Integrated Group Actions and the Banach Gelfand Triple (Linearization of Group Actions)

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Submitted Abstract I

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In this talk we would like to report on progress on a program towards rebuilding parts of Harmonic Analysis (HA) from scratch, based on methods from elementary linear functional analysis (FA), namely the concepts of Banach spaces and their duals, bounded linear operators and strong convergence of operators. After all, for me, HA has not only been at the cradle of FA, and almost has given birth to it a century ago, but a big junk of HA (including applications to engineering and physics) is concerned with the boundedness of certain linear operators between suitable Banach spaces (of functions or distributions). These are typically infinite dimensional vector spaces, which have to be equipped with suitable (families of semi-) norms in order to allow for infinite sums

Submitted Abstract II

The historical development of the subject of Fourier Analysis over the last 200 years has led to quite involved concepts, up to the point where experts in pure mathematics derive "abstract" results which are so complicated that even they themselves are not able (or willing?) to apply these results to concrete examples. But who else should take the pain of digesting such complicated results which are sometimes derived with very little motivation beyond the point that one gains "insight at the general level" and obtains "answer to questions that nobody would pose unless he/she can provide a complicated and involved answer". Right, the community is used to many of the complicated concepts and it appears unavoidable that young people have to learn all these techniques in order to make progress and publish a paper. But on the other hand there are so many easy and natural questions which are not covered by the standard mathematical literature, which would help

Submitted Abstract III

engineers or other applied scientist to make better use of mathematical methods, be they at the structural level (e.g. group theoretical interpretations) or in the mathematical description. Just think of the manifold "mystifying" descriptions of the Dirac Delta in articles and even standard introductory books on applied Fourier Analysis or Systems Theory, which leave those poor students with the impression that "Mathematics is just another form of Black Magic".

There is another, widespread aspect for the development of mathematical theory: The more we know about a subject the more we identify the most efficient notions, and the better we can describe (and use) mathematical insight. Just think of the famous formula $exp(2\pi i) = 1$. Good notation may foster the usability of concepts which appear at first sight esoteric (think of $L^1(R^d)$, based on the Lebesgue integral).

Submitted Abstract IV

The goal of this presentation will be to illustrate (only) certain aspects of a general idea, called Conceptual Harmonic Analysis (CHA), which aims at rebuilding major parts of HA, with an orientation towards usefulness for the applied sciences, but still with correct and simple mathematical methods (from FA). In its final stage this concept should provide a basis for developing good algorithms providing quantitative results complementing abstract estimates. These tools should also help to teach the subject properly to young mathematicians and to applied scientists, giving them a solid basis for their research.

More concretely, the methods used to obtain from a strongly continuous representation of a locally compact Abelian group G on a Banach space to the "integrated group representation" (i.e. the linearization of the group representation), and the ideas around Banach Gelfand Triples which appear to be relevant here will be

Submitted Abstract V

addressed. At the inner level often it is a valid idea to replace sums by integrals (in the Riemann sense), just think of the kernel theorem. The expansion to the outer layers then is realized by taking suitable limits. In this way standard tools (like convolution, Fourier transform, etc.) can be transferred to a rather general setting (e.g. LCA groups). A first outline of this program (which is under further development) has been given in my course at ETH in the autumn of 2020 (www.nuhag.eu/ETH20 for the links).

Some Papers relevant for the talk

[fe17] H. G. Feichtinger:
A novel mathematical approach to the theory of translation invariant linear systems.
In I. Pesenson, et al., editors, *Recent Applications of Harmonic Analysis to Function Spaces, Differential Equations, and Data Science.*, pages 483–516. Birkhäuser, Cham, 2017.

[fegu21] H. G. Feichtinger and A. Gumber.
Completeness of shifted dilates in invariant Banach spaces of tempered distributions.
Proc. Amer. Math. Soc., 149(12):5195–5210., 08 2021.

[fe22] H. G. Feichtinger.
Homogeneous Banach spaces as Banach convolution modules over $M(G)$.
Mathematics, 10(3):1–22, 2022.

Standard Approach to Harmonic Analysis

The classical approach considers $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ as the central object of Fourier Analysis (over LCA groups). For simplicity we reduce the presentation here to the case $G = \mathbb{R}^d$.

First we have to define the Lebesgue space $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, which allows us to define central terms of Fourier analysis: *convolution*, the *Fourier transform* (hence deriving the convolution theorem), and *Plancherel's Theorem*, i.e. the possibility of viewing the Fourier (-Plancherel) Transform as a unitary automorphism of the *Hilbert space* $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.

Since $\boldsymbol{C}_c(\mathbb{R}^d)$ is a dense subspace of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ and furthermore the Riemann-integral coincides with the Lebesgue integral on $\boldsymbol{C}_c(\mathbb{R}^d)$, and thus one can view $\boldsymbol{L}^1(\mathbb{R}^d)$ as completion of the normed space $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$. For general LCA groups one has to replace the Riemann integral by the *Haar measure*!

Conceptual Harmonic Analysis

From my point of view the use of Lebesgue spaces in Fourier Analysis makes *the study of the Fourier transform challenging*, but mostly because (!bold claim) aside from the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ the Banach spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p \leq \infty$ is not well suited for the study of questions concerning the Fourier transform, and even less for Gabor Analysis or Time-Frequency Analysis!

It is much more appropriate to make use of THE Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ (which can defined over general LCA groups) in order to access the key problems of Fourier Analysis: convolution, Fourier transform, sampling, periodization and so on, including approximation of continuous problems by discrete ones (FFT for the FT). The corresponding framework is called Conceptual Harmonic Analysis.

Fourier Analysis from Scratch

Despite the existence of the huge amount of literature built up in the last 100 years (since the invention of Lebesgue integration, and then distribution theory), including the characterization of Besov spaces, Shubin classes or modulation spaces making which use the distributional Fourier transform for the space of tempered distribution, it appears worthwhile to reconsider alternative and *more direct* approaches to the core topics of Fourier Analysis.
The first step is the introduce $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ as the dual to the Banach space (in fact pointwise Banach algebra with bounded approximate units) $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, and then to identify $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ with the Banach space of all "multipliers" of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, i.e. of all bounded, linear operators on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ commuting with (all) translations.
This allows to introduce convolution in $\boldsymbol{M}_b(\mathbb{R}^d)$.

`www.nuhag.eu/ETH20 (different links)`

Signals and Systems for Engineers

If we browse engineering books or courses on "Signals and Systems" we can learn about different types of signals:

- discrete or continuous;
- periodic or non-periodic;
- one-dimensional, multi-dimensional (images);
- corresponding concepts of convolution and
- different forms of Fourier transforms adapted to the different settings, e.g. the DFT/FFT for the (i.e. finite) setting!
- (translation invariant) systems are described as convolution operators which are turned into pointwise multiplications operators by the FT (filters).

Wide range of applications: MP3, JPEG, dig. signal processing,...!

TILS: Lüke/Ohm

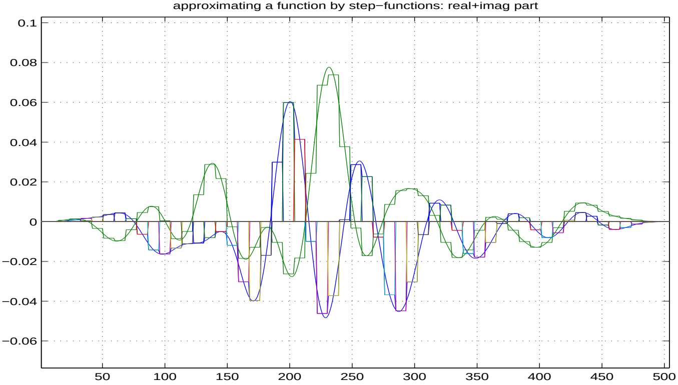

Figure: A typical illustration of an approximation to the input of a TILS T, preparing for the use of the Dirac impulse.

The explanation of this plot I

1. The picture shows (real and imaginary part of a) function, which is approximated by step functions;
2. These step functions are obtained by compressing a boxcar function $\mathbf{1}_{[-1/2,1/2]}$ and shifting it to the correct position;
3. Obviously the step functions get closer to the original function as the spacing gets more and more narrow $(h \to 0)$;
4. On the other hand the compressed rectangular function, all assumed to satisfy $\int_{-\infty}^{\infty} b(x)dx = 1$ tend, in the limit, to δ_0, the Dirac "function", thus justifying the rule

$$\int_{-\infty}^{\infty} \delta(y)dy = 1.$$

One must say, that an attempt to make the statement found in this context mathematically solid claims is a challenging task!

Questions arising from these pictures

- In which sense does the limit of the rectangular functions exist? Maybe the symbol δ or δ_0 is just a *phantom*?
- What kind of argument is given for the transition to integrals? Do we collect (as I learned in the physics course) uncountably many infinitely small terms in order to get the integral?
- In which sense are these step function convergent to the input signal f, e.g. uniformly, or in the $\boldsymbol{L}^1$-sense, and how are the steps determined (samples, local averages)?
- What has to be assumed about the boundedness properties of the operator T? In other words, which kind of convergence of signals in the domain will guarantee corresponding (or different) convergence in the target domain?

Impulse Response and Transfer Function I

The above sketch tries to *intuitively support* the idea of an *impulse response*, we would say $T(\delta_0)$, based on the reported observation that $T(e_\alpha)$ has a limit, which (taken the justification of the following term as granted) is called the impulse response of the system T, since

$$T(\delta_0) = T(\lim_{\alpha\to\infty} e_\alpha) = \lim_{\alpha\to\infty} (T(e_\alpha)) = \mu_0! \tag{1}$$

As it turns out, this formula can be justified under suitable conditions, but not by choosing $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}}) = \left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (as has been observed by I. Sandberg). BUT the main reason is the fact that this space is NOT separable, and that an abstract operator my vanish on $\boldsymbol{C}_0(\mathbb{R}^d)$ but still be non-trivial.
The situation is quite different for $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}}) = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.
Here we can make use of Plancherel's Theorem.

Impulse Response and Transfer Function II

Theorem

Any bounded linear operator $T : \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right) \to \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ *can be characterized via a pointwise multiplication on the Fourier transform side, by some (uniquely determined)* $h \in \left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$, *namely via*

$$\mathcal{F}(T(f)) = \widehat{f} \cdot h, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{2}$$

In fact, this identification is isometric, i.e. satisfies

$$\|h\|_\infty = |||T|||_{\boldsymbol{L}^2(\mathbb{R}^d)} \,.$$

This pointwise multiplier on the Fourier transform is called *transfer function* of the system T. For the case $T(f) = \mu \star f$ with $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ we have of course $h = \widehat{\mu}$.

NEW WAY: Modelling via $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ I

1. We start with the pointwise Banach algebra $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$, of continuous, complex-valued (hence bounded) functions vanishing at infinity, endowed with the sup-norm (written as $\|f\|_\infty$ for $f \in C_b(\mathbb{R}^d)$). It contains $C_c(\mathbb{R}^d)$ as a dense subspace, in fact it coincides with the closure of $C_c(\mathbb{R}^d)$ in $(C_b(\mathbb{R}^d), \|\cdot\|_\infty)$!
2. By (our, justified) definition the bounded linear functionals are called bounded measures, and we use the symbol $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$ for $(C_0'(\mathbb{R}^d), \|\cdot\|_{C_0'})$. Recall that $\mu_0 = w^*\text{-}\lim_{\alpha\to\infty} \mu_\alpha$ if and only for any $f \in C_0(\mathbb{R}^d)$: $\mu_\alpha(f) \to \mu_0(f)$ (w^*—-convergence of measures);
   E.g. $\mu = w^*\text{-}\lim_{|\Psi|\to 0} D_\Psi \mu.$

NEW WAY: Modelling via $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ II

3. Using the simple fact that $\|\delta_x\|_{M_b} = 1$ one observes that the closed linear span of the *Dirac functionals* $\delta_x(f) := f(x)$, for $f \in C_0(\mathbb{R}^d)$, are the discrete [bounded] measures, can be characterized as absolutely convergent series of the form $\nu = \sum_{k=1}^{\infty} c_k \delta_{x_k}$, with $\sum_{k=1}^{\infty} |c_k| < \infty$. We use the symbol $M_d(\mathbb{R}^d)$ for this closed subspace of $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$.

Note that translations acts in a natural way on e.g. $C_b(\mathbb{R}^d)$, given by $T_x(f)(z) = T(z-x)$. The fact, these translations form a commutative group of isometric operators, i.e. satisfy

$$\|T_x f\|_\infty = \|f\|_\infty, \quad f \in C_0(\mathbb{R}^d), x \in \mathbb{R}^d, \tag{3}$$

An object of great interest is the subalgebra of all bounded linear operators which commute with the shift operators! We use the symbol $\mathcal{H}_G(C_0(G))$.

NEW WAY: Modelling via $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ III

Theorem

There is an (natural) isometric identification between translation invariant, linear systems on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ (i.e. bounded linear mappings on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ commuting with translations, and the space $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$.

$$(\mathcal{H}_{\mathbb{R}^d}(C_0(\mathbb{R}^d)), |||\cdot|||) \cong (M_b(\mathbb{R}^d), \|\cdot\|_{M_b}).$$

In fact, every such operator $T \in$ is a convolution operator by a uniquely determined bounded measure μ, we write $C_\mu(f)$ or (later) $\mu * f$, for $f \in C_0(\mathbb{R}^d)$ and $\mu \in M_b(\mathbb{R}^d)$.

The "non-trivial" part is of course to show that $C_\mu(f)$ is not only bounded and (in fact uniformly) continuous, but also still tending to zero at infinity.

Theorem

[Characterization of LTISs on $\boldsymbol{C}_0(\mathbb{R}^d)$]

There is a natural isometric isomorphism between the Banach space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, endowed with the operator norm, and $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, the dual of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, via:

1. *Given a bounded measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the operator C_μ (to be called convolution operator with convolution kernel μ later on) via:*
$$C_\mu f(x) = \mu(T_x f^\checkmark). \tag{4}$$
2. *Conversely we set $\mu = \mu_T$ for a given $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ by*
$$\mu_T(f) = [Tf^\checkmark](0). \tag{5}$$

The claim is that both of these mappings: $C : \mu \mapsto C_\mu$ and the mapping $T \mapsto \mu_T$ are linear, non-expansive, and inverse to each other. Consequently they establish an isometric isomorphism between the two Banach spaces with

$$\|\mu_T\|_{\boldsymbol{M}} = \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \quad \text{and} \quad \|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|\mu\|_{\boldsymbol{M}}. \qquad (6)$$

This is a rather important theorem, allowing to define convolution, and also to derive the convolution theorem, not just in an $\boldsymbol{L}^1$-context, but for general measures!

The previous characterization allows to introduce in a natural way a Banach algebra structure on $\boldsymbol{M}(\mathbb{R}^d)$. In fact, given μ_1 and μ_2 the translation invariant system $C_{\mu_1} \circ C_{\mu_2}$ is represented by a bounded measure μ. In other words, we can define a new (so-called) *convolution product* $\mu = \mu_1 * \mu_2$ of the two bounded measures such that the relation (completely characterizing the measure $\mu_1 * \mu_2$)

$$C_{\mu_1 * \mu_2} = C_{\mu_1} \circ C_{\mu_2} \qquad (7)$$

Of course this observation can be turned into a formal definition of $\mu_1 \star \mu_2$!
It is immediately clear from this definition that $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is a **Banach algebra with respect to convolution**! Associativity is given for free, but *commutativity of the new convolution is not so obvious* (and will follow only later, clearly as a consequence of the commutativity of the underlying group).
The translation operators themselves, i.e. T_z are elements of $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, which correspond exactly to the Dirac measures δ_z, $z \in \mathbb{R}^d$, due to the following simple consideration.

$$C_{\delta_x} f(z) = \delta_x(T_z f^{\checkmark}) = [T_z f^{\checkmark}](x) = f^{\checkmark}(x-z) = f(z-x) = [T_x f](z), \quad (8)$$

and thus $C_{\delta_x} = T_x$, in particular $C_{\delta_0} = T_0 = Id$, resp.

$$f = \delta_0 * f \quad \text{and} \quad T_x f = \delta_x * f, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \quad (9)$$

In the *engineering literature* this (trivial) fact is offered as the *sifting property* of the Dirac delta, and written as

$$\int_{\mathbb{R}^d} f(y)\delta(x-y)dy = f(x).$$

For every $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one has

$$lim_{|\Psi|\to 0} \|\mathrm{D}_\Psi \mu * f - \mu * f\|_\infty = 0. \tag{10}$$

Moreover the limit is uniform for (bounded and) equicontinuous sets $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$.

Here we use BUPU's, i.e. FINE (uniform) partitions of unity, which can be obtained from a given smooth partition of unity, say $\varphi \in \mathcal{D}(\mathbb{R}^d)$, with $\sum_{k \in \mathbb{Z}^d} T_i \varphi(x) = 1$.
Such partitions of unity can be for example the family of cubic B-splines or smooth variants. Also a periodized Gauss function divided by the sum.
For any such BUPU we define a bounded, w^*-w^*- continuous and non-expansive discretization operator

$$\mathrm{D}_\Psi : \mu \mapsto \sum_{i \in I} \mu(\psi_i) \delta_{x_i}.$$

The details of this construction are e.g. in the course notes of my recent ETH20 course: `www.nuhag.eu/ETH20`.

Basic Facts concerning BUPUs

- Given any BUPU $\Psi = (\psi_i)_{i\in I}$ and $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ we have

$$\sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}_b} = \|\mu\|_{\boldsymbol{M}_b}.$$

- $\mathrm{Sp}_\Psi^* = \mathrm{D}_\Psi\mu$, given by

$$\mathrm{Sp}_\Psi(f) := \sum_{i\in I} f(x_i)\psi_i, \quad \mathrm{D}_\Psi\mu = \sum_{i\in I} \mu(\psi_i)\delta_{x_i},$$

with

$$\lim_{|\Psi|\to 0} \|\mathrm{Sp}_\Psi(f) - f\|_\infty = 0, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

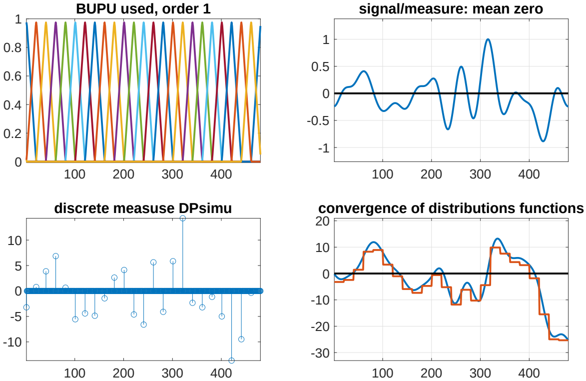

Figure: The (triangular) BUPU, the smooth signal/measure, the discretized measure illustrate by the STEM-command, and the corresponding distribution function, which has a jump of size c_k at the position of the corresponding (positive or negative) Dirac measure.

Alternative Convolutions of Bounded Measures

We have various "kinds of convolutions"

1. linear system C_μ, convolution acting on $\boldsymbol{C}_0(\mathbb{R}^d)$;
2. internal convolution in $\boldsymbol{M}_b(\mathbb{R}^d)$;
3. pointwise convolution in $\boldsymbol{C}_c(\mathbb{R}^d)$ (via Riemannian integrals) or pointwise a.e. for $f, g \in \boldsymbol{L}^1(\mathbb{R}^d)$;
4. the adjoint operators of convolution on $\boldsymbol{C}_0(\mathbb{R}^d)$, also act on $\boldsymbol{M}_b(\mathbb{R}^d) = \boldsymbol{C}_0'(\mathbb{R}^d)$ (coincides with internal convolution with flipped version $\mu^\checkmark$).
5. the classical definition using

$$\mu_1 * \mu_2(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+y) d\mu_1(x) d\mu_2(y).$$

Of course one can show (without too much work) that all these concepts are compatible, i.e. that we have a *commutative* and *associative* convolution, JUST ONE!

The Banach Algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$

Once we have found out that $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is a commutative Banach algebra with *convolution* as multiplication, with identity operator being represented by δ_0, with $\delta_0 * f = T_0(f) = f$, we can define $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ inside of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, where we use the isometric embedding

$$(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1) \hookrightarrow (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}). \qquad (11)$$

For general groups this isometric embedding is provided by the *Haar measure*. It is also compatible with the concept of pointwise multiplication (such as $\mu \cdot k$, with $k \in \boldsymbol{C}_c(\mathbb{R}^d)$) or with translation! We also can go on and define the Fourier (Stieltjes) transform of a bounded measure and verify the *convolution theorem*. Thus we also have $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$, the Fourier algebra, with norm

$$\|\widehat{f}\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)} = \|f\|_{\boldsymbol{L}^1(\mathbb{R}^d)}.$$

Homogeneous Banach Spaces

In a recent paper we have developed an approach to this setting for general LCA groups which does not make use of the Haar measure nor of any kind of measure theory. We only have to provide a method which allows to find "arbitrary fine BUPUs".
In order to derive results related to the well-known estimates

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{L}^p(\mathbb{R}^d) \quad \text{better} \quad \boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{L}^p(\mathbb{R}^d) = \boldsymbol{L}^p(\mathbb{R}^d),$$

together with the corresponding norm estimate we take $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ as a model case.
The defining properties of a homogeneous Banach space are essentially that translation acts as a strongly continuous group of isometric operators on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e.

$$\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad x \in \mathbb{R}^d,$$

$$\lim_{x \to 0} \|T_x f - f\|_{\boldsymbol{B}} = 0, \quad f \in \boldsymbol{B}.$$

For any BUPU $\Psi = (\psi_i)_{i\in I}$ the operator

$$D_\Psi \mu * f = \sum_{i\in I} \mu(\psi_i) T_{x_i} f$$

is well defined and satisfies

$$\|D_\Psi \mu * f\|_{\boldsymbol{B}} \leq \sum_{i\in I} |\mu(\psi_i)| \|T_{x_i} f\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}} \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}_b} \leq \|\mu\|_{\boldsymbol{M}_b} \|f\|_{\boldsymbol{B}}.$$

So it the crucial part for the definition of $\mu * f$ as the limit of $D_\Psi \mu * f$, for $|\Psi| \to 0$. The crucial part is the verification of the property that

$$(D_\Psi \mu * f)_{|\Psi| \to 0}$$

is in fact a *Cauchy net* (a kind of Riemannian sum with values in the Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and "joint refinements".

Wiener Amalgam Spaces

Using this fact we can go on and verify

1. $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$;
2. $\boldsymbol{M}_b * \boldsymbol{W}(\boldsymbol{C}_0, \ell^1) = \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (Wiener's algebra);
3. Also for $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ satisfies

$$\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d).$$

Hence $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$ and $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ are so-called *Segal algebras*. Moreover, these inclusion results come with norm estimates of the form

$$\|\mu * f\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}_b} \|f\|_{\boldsymbol{B}}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d), f \in \boldsymbol{B}.$$

Translation Invariant Spaces

Up to now we assume an isometric action of the group via translation. The same method works more generally:

1. The action does not have to be translation, it is enough to have an isometric, strongly continuous representation ρ of the group $\mathbb{R}^d$ (or any LCA group) on $\boldsymbol{B}$, with

$$\|\rho(x)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \text{and} \quad \lim_{x\to 0} \|\rho(x)f - f\|_{\boldsymbol{B}} = 0.$$

2. Alternatively, we assume only the control of the action by

$$\|\rho(x)f\|_{\boldsymbol{B}} \leq w(x)\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}, x \in \mathbb{R}^d.$$

We will return to this subject later!

Projective Representations

There are different directions how the above setting can be extended further. It suffices that ρ is a *projective representation* of $\mathbb{R}^{2d}$, i.e. the action of *phase space* via *TF*-shifts:

$$\pi(\lambda)(f) = M_s T_t(f), \quad \lambda = (t, s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

The corresponding *integrated group action* is then closely related to the Schrödinger representation of the (reduced) Heisenberg group. For $F \in \boldsymbol{L}^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ one has (synthesis):

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \pi(\lambda) g F(\lambda) d\lambda,$$

This mapping is not only a continuous bilinear mapping from $\boldsymbol{L}^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, but also w^*-continuous, i.e. we can derive the approximation of $V_g^*(F)$ by finite Gabor sums with (uniformly bounded) coefficients in $\left(\ell^1, \|\cdot\|_1\right)$.

The Banach Gelfand Triple I

It is one of the useful facts that among the family of all (non-trivial) Banach spaces of functions or (tempered) distributions which are isometrically invariant under TF-shifts ("cum grano salis") there is a minimal one (namely $(\boldsymbol{S_0}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S_0}}) = (\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1,\ell^1)}) = (\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$ and a maximal one, the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty)(\mathbb{R}^d) = (\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$, the space of so-called mild distributions, or simply (in a certain way) the completion of $\boldsymbol{C_b}(\mathbb{R}^d)$ with respect to locally uniform convergence of the spectrograms.

[fe20-1] H. G. Feichtinger.
A sequential approach to mild distributions.
Axioms, 9(1):1–25, 2020.

The Banach Gelfand Triple II

Together with the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (which can be identified with , the space of mild distributions with a square integrable STFT [defined via Riemann integrals!]), resp. the completion of $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_2)$, we have
THE Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
I guess that in this community my explanations concerning the usefulness of the Banach Gelfand Triple for GABOR ANALYSIS, and more generally TIME-FREQUENCY ANALYSIS, but also for CLASSICAL ANALYSIS are known, at least to some extent.

[fe19] H. G. Feichtinger.
Classical Fourier Analysis via mild distributions.
MESA, Non-linear Studies, 26(4):783–804, 2019.

The Benefits of the Setting I

The benefits of this setting have become visible over the last two decades (actually starting with the first Gabor book), but only slowly we are going to realize the need to rebuild Fourier Analysis from scratch (!), not starting from $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ and measure theory, but rather try to obtain the basic properites of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual as fast as possible, using only basic functional analytic tools ($>>$ Fourier Standard Spaces).

[feko98] H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, pages 233–266. Birkhäuser, Boston, MA, 1998.

[fezi98] H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 123–170. Birkhäuser Boston, 1998.

Fourier Standard Spaces

As pointed out in earlier talks and papers, many of the usual function (and distribution) space are so-called *double Banach modules*, namely both with respect to *convolution* as well as with *pointwise multiplication*

The most natural setting (if we want to avoid weight functions) is to assume that we have Banach spaces in a sandwhich situation

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$$

which are both Banach modules over $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution, but also over the Fourier Stieltjes Algebra $\mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d))$ with respect to pointwise multiplication.

Typically one has isometric action of TF-shifts, as well as the density of $\mathcal{S}(\mathbb{R}^d)$ in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, or simply strongly continuous action of both translation and modulation operators (hence TF-shifts).

Compactness in Fourier Standard Spaces I

Theorem

Given a Fourier Standard Space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, such that $\mathcal{S}(\mathbb{R}^d)$ or $\boldsymbol{S}_0(\mathbb{R}^d)$ is a dense subspace of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then a bounded and closed subset $M \subset \boldsymbol{B}$ is compact if and only if

1. *M is uniformly tight, i.e. for any $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ such that*

$$\|h \cdot f - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M;$$

2. *M is equicontinuous, i.e. for any $\varepsilon > 0$ there exists $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ such that*

$$\|g * f - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M.$$

Compactness in Fourier Standard Spaces II

The proof relies on a few facts:

1. Each of the approximate units, which can be taken from the dense subset $S_0(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$, acts uniformly on a compact subset, but conversely tightness and equicontinuity also imply the same property.
2. Since

$$S_0 \cdot (B * S_0) \subseteq S_0 \cdot (S_0' * S_0) \subset S_0$$

such concatenated approximate units are in fact compact operators from $(S_0'(\mathbb{R}^d), \| \cdot \|_{S_0'})$ to $(S_0(\mathbb{R}^d), \| \cdot \|_{S_0})$.

The weighted case, general Modulation Spaces I

Whereas the assumption of **isometric TF-invariance** restricts the scope of the function spaces which can be discussed significantly, it is still comprehensive enough to allow the treatment of spaces such as $\boldsymbol{L}^p(\mathbb{R}^d), \mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d), \boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ or modulation spaces $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$, or space of Fourier multipliers between $\boldsymbol{L}^q$-spaces, all with their natural norm, it excludes weighted variants of such spaces. So the *key point* in dealing with Fourier Standard Spaces is - aside from a certain level of generality - the simplicity and the restriction to not-so-technical proofs, still providing useful results. Let us just think of the Fourier transform in a natural setting the kernel theorem, or regularization methods.

Readers familiar with the theory of *tempered distributions* (L. Schwartz) know how to adapt the methods to weighted spaces with polynomially moderate weights, using the standard polynomial weight functions $v_s(x) = \langle x \rangle^s = (1 + |x|^2)^{s/2}$, with $s \in \mathbb{R}$.

The weighted case, general Modulation Spaces II

Going even further one can work with *translation and modulation invariant Banach spaces of ultra-distributions*. Essentially this has been the setting of earlier (more abstract) papers, for LCA groups. If one want to have access to the notion of the support of a give (ultra-) distribution one has to restrict the attention to Beurling weights which allow the create compactly supported functions in the Fourier-Beurling algebra. This is known to be equivalent to the (BD) = *Beurling-Domar* (*non-quasi-analyticity*) condition.
From my point of view this setting is the natural setting for distribution spaces over LCA groups, allowing the characterize compactness, take Fourier transforms and so on for the spaces, very much like Banach spaces of tempered distributions satisfying

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

The weighted case, general Modulation Spaces III

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Harmonic analysis based on certain commutative Banach algebras.
Acta Math., 96:1–66, 1956.

[re68] H. Reiter.
Classical Harmonic Analysis and Locally Compact Groups.
Clarendon Press, Oxford, 1968.

[brfe83] W. Braun and H. G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

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Compactness in translation invariant Banach spaces of distributions and compact multipliers.
J. Math. Anal. Appl., 102:289–327, 1984.

The Synthesis of Anti-Wick Operators I

The same arguments can be applied to the synthesis of Anti-Wick operators with a symbol in $\boldsymbol{M_b}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, i.e. for expressions of the form

$$T(f) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} P_\lambda d\mu(\lambda),$$

where we have

$$P_\lambda(f) = \langle f, g_\lambda \rangle g_\lambda = \pi \otimes \pi^*(\lambda)(P_0)(f) := \langle \pi(\lambda)^* f, g_\lambda \rangle \pi(\lambda)(g).$$

Although $\lambda \mapsto \pi(\lambda)$ is only a projective representation, the mapping $\lambda \mapsto \pi \otimes \pi^*(\lambda)$ is a true representation, which acts strongly continuous on the Hilbert-Schmidt operators and other operator spaces.

The Synthesis of Anti-Wick Operators II

The fact that $\pi \otimes \pi^*$ defines a strongly continuous, isometric representation on many Banach spaces of operators (it is a unitary representation on the $\mathcal{HS}$-operators) calls for an equivalent description, which is possible in terms of the Kohn-Nirenberg symbol $\sigma(T)$ of an operator T, which satisifies

$$\sigma[\pi \otimes \pi^*(\lambda) T] = T_\lambda[\sigma(T)], \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

This allows to implement e.g. a Gabor multiplier as a convolution operator via Kohn-Nirenberg calculus.

[sk20] E. Skrettingland.
Quantum harmonic analysis on lattices and Gabor multipliers.
J. Fourier Anal. Appl., 26(3):1–37, 2020.

[de21] M. A. de Gosson.
Introduction to Quantum Harmonic Analysis.
De Gruyter, 2021.

Discretization of Convolution and Coorbit Theory I

Discretization of convolution and the approximation of a convolution product via discretization plays the crucial role in *coorbit theory*. The starting point is the reproducing kernel property of the range of the voice transform, respectively the STFT in the case of *modulation spaces*. For a fixed window $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ we can use for $f, g \in \mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$ capital letters for the transform, i.e. $F := V_g(f)$ and $G = V_g(g)$, which are functions on a LC group (e.g. reduced Heisenberg group $\mathbb{R}^d \times \widehat{\mathbb{R}}^d \times \mathbb{T}$). The key fact is then

$$F = F * G \approx \mathrm{D}_\Psi F * G \approx \mathrm{D}_\Psi{}^+ F * G. \tag{12}$$

Written out in more detail see the adaptive weights concept arising:

$$F * G \approx \sum_{i \in I} \langle F, \psi_i \rangle T_{x_i} G \approx \sum_{i \in I} w_i F(x_i) T_{x_i} G, \quad w_i = \|\psi_i\|_1. \tag{13}$$

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Monatsh. Math., 108(2-3):129–148, 1989.

[fegr92-1] H. G. Feichtinger and K. Gröchenig.
Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view.
In C. K. Chui, editor, *Wavelets :a tutorial in theory and applications*, volume 2 of *Wavelet Anal. Appl.*, pages 359–397. Academic Press, Boston, 1992.

Connection to the Irregular Sampling Problem

Following the "historical path" the ideas first developed in the context of coorbit theory was then transferred to the problem of recovering a band-limited function F from a set of given samples taken at a discrete family of points $X = (x_i)_{i \in I}$. The regular case, i.e. $X = \Lambda \lhd G$ we talk about the Shannon Sampling Theorem.

The PROBLEM with this analogy is that it is not possible to find some $G \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $G * G = G$!! However, one can find (e.g. Schwartz) functions G, H such that $F = F * G$ for F with $\text{supp}(\widehat{F}) \subset \Omega$ (compact) and a second band-limited function H such that $G = G * H$. This suffices in order to establish iterative reconstruction algorithms, which starts from

$$F = F * H \approx F * H \Rightarrow F \approx \mathrm{D}_\Psi F * G, \tag{14}$$

with the benefit that the first approximation is still reproduced under convolution with H !

Literature Concerning Irregular Sampling

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Multidimensional irregular sampling of band-limited functions in L^p-spaces.
Conf. Oberwolfach Feb. 1989, pages 135–142, 1989.

[fe92] H. G. Feichtinger.
New results on regular and irregular sampling based on Wiener amalgams.
In K. Jarosz, editor, *Function Spaces, Proc Conf, Edwardsville/IL (USA) 1990*, volume 136 of *Lect. Notes Pure Appl. Math.*, pages 107–121, New York, 1992. Marcel Dekker.

[fegr93] H. G. Feichtinger and K. Gröchenig.
Error analysis in regular and irregular sampling theory.
Applicable Analysis, 50(3-4):167–189, 1993.

Changing the functions discretized!

Given the discretization of the action of $\boldsymbol{L}^1(\mathbb{R}^d)$ (or better $\boldsymbol{M}_b(\mathbb{R}^d)$) or, in the weighted case of the *Beurling algebra* $(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w})$, acting via convolution on a translation invariant Banach space of distributions, or - in a similar way - a Fourier Beurling algebra $\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d)$ acting via pointwise multiplication on a modulation invariant Banach space, we have to mention that in the case of coorbit theory not the $\boldsymbol{L}^1$-factor in the convolution product is discretized, but so-to-say the $\boldsymbol{L}^p$-factor in the relation $\boldsymbol{L}^1 * \boldsymbol{L}^p \subset \boldsymbol{L}^p$. This requires a bit more of technical details and thus the role of Wiener amalgam spaces (over non-commutative locally compact groups, for the case of coorbit theory) becomes more involved and requires pointwise estimates using the oscillation function of a smooth (say band-limited) function!

Some References to related papers

[dipivi15-1] P. Dimovski, S. Pilipovic, and J. Vindas.
New distribution spaces associated to translation-invariant Banach spaces.
Monatsh. Math., 177(4):495–515, 2015.

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Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces.
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[fegu21] H. G. Feichtinger and A. Gumber.
Completeness of shifted dilates in invariant Banach spaces of tempered distributions.
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Summary of Relevant Tools

As a summary of all the topics raised we can say:

- The use of BUPUs as a general tool is not yet fully exploited. It is the basis for numerical integration methods, but also at the basis of coorbit theory and most algorithms for irregular sampling (for *band-limited functions* or functions in *spline-type spaces*).
- BUPUs are also a crucial tool for estimates involving *Wiener Amalgam Spaces* $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$, because they allow to give a fine description of the *global behaviour of local properties*.
- For illustration: For compact Ω the $\boldsymbol{L}^p$-norm and the $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$-norm are equivalent on the space of Ω-bandlimited functions $(1 \leq p \leq \infty)$. Similarly the assumption $V_g(f) \in \boldsymbol{L}^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is equivalent to the assumption $V_g(f) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^{2d})$.

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