one can use them in order to obtain surprising properties of smooth functions, where the global behaviour completely describes the membership in L^p-spaces (Plancherel-Polya inequalities!)

STFT and Gabor Analysis I

Clearly *Time-Frequency Analysis* and *Gabor Analysis* as well as *Wavelet Theory* has been one of the key problems in my work starting around 1980, where function spaces played an important role, namely for the development of *coorbit theory*, together with K. Gröchenig. In this approach the connection between group representation theory and function spaces (on the group) and function spaces (like modulation spaces) became appearant.

To connect the example above let us first look at the construction of compactly supported wavelets of a given degree of smoothness. Why has this been such a tremendous break-through at this time, and with many consequences.

Here the basic observation is the fact, that a good wavelet system is not only an orthogonal expansion for the Hilbert space, but also provides an *unconditional basis* for the surrounding family of function spaces!

Wavelet Thoughts I

- First of al there was a good match between the coefficients which one obtains from such an (orthogonal) wavelet expansions and the classical function spaces, e.g. $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$;
- Secondly the Calderon-Zygmund operators have a nice characterization via their matrix representation: they are characterized via good off-diagonal decay.
- Functions in certain Besov spaces can be characterized by the *sparsity* resp. decay properties of their wavelet coefficients (for good wavelets), which lead to compressed sensing, etc..

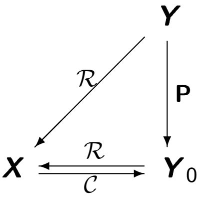
Frames and Banach Frames I

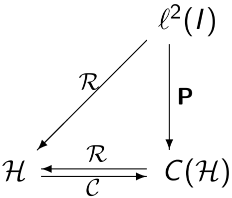
There is a clear connection between the irregular sampling problem for band-limited functions, or functions in spline-type spaces (often called principal shift-invariant spaces, etc.) and the existence of Gabor families (via sampling of the STFT) or wavelet frames (via sampling of the CWT (continuous wavelet transform). *Coorbit Theory* describes the analogy between the two situations (or also *Shearlet Expansions*).
The big difference between STFT and CWT however it the Balian-Low Theorem, which prohibits the existence of even Riesz bases of Gaborian type for $\boldsymbol{L}^2(\mathbb{R}^d)$ and thus unconditional basis for the "surrounding spaces", which are in this case the modulation spaces (see **fegr89:[4]**, **gr91:[5]**, **gr01:[6]**).

Frames via diagrams I

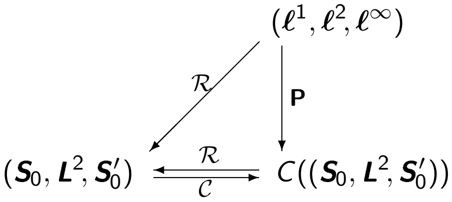
When we think of linear algebra and take an abstract view on *frame theory* then a frame is just a generating system of vectors. By taking coefficients with respect to such a system one obtains an identification of the given finite dimensional space with a closed subspace of a larger index set. There is (a particular) inverse, via the PINV (if we use matrix notation).
For an abstract Hilbert spaces $\mathcal{H}$ a frame provides an identification of the given space $\mathcal{H}$ (e.g. $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$) with a closed subspace of a sequence space, namely $\ell^2()$, for a suitable index set. This can be expressed by a simple double inequality. Every closed subspace is complemented, and thus reconstruction (even from slightly perturbed) coefficients is possible.
$\mathcal{R}$ being a left inverse of $\mathcal{C}$ implies that $\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.

Frames via diagrams II





Frames via diagrams III



Abstract Banach Frames

When extending this concept one has to learn that it is not enough to replace $p = 2$ by a general $p \in [1, \infty)$ (say) in order to have a concept of Banach frames (or atomic decompositions), it is this diagram that has to be completed.
Again, we can learn from the simple cases, that one should first look at the Hilbert space case, to know what one has to do (e.g. construct the canonical dual frame or provide some constructive way to obtain dual frames, as we have done in the context of coorbit theory), instead of just looking at an individual case.

Lesson 2

1. Banach frames should be considered for *families of alike space* (e.g. modulation spaces, shearlet spaces, and so on).
2. One has to make sure that the constructed reconstruction operators are uniformly bounded for (large) subfamilies of spaces, not just individually adapted.
3. In some cases it is possible to prove that the canonical reconstruction operator (using the canonical dual frame) has already nice properties, but sometimes one has other methods and other criteria for suitability (e.g. numerical efficiency).

An experiment of thought

Think of the extension of the Shannon sampling theorem (e.g. the irregular version) to weighted $\boldsymbol{L}^p$-spaces on $\mathbb{R}^d$ with polynomial weights.

How would you rate too different options to extend the Shannon Theorem, formulated in plan words:

VERSION 1: Given a family of weighted spaces the use can give us the data of any band-limited function in any of these spaces, and we are able to reconstruction the function from these samples, if they are dense enough.

VERSION 2: We claim that we have an iterative algorithm which allows to start from the samples of a band-limited functions in any of the spaces from a given family of spaces, and we can guarantee that it will converge with respect to the norm in the corresponding space.

Practical considerations

This simple example shows us, from a practical of view, that despite the similarity of the formulations there is a huge difference in the usefulness and applicability of the two version:

1. In the first case, would you ask the user to tell you from which space he has drawn the samples? What could the answer be? Probably: What do you mean by function spaces??
2. In the first formulation both the algorithm and the required sampling density might depend on the function spaces which you use!
3. So if a function f belongs to different function spaces, which one should the user prefer?
4. In the second case (actual results) we can claim, that a certain density of sampling points guarantees even a joint convergence rate in any of the discussed spaces!!!

Quiz answer!

So why is the more general (??, really) version of Hausdorff-Young less useful than the classical one??

- It is more confusing, because it has many more (useless) parameters;
- !The constants might depend on the pair r, s and p used in the problem description !!

Note: In some other cases the introduciton of new parameters might *not lead to actually new spaces*, but just to a more complicated description of the same spaces that are already established. But this is a different story!!
Sometimes, however, looking for *optimal domains* or *minimal target spaces* gives challenging problems and interesting results.

Usefulness of results

As a final joke let me compare the situation with a real life situation. Would you prefer to obtain a booklet with all the possible connections from A to B, or would you prefer a simple description if the train connections from A to C, knowing what the bus schedule is from C to B??

SO FINAL LESSON: Formulate your results in a way which allows users to memorize the core statement, and leave it the them to derive more complex claims at the time of use. Make it SIMPLE!

Happy Birthday, Lubos!