NuHAG Numerical Harmonic Analysis Group

THE Banach Gelfand Triple and its role in Classical Fourier Analysis and Operator Theory

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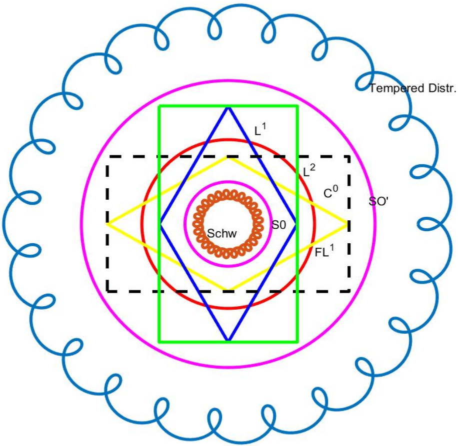
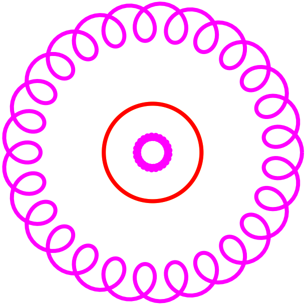
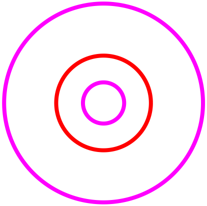
A Zoo of Banach Spaces for Fourier Analysis



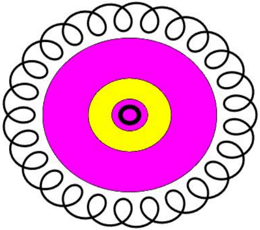
tempered distributions



mild distributions



The Essence: Reducing to the Banach Gelfand Triple



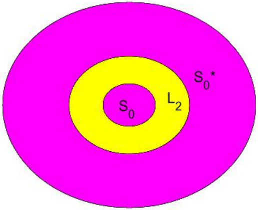
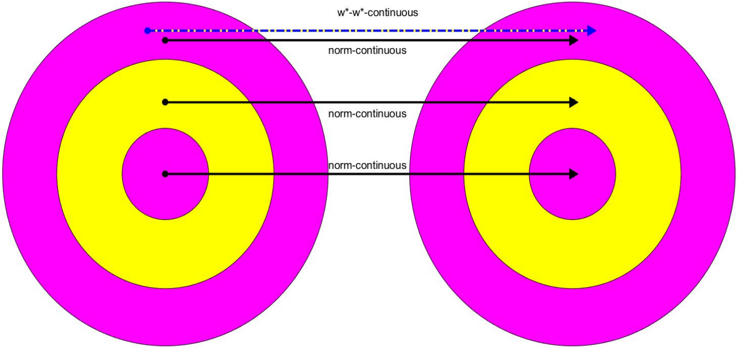


Abbildung: Our GOAL: COMPARISON with $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Banach-Gelfand-Triple-Homomorphisms



The History of Fourier Analysis

From individual function to function spaces

- Classical Fourier Series and Fourier transforms;
- Questions of pointwise (a.e.) convergence;
- Mapping properties, e.g. Hausdorff Young;
- Abstract Harmonic Analysis (LCA groups), convolution;
- Distribution theory (microlocal analysis);
- Computational HA (FFT, FFTW);
- Systems Theory, impulse response, transfer function.

GOAL: Moving from the consideration of the individual function, operator or function spaces to the relationship between functions on different groups and their connections.

Conceptual Harmonic Analysis

AHA (Abstract Harmonic Analysis) is able to understand the similarity between different notions of the Fourier transforms from an axiomatic viewpoint, mostly by *analogy*.
Given a LCA group G it is well known that there is a translation invariant linear functional on $\boldsymbol{C}_c(G)$, called the *Haar measurer*.
Consequently there is an associated Banach space $(\boldsymbol{L}^1(G), \|\cdot\|_1)$, which in fact is a Banach algebra with respect to *convolution*, turning $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ into a commutative Banach algebra.
There is the *dual group* $\widehat{G}$, of all *characters* of G, and once more $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ appears as a natural domain for the Fourier transform, which maps $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ into $(\boldsymbol{C}_0, \|\cdot\|_\infty)$ (by the Riemann-Lebesgue Lemma). In contrast, (Numerical Harmonic Analysis) NHA provides efficient code to realize the FT numerically in the finite, discrete setting (FFT).

The idea of CONCEPTUAL HARMONIC ANALYSIS

The observed *gap between the use of Fourier Analysis in the world of pure mathematics and on the other hand in physics or specifically engineering is motivating my activities towards the establishment of a new look at Fourier analysis, called*

Conceptual Harmonic Analysis (CHA).

On the one hand it tries to unify methods and tools established on both sides in order to help contribute to the solution of real world problems. Among others this includes

the study the relationships between different groups

(how can we approximate the real line, resp. function on the real line, by finite vectors, viewed as functions on $\mathbb{Z}_n$?) and secondly:

How to derive effectively quantitative estimates?

A Look at the Typical Setting

We add to the *functional analytic viewpoint* a *numerical component*. In the sequel T could be some linear operator, which maybe viewed over different domains, e.g. the FT, convolution operator, pseudo-differential operators :

1. Let T be a bounded linear operator on some Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Can we find for any $\varepsilon > 0$ and $f \in \boldsymbol{B}$ a suitable approximation A of T, which can be numerically realized (typically: which can be factorized through some finite dimensional vector spaces) such that

$$\|Tf - Af\|_{\boldsymbol{B}} \leq \varepsilon.$$

2. Given the equation $T(f) = h$, find a computable f_a, with

$$\|f - f_a\|_{\boldsymbol{B}} \leq \varepsilon.$$

Basic Properties of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

QUICK summary of basic properties of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$:

Theorem

1. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach space;*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is continuously embedded into* $L^1 \cap C_0(\mathbb{R}^d)$*;*
3. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra with respect to pointwise multiplication; compactly supported elements are dense;*
4. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is (isometrically) Fourier invariant;*
5. *The norm of* $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is isometrically translation invariant:* $\|T_x f\|_{S_0} = \|f\|_{S_0}, \quad f \in S_0(\mathbb{R}^d), x \in \mathbb{R}^d$.

A compactly supported, continuous function belongs to $S_0(\mathbb{R}^d)$ *if and only if* $\widehat{f} \in L^1(\mathbb{R}^d)$*, e.g. piecewise linear in* $L^1(\mathbb{R})$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x) \quad \textbf{TIME-SHIFT}$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \qquad \textbf{Modulation = FR-Shift}.$$

Behavior under Fourier transform

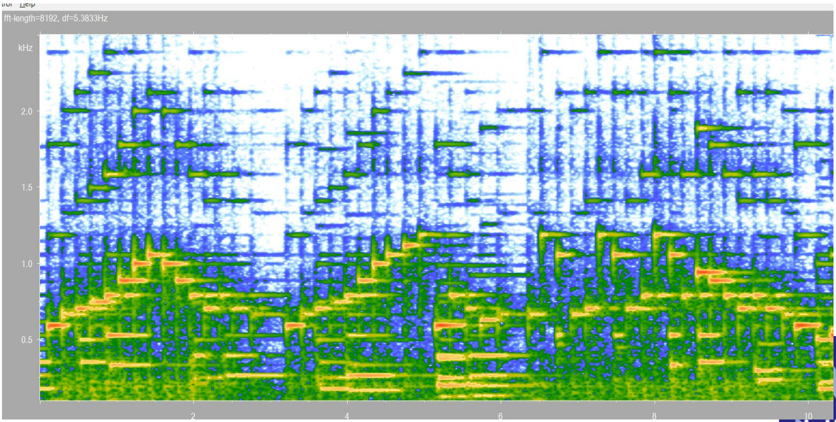
$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

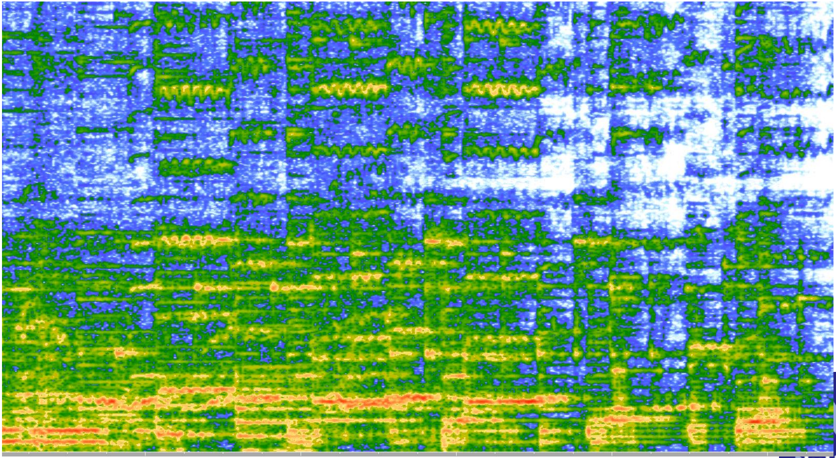
$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \; \lambda = (t, \omega);$$

A Typical Musical STFT

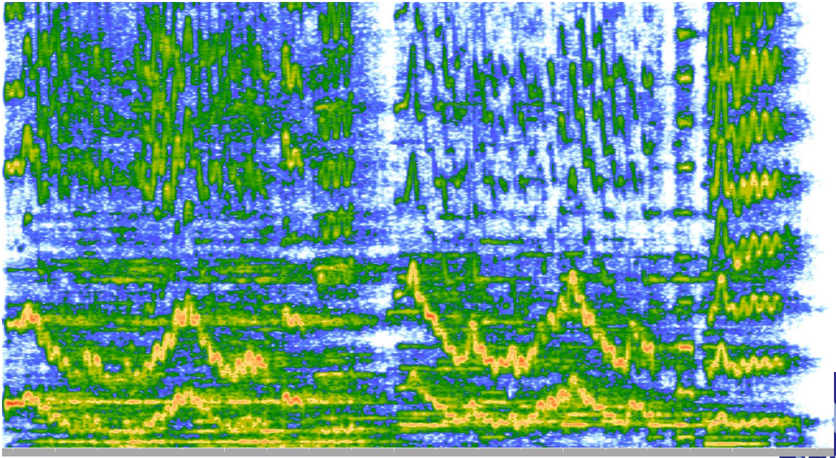
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello

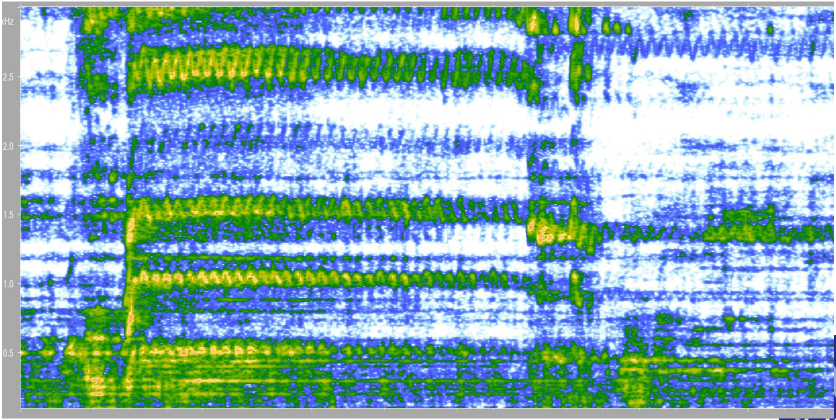


A Musical STFT: Maria Callas



A Musical STFT: Tenor: VINCERA!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

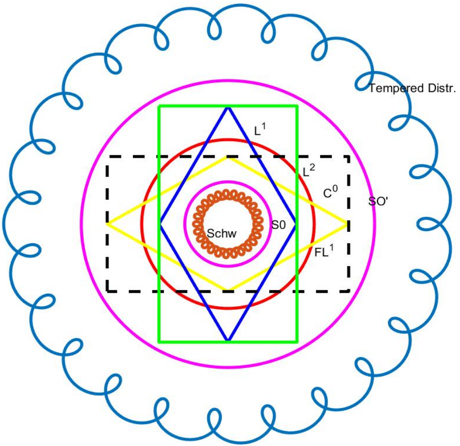
Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|M_u T_\eta f\|_{S_0} = \|f\|_{S_0}, \quad f \in S_0(\mathbb{R}^d). \in S_0(\mathbb{R}^d),$ *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

A Zoo of Banach Spaces for Fourier Analysis



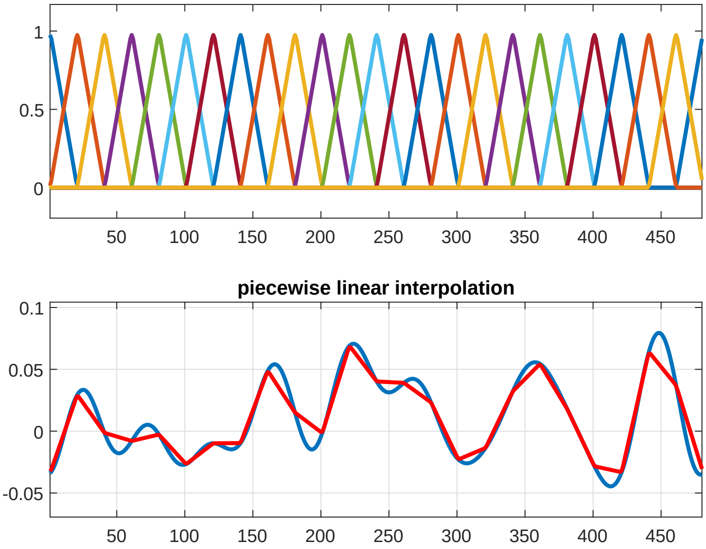
Continuous FT versus FFT/DFT

Engineers simply argue: The computer does not accept continuous functions as input, but only finite vectors. This requires to invoke the DFT/FFT whenever a FT is needed!

Theorem (Feichtinger/Kaiblinger, 2007)

Given any partition of unit, i.e. a family $(T_n\psi)_{n\in\mathbb{Z}^d}$, *with* $\psi \in \boldsymbol{C}_c(\mathbb{R}^d)$ *and* $\widehat{\psi} \in \boldsymbol{L}^1(\mathbb{R}^d)$, *with* $\sum_{n\in\mathbb{Z}^d} \psi(x-n) \equiv 1$ *on* $\mathbb{R}^d$, *then for any* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *one has norm convergence (hence uniform convergence and* $\boldsymbol{L}^p$*-convergence for* $p \in [1,\infty)$ *of the quasi-interpolation operators, for* $\rho \to 0$:

$$Q_\rho f(x) = \sum_{n\in\mathbb{Z}^d} f(\rho n)\psi((x-n)/\rho).$$



(2)

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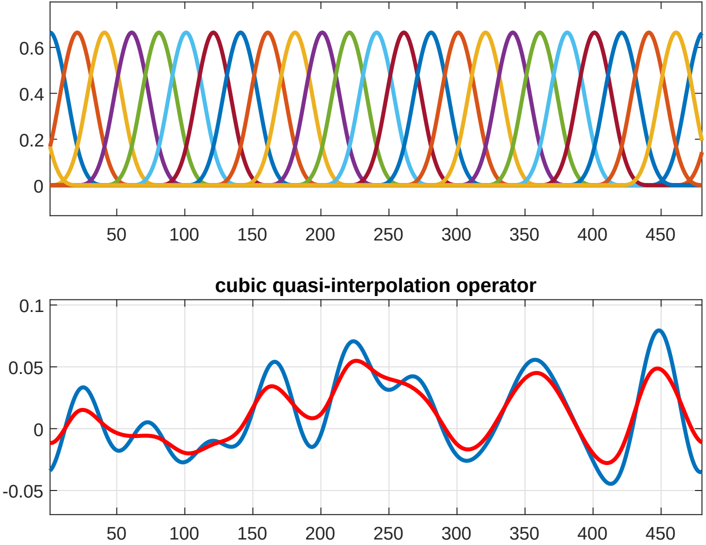

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Applications to the Fourier Transform

One of the basic problems of the FT defined on $\boldsymbol{L}^1(\mathbb{R}^d)$ is the fact that the inversion theorem is *not universally applicable* because $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ does not imply that $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$. The usual way out is to multiply with some decaying function, a so-called summability kernel, and to show that

$$f(x) = \lim_{\rho \to 0} \int_{\mathbb{R}^d} h(\rho s)\widehat{f}(s)e^{2\pi i s t}\, ds.$$

It turns out that all the classical summability kernels (Fejer, De la Vallee Poussin, Gauss), with $h(0) = 1$ belong to $\boldsymbol{S}_0(\mathbb{R}^d)$! And thus $h(\rho s)\widehat{f}(s) \in \boldsymbol{L}^1(\mathbb{R}^d)$ and the inverse Fourier transform works as integral. One has $h = \widehat{g}$, and dilation of h corresponds to $\boldsymbol{L}^1$-compression of g (resulting in a *Dirac sequence*. Hence the limit (for $\rho \to 0$) works for many different function spaces (so called *homogeneous function spaces*).

Poisson's Formula

The validity of Poisson's formula, namely

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n)$$

requires more than just absolute convergence of both sides.

For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ one has $\widehat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, the restriction of f to $\mathbb{Z}^d$ is in $\ell^1(\mathbb{Z}^d)$. The left hand side is the value of the $\mathbb{Z}^d$-periodized version of f. The right hand side is the Fourier series of that periodic function. So all one has to show that the Fourier coefficients of the periodized version of f are just the samples of $\widehat{f}$ at $\mathbb{Z}^d$.

Dirac Combs

The dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space (of tempered distributions), which is also Fourier invariant, by the rule

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), \quad f \in \boldsymbol{S}_0, \sigma \in \boldsymbol{S}_0'.$$

The Dirac Comb belongs to $\boldsymbol{S}_0'(\mathbb{R}^d)$, given as $\sqcup\!\sqcup_{\mathbb{Z}^d} := \sum_{k \in \mathbb{Z}^d} \delta_k$ Poisson's formula translates to

$$\mathcal{F}(\sqcup\!\sqcup_{\mathbb{Z}^d}) = \sqcup\!\sqcup_{\mathbb{Z}^d}.$$

Sampling and Periodization

An important argument for applications in digital signal processing is the use of *regular sampling*, i.e. the multiplication by a Dirac comb.
Note that

$$⧢_{\mathbb{Z}^d} \cdot f = \sum_{n \in \mathbb{Z}^d} f(n)\delta_n, \quad f \in \boldsymbol{C}_b(\mathbb{R}^d),$$

and that it will be a bounded measure for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, since $(f(n))_{n \in \mathbb{Z}^d} \in \boldsymbol{\ell}^1(\mathbb{Z}^d)$. In both cases the sampled version belongs to $\boldsymbol{S}_0'(\mathbb{R}^d)$ and thus has a Fourier transform:

$$\mathcal{F}(⧢_{\mathbb{Z}^d} \cdot f) = \mathcal{F}(⧢_{\mathbb{Z}^d}) * \widehat{f} = ⧢_{\mathbb{Z}^d} * \widehat{f},$$

which is just the $\mathbb{Z}^d$-periodized version of $\widehat{f}$!

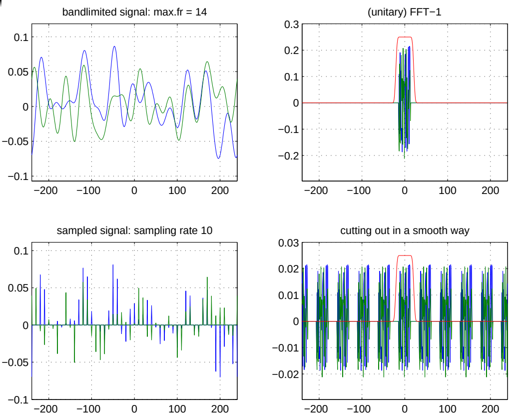

Abbildung: If there is a bit of oversampling, one can choose a better localized reconstruction atom (than SINC).

Generalized Fourier Transform I

The dense embedding of the Schwartz space $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ implies that $\boldsymbol{S}_0'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions.

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0'(\mathbb{R}^d)$ if and only if its STFT (with respect to a fixed, non-zero Schwartz window g, e.g. a Gaussian) is a bounded (and continuous) function on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Convergence in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is equivalent to uniform convergence in the STFT-domain. On the other hand, w^*-convergence corresponds to uniform convergence over compact subsets $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Generalized Fourier Transform II

Of course the generalized FT on $\boldsymbol{S}_0'(\mathbb{R}^d)$ CAN be viewed as the restriction of the FT as defined for tempered distributions. It corresponds (up to phase factors) to a rotation of the spectrogram by 90 degrees, if one takes a Gaussian (Fourier-invariant) window! Since $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ for any $p \in [1, \infty]$ it is also true that $\left(\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_F Lpsp\right)$ is well defined as a subspace of $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.

Why should we care about convolutions?

In engineering, especially in communication theory, the *theory of time-invariant channels*, i.e. operators which are linear and commute with translations (motivated by the time-invariance of physical laws!), also called TILS (time-invariant linear systems) is one of the cornerstones. The collection of all these operators form a commutative algebra, with the pure frequencies as common eigenvalues.

Still in an engineering terminology: Every such system T is a *moving average* or (equivalently) a *convolution operator*, described by the *impulse response* of the system, or alternatively, it can be described as a *Fourier multiplier*, being understood as a pointwise multiplier on the Fourier transform side, by the so-called *transfer function*.

The concrete case of BIBOs systems

From my point of view the most natural setting is that of so-called *BIBOS*, bounded-input-bounded-output systems, or more precisely the bounded linear operators on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ which commute with translations.
It is not hard to show that they are exactly the *convolution operators* which are arising from a given linear functional $\mu \in \boldsymbol{M}_b(\mathbb{R}^d) := \left(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\right)$:

$$\mu * f(x) = \mu(T_x f^{\checkmark}), \quad x \in \mathbb{R}^d, \tag{1}$$

where $f^{\checkmark}(x) = f(-x), x \in \mathbb{R}^d$.

Any TILS on $\boldsymbol{C}_0(\mathbb{R}^d)$ is if this form!

Given T choose $\mu(f) := [T(f^{\checkmark})](0)$, $f \in \boldsymbol{C}_0(\mathbb{R}^d)$.

More general "multipliers"

According to R. Larsen (his book on the multiplier problem appeared in 1972) it is meaningful to ask for a characterization of the space of "multipliers", i.e. all linear, bounded operators from one translation invariant Banach space into another one, which commute with translations. You may think of such operators from $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ to $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$, for two values $p, q \in [1, \infty]$. Using the concept of *quasi-measures* introduced by G. Gaudry he demonstrates that each such operator is a convolution operator by some quasi-measure, which also has an alternative description as a Fourier multiplier. The drawback of the concept of quasi-measures is the fact, that they do not involve any global condition and thus a general quasi-measure does NOT have a well defined Fourier transform. So the expected claim that the transfer function is the FT of the impulse response (and vice versa) cannot be formulated in that context.

Multipliers between L^p-spaces

The key result for multipliers be summarized as follows:

Theorem

Any bounded linear operator from $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ into $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ which commutes with translations can be described by the convolution with a uniquely $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$, i.e. via

$$Tf(x) = \sigma(T_x[f^{\checkmark}]), \quad f \in \mathbf{S}_0(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Thus in fact, T maps $\mathbf{S}_0(\mathbb{R}^d)$ even into $\left(\mathbf{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$, with norm equivalence between the operator norm of the operator T and the functional norm (in $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$) of σ.
If σ is a regular distribution, induced by some test function $h \in \mathbf{S}_0(\mathbb{R}^d)$, then we even have that $f \mapsto h * f$ (equivalently given in a pointwise sense) maps $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ into $\left(\mathbf{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$.

Spectrum of a Bounded Function

The fact that $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ for any $1 \leq p \leq \infty$ implies that one go beyond Hausdorff-Young, which shows that for $1 \leq p \leq 2$ on has $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{L}^q(\mathbb{R}^d)$ (with $1/p + 1/q = 1$). Even for $p > 2$ there is a Fourier transform (locally in $\boldsymbol{PM} = \mathcal{F}\boldsymbol{L}^\infty$).

The (known) statement that $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ is of course much weaker than the claim $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d)$.

This fact allows to directly define the *spectrum* of $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$ using the general definiton:

$$\operatorname{spec}(\sigma) := \operatorname{supp}(\widehat{\sigma}), \quad \sigma \in \boldsymbol{S}_0'.$$

If $\operatorname{supp}(\sigma)$ is a finite set it can be shown to be just a finite sum of Dirac measures, i.e. $\sigma = \sum_{k=1}^K c_k \delta_{x_k}$.

Fourier Analysis and Synthesis

Putting ourselves in the setting of $\boldsymbol{S}_0'(\mathbb{R}^d)$ (which contains all of $\boldsymbol{L}^\infty(\mathbb{R}^d)$, hence $\boldsymbol{C}_b(\mathbb{R}^d)$ and even more special the *pure frequencies*, i.e. the exponential functions $\chi_s(t) = exp(2\pi is \cdot t)$, for $t, s \in \mathbb{R}^d$), we may ask ourselves the following two questions:

1. What are the *pure frequencies* which can be "filtered out of the signal" ($h \in \boldsymbol{C}_b(\mathbb{R}^d)$ or $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$);
2. Can one resynthesize the signal from those pure frequencies (as it is obvious of the case of classical Fourier series expansions of periodic functions);

The KERNEL THEOREM for $\mathcal{S}(\mathbb{R}^d)$

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{2}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (2) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)

One of the corner stones for the kernel theorem is: One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert, showing a kind of uniqueness of the functor $G \mapsto \boldsymbol{S}_0(G)$) is the tensor-product factorization:

The KERNEL THEOREM for S_0 II

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \quad (3)$$

with equivalence of the corresponding norms.

A few more details

Given two functions f^1 and f^2 on $\mathbb{R}^d$ respectively, we set $f^1 \otimes f^2$

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1)f^2(x_2), \;\; x_i \in \mathbb{R}^d, i = 1, 2.$$

Given two Banach spaces $\boldsymbol{B}^1$ and $\boldsymbol{B}^2$ embedded into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, $\boldsymbol{B}^1\hat{\otimes}\boldsymbol{B}^2$ denotes their *projective tensor product*, i.e.

$$\left\{ f \mid f = \sum f_n^1 \otimes f_n^2, \; \sum \|f_n^1\|_{\boldsymbol{B}^1}\|f_n^2\|_{\boldsymbol{B}^2} < \infty \right\}; \quad (4)$$

The KERNEL THEOREM for S_0 III

Any such representation of f will be called an *admissible representation*, and of course this is not unique, because one can add certain terms and subtract part of it later on. There is also often no optimal or canonical representation (as we have it for finite, discrete measures). It is easy to show that (4) defines a Banach space of tempered distributions on $\mathbb{R}^{2d}$, in our case a Banach space of mild distributions or even with respect to the folowing (natural quotient) norm:

$$\|f\|_{\hat{\otimes}} := inf \left\{ \sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2}, \right\}, \qquad (5)$$

where the infimum is taken over all *admissible representations*.

The KERNEL THEOREM for S_0 IV

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g \rangle = \langle k, g \otimes f \rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ FORMALLY (!) as

$$Kf(g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The KERNEL THEOREM for S_0 IV I

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S \rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

The KERNEL THEOREM for S_0 IV II

In analogy to the matrix case we have for $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ for the corresponding operator $T = T_K$:

$$K(x,y) = \delta_x(T(\delta_y)), \quad x,y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism I

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $\left(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')\right)$, *where the first set is understood as the* w^* *to norm continuous operators from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\mathbb{R}^d)$, *the so-called regularizing operators.*

Spreading function and Kohn-Nirenberg symbol I

1. For $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$$

The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \sigma(x,\omega)e^{-2\pi i (y-x)\cdot\omega}\,d\omega\right)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

Spreading function and Kohn-Nirenberg symbol II

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) M_\eta T_{-u} f(x) du d\eta.$$

$\widehat{\sigma}$ is called the spreading function of T_σ.

Further details concerning Kohn-Nirenberg symbol I

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

$$k(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x) = \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y-x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

The kernel $k(x,y)$ can be described as follows (imitating our MATLAB viewpoint: first reshape the matrix to side-diagonal form, then take the FT along side-diagonals).

Kohn-Nirenberg symbol and spreading function II I

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x) f(x-t)\, dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu) f(x-t) e^{2\pi i x\cdot\nu}\, dt\, d\nu$
	$=$
	$\int\int \eta_H(t,\nu) M_\nu T_t f(x), dt\, d\nu\,,$

Spreading representation and commutation relations I

The description of operators through the spreading function and allows to understand a number of commutation relations.
If an operator is a limit (in the strong operator topology) of translation operators it is just a convolution operator with some $\tau \in \boldsymbol{S}_0'(\mathbb{R}^d)$, resp. its spreading representation is just an element concentrated on the *time axis* (more or less representing $\hat{\tau}$, the "individual frequency contributions".

Similarly, multiplication operators require just the use of modulation operators, so their spreading function is concentrated in the frequency axis of the TF-plane.
Finally typical *Gabor frame operators* arising from a family of Gabor atoms (g_λ), where $\lambda \in \Lambda$, some lattice within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ typically commute with TF-shift operators, one can say that they are obtained by periodizing the projection operator $f \mapsto \langle f, g \rangle g$ along the lattice.

The symplectic Fourier transform I

The *symplectic Fourier transform* connects the Kohn-Nirenberg symbol with the spreading function, i.e.

$$\mathcal{F}_s(\sigma(T)) = \eta(T) \quad \text{resp.} \quad \mathcal{F}_s(\eta(T)) = \sigma(T). \tag{6}$$

$$(\mathcal{F}_{symp} f)(k,l) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y) e^{-2\pi i (k\cdot y - l\cdot x)}; \quad f \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d). \tag{7}$$

It is completely characterized by its action on elementary tensors:

$$\mathcal{F}_{symp}(f \otimes \hat{g}) = g \otimes \hat{f}, \quad f,g \in \boldsymbol{S}_0(\mathbb{R}^d), \tag{8}$$

and extends from there in a unique way to a $w^* - w^*$ continuous mapping from $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ to $\boldsymbol{S}_0'(\mathbb{R}^{2d})$, also ${\mathcal{F}_s}^2 = Id$.

The symplectic Fourier transform II

Best approximation by Gabor Multipliers

The Kohn-Nirenberg setting is also the best place, not only to compute a Gabor multiplier (given the ingredients, so ideally a pair (gt, Λ), with $gt \in \boldsymbol{S}_0(\mathbb{R}^d)$, such that the Gabor family $(\pi(\lambda)(gt))_{\lambda \in \Lambda}$ defines a tight Gabor frame) but also to find for a given Hilbert-Schmidt operator T (in the MATLAB situation: a given matrix) the best approximation by a Gabor multiplier based on the pair (gt, Λ), i.e. a collection of coefficients $(m_\lambda)_{\lambda \in \Lambda}$ such that

$$\left\| \sum_{\lambda \in \Lambda} m_\lambda P_\lambda - T \right\|_{\mathcal{HS}} := \min!! \tag{9}$$

In the KNS setting this is like the problem to approximate a given function $f \in \boldsymbol{L}^2(\mathbb{R})$ by linear combinations of shifted B-splines (a

The symplectic Fourier transform III

task which has a nice, FFT-based solution), since the unitary equivalence between Hilbert Schmidt operators (resp. their integral kernels in $\boldsymbol{L}^2(\mathbb{R}^{2d})$) and the corresponding KNS symbol gives the following equivalent task:

$$\left\| \sum_{\lambda \in \Lambda} m_\lambda T_\lambda(\sigma(P_g)) - \sigma(T) \right\|_{\mathcal{HS}} := \min!! \qquad (10)$$

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Aside from being a nice and very useful function space, originally (already in 1979) introduced as the minimal Segal algebra with the property of being isometrically invariant under TF-shifts, the space $\boldsymbol{S}_0(\mathbb{R}^d)$ (and more generally *modulation spaces* (introduced by the author in 1983) turn out to provide the “right setting” for questions of TF-analysis, Gabor Analysis or the theory of *pseudo-differential operators*.
The theory of *coorbit space* developed together with K. Gröchenig (published 1988/1989) provides a unified framework, exposing also the analogy between wavelet theory and time-frequency analysi, resp. the classical Besov-Triebel-Lizorkin spaces and modulation spaces and Wiener amalgam spaces.

As formulated in a series of papers by Jens Fischer, "there is just one Fourier transform".

This statement can be supported by the fact that all the usual forms of Fourier Analysis can be realized inside of $\boldsymbol{S}_0'(\mathbb{R}^d)$, and approximate each other mutually in the w^*-sense.

Thus one can approximate the non-periodic case by the periodic one (the usual trick of letting the period go to infinity), or to approximate a given function (e.g. $f \in \boldsymbol{S}_0(\mathbb{R}^d)$) by a discrete and periodic signal, i.e. a finite linear combination of shifted Dirac combs!

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