NuHAG Numerical Harmonic Analysis Group

# The Role of Finite-dimensional Approximation for Applications

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## Official Abstract

The goal of this talk is to point out how some of the soft and heuristic considerations found in engineering books and courses can be described in a mathematically precise way, based on methods from linear *functional analysis* and *approximation theory*. While the correct term is that of $w^*$-convergence in dual Banach spaces, I will present these concepts in the more concrete setting of so-called *mild distributions* over $R^d$.
Together with the Segal algebra $S_0(R^d)$ and the Hilbert space $L^2(R^d)$ the mild distributions form the so-called *Banach Gelfand Triple* $(S_0, L^2, S_0')$, often compared with the number system $(Q, R, C)$, of rational, real and complex numbers. A natural concept of convergence in $S_0'$ is in fact the $w^*$-convergence, which can be expressed equivalently as the uniform convergence of the spectrograms of such distributions over any compact subset of the *time-frequency plane*.

Alternatively, one can describe it as the norm convergence for any of the projection onto a finite-dimensional subspace of $S_0'$.
We will illustrate the usefulness of this approach (THE Banach Gelfand Triple, and "mild convergence" as I call it now) for the description of transitions between the different worlds of signals discussed in the engineering world (periodic versus non-periodic, discrete versus continuous, etc.).
Classical Fourier Theory starts from integrable function on the torus, and then extends this notation to functions on the real line or $R^d$ (using Lebesgue integration). Numerical computation of the Fourier transform of a decent function is typically realized with the help of the FFT-algorithm, applied to suitable sampling values of the given function. Finally, an elegant proof of the Shannon-Sampling Theorem can be given using the fact that the standard Dirac comb on the real line is invariant under the distributional Fourier transform, in the sense of mild distributions.

In each of these situations, it makes a lot of sense to ask for the mutual dependence of the different methods, and *mild convergence* (another word for distributional convergence) appears as the most suitable unifying concept allowing good answers to those questions. In a way this talk provides one more element in a long series of papers and talks by the speaker, suggesting to make use of the Banach Gelfand Triple in engineering courses, but also for application oriented courses on the Fourier transform and modern applications in *time-frequency analysis*. Course material in this direction can be found at

www.nuhag.eu/ETH20

# Signals in Engineering Applications I

In applied courses one often distinguishes between

1. *periodic* and *non-periodic* functions;
2. discrete versus continuous signals;
3. discrete and periodic signals can be viewed as *finite vectors*;
4. signals can be one-dimensional (audio) or multi-dimensional (e.g., images, tensors, etc.);
5. Sometimes "mysterious" objects like the *Dirac Delta* appears, satisfying the *sifting property*.
6. Maybe one talks about stochastic processes and so on.

# A First Mathematical Answer: I

In applied mathematics courses one often finds some terminology from Abstract Harmonic Analysis (AHA) and linear *functional analysis*.

1. *periodic* functions are actually function on the unit circle (torus) [kind of wrapped up];
2. discrete signals correspond to functions on $\mathbb{Z}$;
3. discrete and periodic signals correspond to function on the *cyclic group* $\mathbb{Z}_N$ of order $N \in \mathbb{N}$.
4. multi-dimensional signals are just functions on product groups, e.g., an $m \times n$ pixel-image is viewed as a function on $\mathbb{Z}_m \times \mathbb{Z}_n$.
5. The *Dirac Delta* is just a bounded measure resp. a *generalized function* (distribution);

# The Mathematical Background I

The mathematical description of signals and their transformations seems to require the following tools:

- Integrability (in order to properly define convolution and the Fourier transform, up to the *convolution theorem*!); thus *Lebesgue integration*;
- At least the Hilbert spaces $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, in order to show that the Fourier transform on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ is a *unitary automorphism*;
- Some theory of *operators* on *Banach spaces*;
- Topological vector spaces, for the theory of *distributions*.

# Why do we need Functional Analysis? I

What are the ingredients needed for clean mathematical treatment of signals and their transformations?

- Many of the relevant transformations $T$ are *linear*, meaning basically that they satisfy the *superposition* principle

$$T(f+g) = T(f) + T(g).$$

  This means that both the domain and the range of $T$ should be *linear spaces* (or vector spaces)!

- In most cases these spaces are so rich that no finite basis can be used (which would allow to describe linear operators by their matrices), meaning that these signal spaces can be *infinite dimensional*[1].
- By consequence one has to invoke methods from *linear functional analysis* (FA).

[1] Just another name for NOT FINITE DIMENSIONAL.

# Banach Spaces I

In order to give - in an infinite dimensional space - also infinite linear combinations or *series* a proper meaning one resorts to *Banach spaces* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. to vector spaces with a *norm*:

$$\|rf + sg\|_{\boldsymbol{B}} \leq |r|\|f\|_{\boldsymbol{B}} + |s|\|g\|_{\boldsymbol{B}}, \quad r, s \in \mathbb{C}.$$

We just require *completeness*, i.e., every *Cauchy sequence* in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is supposed to be convergent.
Linear mappings (transformations) between two such Banach spaces, i.e. $T : (\boldsymbol{B}^1, \|\cdot\|^{(1)}) \to (\boldsymbol{B}^2, \|\cdot\|^{(2)})$ are called bounded if one has

$$\|T(f)\|_{\boldsymbol{B}^2} \leq C\|f\|_{\boldsymbol{B}^1}, \quad f \in \boldsymbol{B}^1.$$

Equivalently, they are continuous, thus mapping convergent sequences in $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to convergent sequences in $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$.

# The Dual Space I

Since obviously $\mathbb{C}$ is such a Banach space with the norm $z \mapsto |z|$ (absolute value) one can consider all the linear, bounded mappings from a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ into $\mathbb{C}$. We call them *linear functionals* on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.
With obvious addition one can turn such a space into a normed space with the norm

$$\|\sigma\|_{\boldsymbol{B}'} := \sup_{f \in \boldsymbol{B}, \|f\|_{\boldsymbol{B}} \leq 1} |\sigma(f)|$$

With this norm $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ is a Banach space, but often too large to allow good finite-dimensional approximations. This will be good enough for the description of the cases, where $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a separable Banach space.

# The Dual Space II

This is one of the main reasons why one has to introduce the $w^*-$topology. We explain only a simple variant of it by declaring

## Definition

A sequence $(_n)_{n\geq 1}$ in $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ is $w^*-$*convergent* with limit $\sigma_0$ if one has for any $f \in \boldsymbol{B}$:

$$\lim_{n\to\infty} \sigma_n(f) = \sigma_0(f), \quad f \in \boldsymbol{B}.$$

By the well-kown *Banach-Steinhaus* principle it is enough t assume that the limit exists for every $f \in \boldsymbol{B}$. Then the limit will be automatically an element of $\boldsymbol{B}'$.

# The Dual Space III

Whereas the unit ball $B := \{f \in \boldsymbol{B} \mid \|f\|_{\boldsymbol{B}} \leq 1\}$ is never compact in the norm topology (expect for the finite dimensional case), it is compact in the $w^*-$-topology. For the dual space $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ of a separable Banach space we can reformulate the Banach-Alaoglou Theorem as follows:

**Theorem**

*For any bounded sequence $(\sigma_n)_{n\geq 1}$ in $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ there exists a subsequence which is $w^*-$convergent to some $\sigma_0 \in (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Moreover*

$$\|\sigma_0\|_{\boldsymbol{B}'} \leq \limsup_{n\to\infty} \|\sigma_n\|_{\boldsymbol{B}'}.$$

Just recall that $\mathbb{R}^n$ with the usual Euclidean structure, is a *Hilbert space*, i.e., it is a Banach space with respect to the ordinary Euclidean norm $\|\mathbf{x}\|_2 := \sqrt{\sum_{k=1}^n |x_k|^2}$, with the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^n} := \sum_{k=1}^{n} x_k y_k.$$

Linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$ are exactly those of the form $T(\mathbf{x}) = \boldsymbol{A} * \mathbf{x}$, for some (real) $m \times n$-matrix $\boldsymbol{A}$. Each row (resp. the choice $m = 1$) shows the dual space of $\mathbb{R}^n$ (thought as column space) is $\mathbb{R}^n$ (the row space), resp. the same norm.
For the standard Banach spaces $\ell^p$ the situation is similar, but for $1 \leq p < \infty$ the dual space is known to be $\ell^q$, with $1/p + 1/q = 1$. We have the reflexive case, unlike $p = 1$. For $\ell^\infty = \ell^{1'}$ the dual space is huge, but in addition to norm convergence we may consider *coordinate-wise convergence* there, e.g. to approximate the constant sequence $\mathbf{1}$ by "finite sequences".

While in the above situation it looks as if the dual space should always be closely related to the original space, let us look at a linear algebra example: the space $P_3(\mathbb{R})$ of *cubic polynomial functions*. It is a 4-dimensional Banach space, with "natural norms", such as

$$\|p(x)\|_{\boldsymbol{B}} := \int_a^b |p(x)| dx$$

for an arbitrary pair of numbers $0 < a < b < \infty$.
The dual space allows *Dirac measures*, i.e. point evaluations $\delta_x$, hence linear combinations of such Dirac measures, of the form

$$p(x) \mapsto \sum_{k=1}^{n} c_k p(x_k).$$

The very definition of the Riemann integral requires to look at the $w^*-$limit of such expressions in order to define $\int_a^b p(x) dx$.

# The key-players for time-frequency analysis

## Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

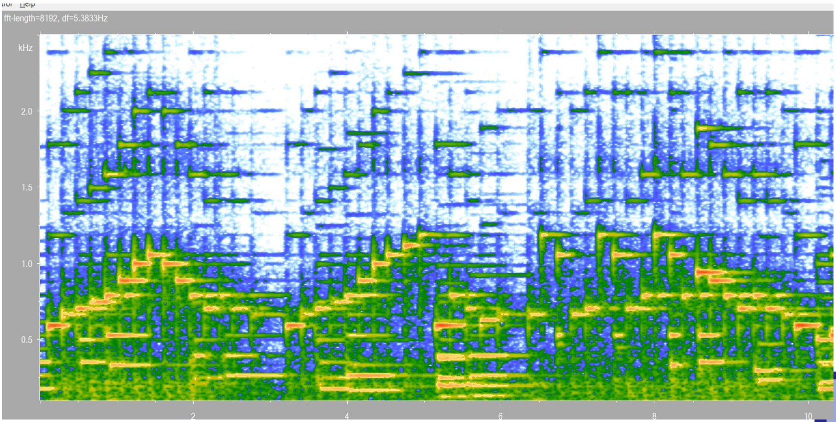
$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

## The Short-Time Fourier Transform

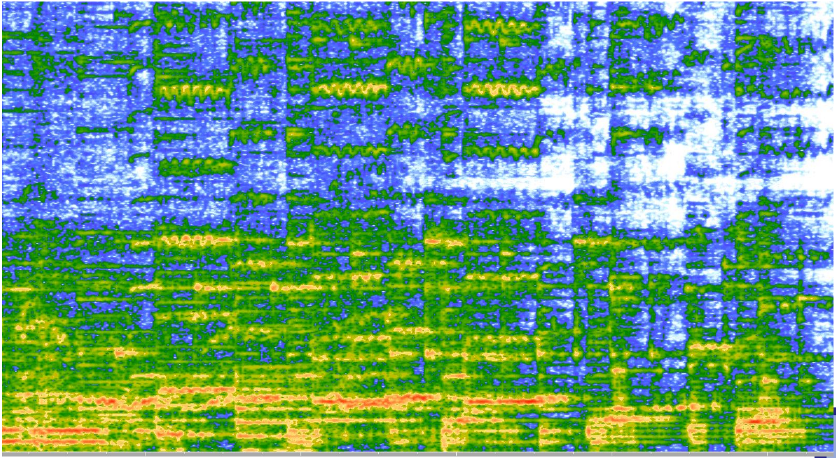
$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

# A Typical Musical STFT

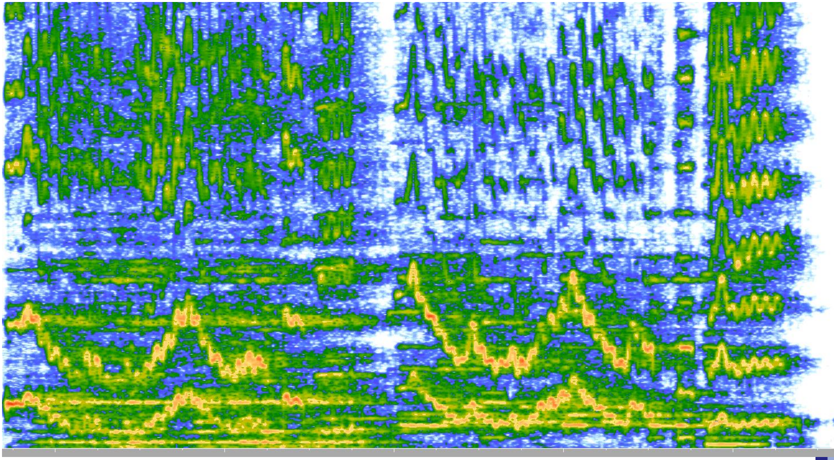
A typical piano spectrogram (Mozart), from recording



# A Musical STFT: Brahms, Cello

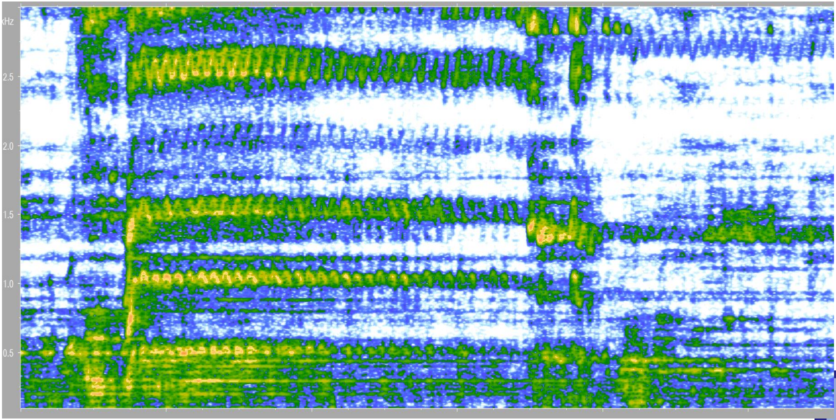


# A Musical STFT: Maria Callas



# A Musical STFT: Tenor: VINCERA!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



# A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero $g$ (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows $g$ define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$x contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

# A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of $V_g$ is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$.
In addition it gives a nice reconstruction formula

$$f = \frac{1}{\|g\|_2{}^2} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \pi(\lambda) g.$$

# Basic properties of $M^1 = S_0(\mathbb{R}^d)$

## Lemma

*Let $f \in S_0(\mathbb{R}^d)$, then the following holds:*

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $L^p$-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

The space $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest member in the family of *modulation spaces* $\left(M^{p,q}(\mathbb{R}^d), \|\cdot\|_{M^{p,q}}\right)$, which can be characterized by suitable mixed norm conditions on the Short-Time Fourier transform of a given function. For $p, q \leq 2$ one can start within $L^2(\mathbb{R}^d)$, while one has to resort to larger spaces (of distributions) otherwise. One can think of $M^{p,q}(\mathbb{R}^d)$ as the (inverse) Fourier image of the amalgam space $W(\mathcal{F}L^p, \ell^q)(\mathbb{R}^d)$. In fact $S_0(\mathbb{R}^d) = M_0^{1,1}(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ in Gröchenig's book. Each of these spaces are in addition *invariant under linear transformations*, and thus especially (anisotropic) dilations. One then goes on to verify that the dual space of $M^{p,q}(\mathbb{R}^d)$ it the modulation space with the conjugate parameters (if $p, q < \infty$), and that they are reflexive Banach spaces for $1 < p, q < \infty$. Any of these spaces (and their Fourier transforms) have nice characterizations via Gabor expansions, with coefficients in suitable weighted, mixed-norm $Lp - Lq$-spaces.

The dual space $S_0'(\mathbb{R}^d) = M^\infty(\mathbb{R}^d)$ (the space of *mild distributions* can be characterized as the space of tempered distributions with bounded spectrograms, and norm (for fixed $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$):

$$\|\sigma\|_{S_0'(\mathbb{R}^d)} = \sup_{(t,s)\in\mathbb{R}^{2d}} |V_g(\sigma)(t,s)|.$$

Obviously the norm convergence in $S_0'(\mathbb{R}^d)$ corresponds to uniform convergence of $V_g(\sigma_n)$ to $_g(\sigma_0)$. In addition, a slight weakening of this concept is very helpful. We summarize a very useful observation (requiring some argument) as follows.

## Lemma

*A bounded net* $(\sigma_\alpha)_{\alpha \in I}$ *of mild distributions in* $(\boldsymbol{S}_0'(\mathbb{R}^d), \| \cdot \|_{\boldsymbol{S}_0'})$ *converges in the* $w^*-$*sense of* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\sigma_0 \in \boldsymbol{S}_0'(\mathbb{R}^d)$ *if and only if we have for any compact subset of* $Q \subset \mathbb{R}^{2d}$, *e.g.* $Q = B_R(0)$, *for some* $R > 0$: *Given* $\varepsilon > 0$ *there exists* $\alpha_0$ *such that* $\alpha \succeq \alpha_0$ *implies*

$$|V_g(\sigma_\alpha)(\lambda) - V_g(\sigma_0)(\lambda)| < \varepsilon, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Instead of discussing the mathematical details (which are not so deep, given the knowledge that we have concerning the involved spaces, just elementary functional analysis is required) we try to use the principle to build intuition concerning concrete cases of $w^*-$convergence.

# BANACH GELFAND TRIPLES: a new category

## Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

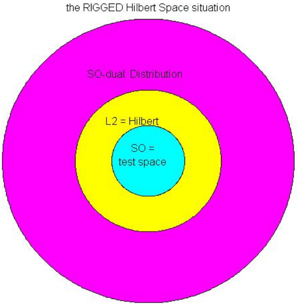
## Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator $T$ is called a [unitary] Gelfand triple isomorphism if

1. $A$ is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. $A$ is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. $A$ extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

# A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and $w^*$)!



# The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

# The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{1}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

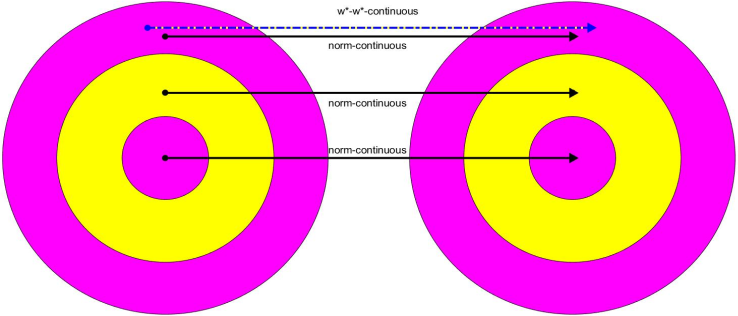



Figure: Banach Gelfand Triple morphism: 3 levels at each side, but FOUR different continuity conditions, due to the extra $w^*$-$w^*$-continuity at the outer level.

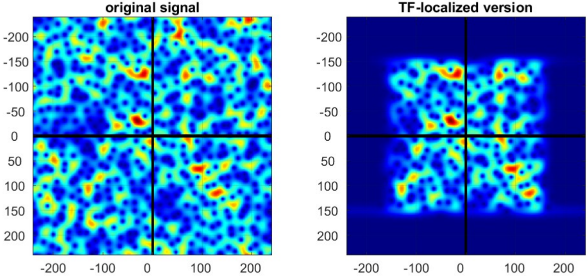



Figure: TFlocrand01.jpg: showing the spectrogram of a random signal and of some $w^*-$approximation, obtained by TF-localization.

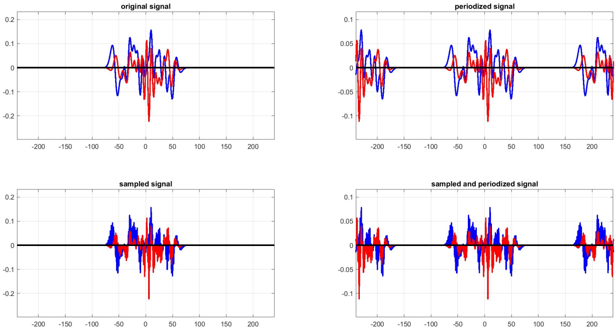



Figure: samper11s.jpg

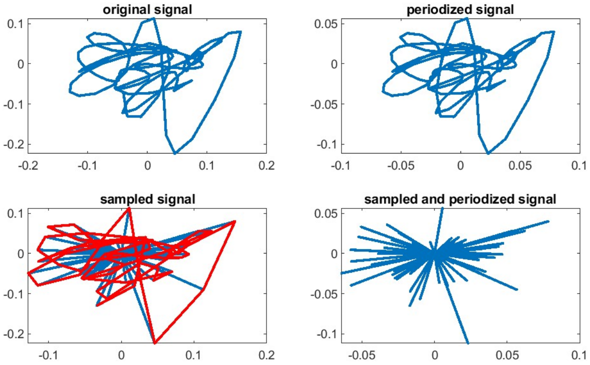



Figure: sampper11sp.jpg: The graphs of different coarse version of a complex-valued, continuous function.

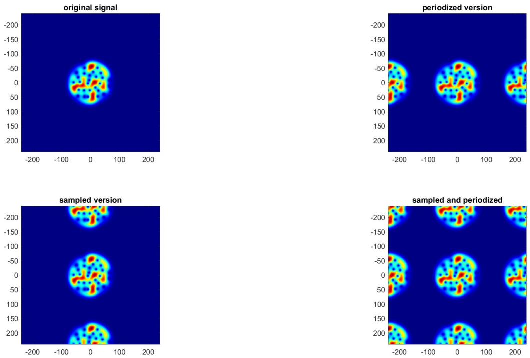



Figure: sampper11.jpg: A

The above pictures suggests that:

- A random signal can be approximated from localized information, very much like a CD-recording captures a piece of music performed on stage, for a certain duration and up to 20kHz.
- Coarse periodization or fine sampling creates periodic patters in the TF-plane, so that the "local observer" (in phase space) finally just sees the central part.
- One may even combine the two procedures, in other words: One can create a discrete and periodic (same as periodic and discrete!) signal which is $w^*-$convergent to the original signal.

There are quite a few obstacles to this viewpoint, if one starts from the classical function spaces, such as $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, not even for $\boldsymbol{L}^2(\mathbb{R}^d)$ (i.e. $p = 2$):

- For $p > 1$ *neither sampling nor periodization makes sense for functions* $f \in \boldsymbol{L}^p(\mathbb{R}^d)$;
- Although Shannon's Sampling Theorem allows to obtain the original, band-limited signal from its sampled version (at the Nyquist rate), the corresonding reconstructions from low-pass versions need not converge in the proper norm (not even in $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$) to the original function.
- For $p = 1$ the classical sampling theorem involving the SINC-function cannot be convergent, because SINC $\notin \boldsymbol{L}^1(\mathbb{R})$.

# $w^*$—convergence of Dirac combs I

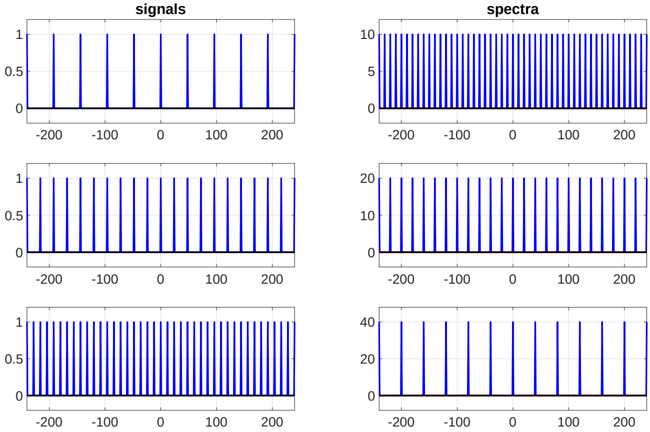



Figure: shahdilates1.eps: Fourier Pairs of Dirac Combs

# $w^*-$convergence of Dirac combs II I

The plot describes Fourier pairs of Dirac combs. We define them as $\sqcup\!\sqcup_b := \sum_{k\in\mathbb{Z}^d} \delta_{bk}$, i.e., $\sqcup\!\sqcup_b(f) = \sum_{k\in\mathbb{Z}^d} f(bk)$, which in fact defines a bounded linear functional on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. In fact we have an estimate of the form

$$\sum_{k\in\mathbb{Z}^d} |f(bk)| \leq C_b \|f\|_{\boldsymbol{S}_0}, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

Sampling can be described as a multiplication with $\sqcup\!\sqcup_b$, via $f \mapsto \sum_{k\in\mathbb{Z}^d} f(bk)\delta_{bk}$, and convolution by $\sqcup\!\sqcup_p$ corresponds to periodization with period $p$. We write $\sqcup\!\sqcup$ for $\sqcup\!\sqcup_1$ in the sequel. Any $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ has a Fourier transform. Poisson's formula, which is valid for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, namely

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

# $w^*-$convergence of Dirac combs II II

is equivalent to the statement that

$$\widehat{\sqcup\!\sqcup} = \sqcup\!\sqcup. \qquad (2)$$

Using the density of $\boldsymbol{C}_c(\mathbb{R}^d) \cap \boldsymbol{S}_0(\mathbb{R}^d)$ in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ we can easily see that

$$w^*\text{-}\lim_{b\to\infty} \sqcup\!\sqcup_b = \delta_0. \qquad (3)$$

On the Fourier transform side we obtain (by dilation) for $a = 1/b$: $\mathcal{F}(\sqcup\!\sqcup_b) = a^d \sqcup\!\sqcup_a$, and by consequence

$$\lim_{a\to 0} a^d \sqcup\!\sqcup_a(g) = \int_{\mathbb{R}^d} g(y)dy = \int_{\mathbb{R}^d} g(y)\mathbf{1}(y)dy, \quad g \in \boldsymbol{S}_0(\mathbb{R}^d),$$

which is more or less the claim that the discrete measures $a^d \sqcup\!\sqcup_a$ converge to the function constant one (viewed as element of

# $w^*$–convergence of Dirac combs II III

$\mathbf{S}_0'(\mathbb{R}^d)$), or in more practical terms: functions $g \in \mathbf{S}_0(\mathbb{R}^d)$ are (absolutely) Riemann integrable and (infinite or finite) Riemannian sums converge to the Riemann integral of $g$ over $\mathbb{R}^d$.
Applying (3) we come up with the claim:

**Lemma**

*For any $g \in \mathbf{S}_0(\mathbb{R}^d)$ we have*

$$w^*\text{-}\lim_{b \to infty} ⊔⊔_b * g = g.$$

It tell us, that every (decent) function in $\mathbf{S}_0(\mathbb{R}^d)$ is the $w^*$–limit of periodic

# Illustrating the localization operators

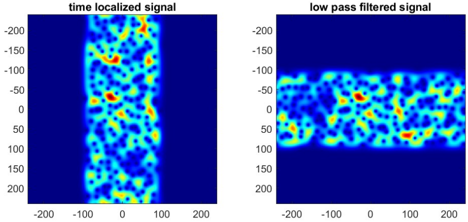



Figure: TFfiltband.jpg

# Double resp. TF-Localization effects

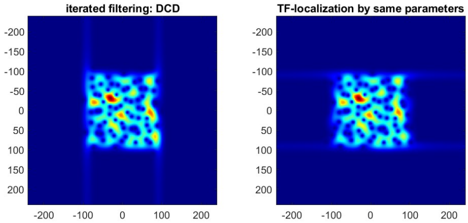



Figure: TFlocalCDC.jpg

# Sampling corresponds to periodization on the FT side

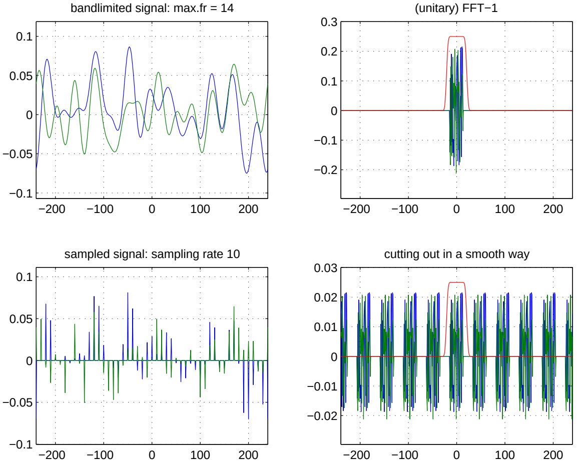



Figure: If there is a bit of oversampling, one can choose a better localized reconstruction atom (than SINC).

# Distributional FT and FFT

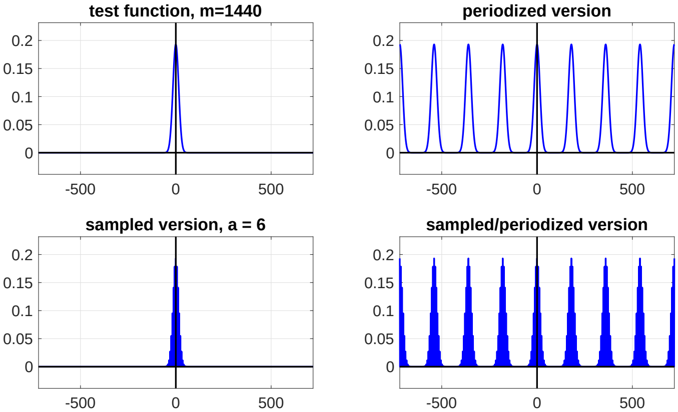



Figure: original(top left), periodize (top right), or sample (left lower corner), or both (right lower corner).

# The Spline Quasi-interpolation operators I

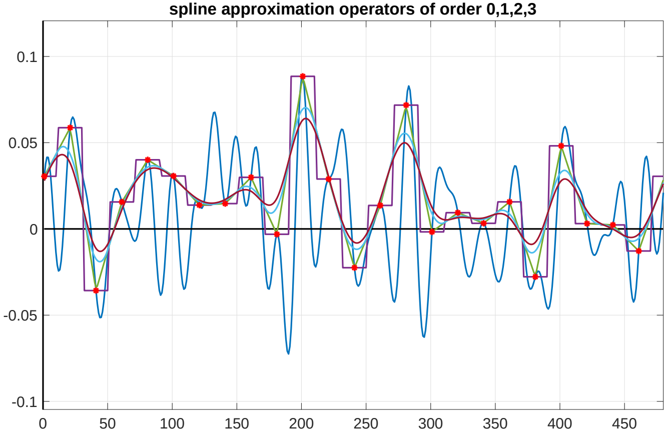



Figure: Approximation by spline functions of order $1, 2, 3, 4$

# Recovery from samples I

The information contained in the samples of $f \in \mathbf{S}_0(\mathbb{R}^d)$ is getting more and more. We know, that for any $f \in \left(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ one has norm convergence of $\mathrm{Sp}_\Psi f$ to $f$ with respect to the sup-norm, as $|\Psi| \to 0$ (think of piecewise linear interpolation over $\mathbb{R}$).
Since the sequence of compressed triangular functions $\mathrm{St}_\rho\Delta, \rho \to 0$ forms a Dirac family one may expect that

$$\mathrm{St}_\rho\Delta * [\alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k)\delta_{\alpha k}] \to \delta_0 * (\mathbf{1}\cdot f) \to f, \quad f \in \mathbf{S}_0. \tag{4}$$

We have to make two observations: First of all this is in fact an alternative description of piecewise linear interpolation, since $\mathrm{D}_{1\alpha}\Delta$ is just a triangular function with basis $[-\alpha, \alpha]$.

$$\mathrm{St}_\rho\Delta * [\alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k)\delta_{\alpha k}] = \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\mathrm{D}_\rho(\Delta). \tag{5}$$

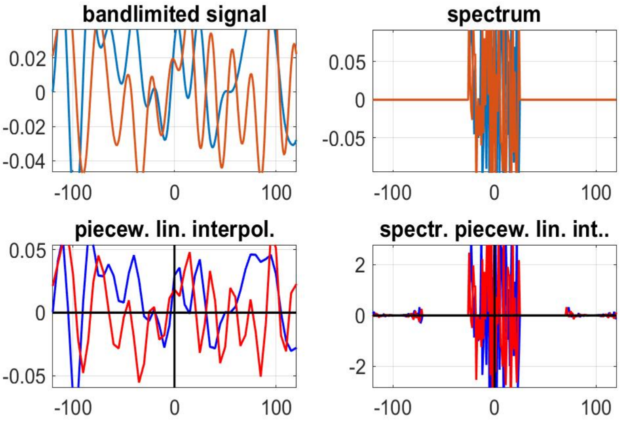



Figure: piecewise linear interpolation of a smooth, i.e. here band-limited signal (partial zoom in, to show the lack of smoothness).

### Theorem

*Assume that* $\Psi = (T_k\psi)_{k\in\mathbb{Z}^d}$ *defines a BUPU in* $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$[a] *and write* $\mathrm{D}_\rho\Psi$ *for the family* $\mathrm{D}_\rho(T_k\psi) = (T_{\alpha k}\mathrm{D}_\rho\Delta)_{k\in\mathbb{Z}^d}$, *with* $\alpha = 1/\rho \to 0$. *Then* $|\mathrm{D}_\rho\Psi| \leq r\alpha \to 0$ *for* $\alpha \to 0$, *and*

$$\|f - \alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\, \mathrm{St}_\alpha\, \psi\|_{\mathbf{S}_0} \to 0, \qquad \text{for } \alpha \to 0, \forall f \in \mathbf{S}_0. \quad (6)$$

[a] As it is required for the definition of $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$, for some $\psi \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ with $\mathrm{supp}(\psi) \subset B_r(0)$.

This result was the cornerstone for the subsequent result published by Norbert Kaiblinger, concerning the approximate computation of the Fourier transform of a “nice function” $f \in \mathbf{S}_0(\mathbb{R}^d)$ via an FFT routine, applied to a sequence of samples of the original function! (a claim which is “obvious” to engineers).

# The role of the FFT I

It is well known that the fast Fourier transform is the workhorse for many of the currently used signal processing tools and behind many applications in digital signal processing.
However it is rarely discussed how it is related to the standard Fourier transform, most often defined as an integral transform, which is valid (in both directions) for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$:

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i s x} dx \quad \text{with inverse} \quad f(t) = \int_{\mathbb{R}^d} \widehat{f}(s) e^{2\pi i t s} ds$$

Both the signals with density in $\boldsymbol{S}_0(\mathbb{R}^d)$, but also the discrete and periodic (DP!) signals (or periodic and discrete signals) have a Fourier transform in the context of *mild distributions*.

# The role of the FFT II

IMPORTANT FACT: The FFT realizes (applied in a suitable way) the transition from a PD-signal to PD=DP signals. Since these signals are $w^*-$dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$ one comes to the conclusion, that the ONLY $w^*$-$w^*$-continuous extension of the FFT algorithm (which applies to all the DP-signals) is the usual Fourier transform in the context of $\boldsymbol{S}_0(\mathbb{R}^d)$, and thus even for $\boldsymbol{S}_0(\mathbb{R}^d)$.

# THANKS to the audience

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