How to define convolution?

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WORKSHOP April 2023
Novi Sad

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Title: How to define convolutions?

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This talk is supposed to provide a panoramic view on different ways of defining the convolution of function, pseudo-measures or distributions, starting from the very classical setting of the Lebesgue space $l^1(R^d)$ where it (still) can be defined in the pointwise sense (almost everywhere), thus turning this Banach space into a commutative Banach algebra with bounded approximate units. This is often taken as a starting point for Fourier Analysis (in particular for the study of the question of spectral analysis via closed ideals of L^1, as outlined in the book of Hans Reiter from 1968 end elsewhere).

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There have been many attempts to extend this notion beyond the Lebesgue setting, e.g. for bounded measures, or for pseudo-measures (the elements of the space FL^∞ in a distributional setting), where one can resort to the pointwise multiplication on the Fourier transform side. On the other hand there have been long-standing attempts to define (at an individual level) the convolution of distributions, with the serious drawback that one may loose the expected rules of associativity.
As a short summary one can say that it is better to avoid pointwise considerations for the definition of convolution, and better connect the possible definitions of convolution with a distributional setting, e.g. in the context of mild distributions.

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Moreover, most if the time good definitions depend on (or can at least be related to) the identification of the distributions which "can be convolved with each other" with corresponding convolution operators between well defined (Banach) spaces of operators, where composition of operators makes sense. Commutativity of convolution can then often be derived via the strong operator topology for such convolution operators by more conventional convolution kernels, e.g. by test functions.

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If time permits we will also shortly discuss the new approach to integrated group actions promoted by the author, which allows to introduce the definition of convolution of bounded measures over LCA groups plus the derivation of the convolution theorem (the Fourier-Stieltjes transform converts convolution into pointwise multiplication of bounded and continuous functions on the frequency domain) without the use of classical measure theory (rather by introducing bounded measures as linear functionals on $C_0(G)$), by identifying this dual space with the space of "multipliers", i.e. bounded linear operators commuting with translations (so-called BIBOS in the engineering literature).
Hans G. Feichtinger 20.03.2023

Standard Approach to Convolution using L^1

The usual approach to convolution (e.g. in Hans Reiter's book) is to first introduce the Lebesgue space $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ and demonstrate that the pointwise definition via

$$f * g(x) = \int_G g(x-y)f(y)dy, \quad x \in \mathcal{G}, \tag{1}$$

makes sense (in the a.e. sense) and that the resulting element of $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ (!equivalence class is well defined) satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in \boldsymbol{L}^1(G). \tag{2}$$

For $\mathcal{G} = \mathbb{R}^d$ one just has to take the usual Lebesgue measure on the Euclidean space $\mathbb{R}^d$, while for general LCA groups one starts from the *Haar measure* (created from the translation invariant, linear functional on $\boldsymbol{C}_c(G)$).

Convolution Theorem

This approach has many advantages, among other because $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ (taken with respect Haar measure) allows to define the Fourier transform as an integral transform

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx, \quad \chi \in \widehat{\mathcal{G}}. \tag{3}$$

Since $\chi : x \mapsto \chi(x)$, a *character*, is a homomorphism from $\mathcal{G}$ into $\mathbb{U} \subset \mathbb{C}$ (unit circle with multiplication) it is clear that $\chi \in \boldsymbol{C}_b(G)$ and thus the integral is well defined, since $|f(x)| = |f(x)\chi(x)|$, $\forall \chi \in \widehat{\mathcal{G}}$. Moreover one can derive the *Convolution Theorem*: Convolution goes to Multiplication.
All these facts can be compressed into one big statement (using the Riemann-Lebesgue Lemma):

The Convolution Theorem

Theorem

$\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ is a Banach algebra with respect to convolution, and the Fourier transform defined via (3) describes an injective and non-expansive homomorphism from the Banach convolution algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ into the pointwise Banach algebra $\left(\boldsymbol{C}_0(\widehat{G}), \|\cdot\|_\infty\right)$, i.e. $\|\widehat{f}\|_\infty \leq \|f\|_1$, and

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g), \quad f, g \in \boldsymbol{L}^1(G).$$

Moreover both $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ and $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$ have (natural) bounded approximate units (Dirac sequences or summability kernels), and the involution $f \mapsto f^* = \overline{f^\checkmark}$, with $f^\checkmark(x) = f(-x)$, goes to conjugation in $\left(\boldsymbol{C}_0(\widehat{G}), \|\cdot\|_\infty\right)$.

Convolution of Bounded Measures I

By the Theorem of Radon-Nikodym we can consider $(\mathbf{L}^1(G), \|\cdot\|_1)$ as a closed subspace of $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$, the space of *bounded, regular Borel measures* on $\mathcal{G}$. In fact, such a measure is *absolutely continuous* if and only if it has a density in $g \in \mathbf{L}^1(G)$, i.e. the measure μ_g is given by:

$$\mu_g(f) = \int_G f(x)\, g(x) dx, \quad f \in \mathbf{C}_c(G).$$

Here we can also define (a more general!) convolution by setting

$$\mu_1 \star \mu_2(f) = \int_G \int_G f(x+y) d\mu_1(x) d\mu_2(y), \quad f \in \mathbf{C}_c(G). \tag{4}$$

Convolution of Bounded Measures II

With some measure theoretic arguments, and using the total variation norm one can show that

$$\|\mu_1 \star \mu_2\|_{\boldsymbol{M}_b} \leq \|\mu_1\|_{\boldsymbol{M}_b} \|\mu_2\|_{\boldsymbol{M}_b}, \quad \mu_1, \mu_2 \in \boldsymbol{M}_b(G).$$

$$\mathcal{F}(\mu_1 * \mu_2) = \mathcal{F}(\mu_1)\mathcal{F}(\mu_2).$$

The F-St-transform naturally extends the FT on $\boldsymbol{L}^1(G)$.

In the theory of *Segal algebras*, or more generally in the context of *homogeneous Banach spaces* over $\mathcal{G}$ convolution is interpreted as a vector-valued integral.
A sloppy argument is the following one: For nice functions f, g also the convolution product is a nice function, and thus we can write

$$g * f(x) = \int_G f^{\checkmark}(y - x)\, g(y) dy = \mu_g(T_x f^{\checkmark}),$$

or alternatively (and it heuristically!):

$$\delta_x(g * f) = (g * f)(x) = \int_G \delta_x(T_y f) g(y) dy = \delta_x\left(\int_G T_y f g(y) dy\right).$$

For a homogenous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (a la Katznelson) the integral exists (at least weakly) because $y \mapsto T_y f$ is a bounded and continuous mapping with values in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, using Bochner integrals.

Invariant Systems I

Engineering students are taught that convolution operators are important because any *time-invariant* linear system T has a representation as a convolution operator (convolution by the so-called *impulse response* μ):

$$T(f) = \mu * f.$$

or as a pointwise multiplier by the *transfer* function τ:

$$\mathcal{F}(Tf) = \tau \cdot \widehat{f}, \quad \text{or} \quad Tf = \mathcal{F}^{-1}(\tau \cdot \widehat{f}).$$

Comparing these formulas one expects of course that

$$\tau = \mathcal{F}(\mu) \quad \text{and} \quad \mu = \mathcal{F}^{-1}(\tau).$$

Obviously the question is whether $\mathcal{F}^{-1}$ and $\mathcal{F}$ are well defined in such cases (e.g. not possible for *quasi-measures*).

Chirp signals

An interesting and challenging case is the family of *characters of second degree* or *chirp signals*. Let us choose

$$Ch(t) = exp(-i\pi t^2), \quad t \in \mathbb{R},$$

which is one of the (distributional) eigenvectors of the Fourier transform, namely satisfies

$$\mathcal{F}(Ch) = Ch.$$

Via *Plancherel's Theorem* the Fourier transform extends from $\boldsymbol{L}^1 \cap \boldsymbol{L}^2(\mathcal{G})$ to a unitary transformation from $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$ to $\left(\boldsymbol{L}^2(\widehat{G}, \|\cdot\|_2)\right)$, here $\widehat{\mathbb{R}} = \mathbb{R}$ and obviously the bounded and continuous function $\tau = Ch$ defines a bounded, linear multiplication operator on $\boldsymbol{L}^2(\widehat{\mathbb{R}})$. BUT!

Problem with pointwise definition

Going to the "time-side" and recalling that

$$\mathcal{F}^{-1}(\mathbf{1}_{[-1/2,1/2]}) = \mathrm{SINC} \in \boldsymbol{L}^2(\mathbb{R}) \setminus \boldsymbol{L}^1(\mathbb{R})$$

we find that the pointwise product, defined as an integral for

$$Ch * \mathrm{SINC}(x) = \int_{\mathbb{R}} \mathrm{SINC}(x-y)Ch(y)dy$$

does not exists as Lebesgue integral, because

$$|\mathrm{SINC}(x-y)Ch(y)| = |\mathrm{SINC}(x-y)| = |T_x(\mathrm{SINC})(y)| \notin \boldsymbol{L}^1(\mathbb{R})$$

for *any* $x \in \mathbb{R}$. This despite the fact that both integrands are continuous, in fact smooth functions!

Tempered Elements in $L^p(\mathbb{R}^d)$

A similar problem appeared in the work of Kelly McKennon in the 70th (subsequent paper by HGFei ([fe77]) where the space of *tempered elements in* $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ is discussed, for $p > 1$. Note that for non-compact groups one can show that $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ is NOT a Banach algebra with respect to convolution, using whatever kind of convolution one takes, or in other words, there does not exist a constant $C_p > 0$ such that

$$\|f * g\|_p \leq C_p \|f\|_p \|g\|_p, \quad f, g \in \boldsymbol{C}_c(G).$$

Note that for $p = 2$, and consequently for $\boldsymbol{L}^p(G) \subset \boldsymbol{L}^1(G) + \boldsymbol{L}^2(G)$ the pointwise product defining the (pointwise) convolution product is (x-a.e.) in $\boldsymbol{L}^1(G)$.

Tempered Elements in $L^p(G)$

Let us describe the set of *tempered elements* in $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ as the set of all functions (equivalence classes) in $\boldsymbol{L}^p(G)$ which at the same time define bounded convolution operator on $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$. Since the pointwise definition may cause problems (e.g. with associativity etc.) one assumes that such a tempered function $g \in \boldsymbol{L}^p(G)$ satisfies:

$$\|g * k\|_{\boldsymbol{L}^p(G)} \leq C_g \|k\|_{\boldsymbol{L}^p(G)}, \quad k \in \boldsymbol{C}_c(G).$$

We can write $\boldsymbol{L}^t_p(G) := \boldsymbol{L}^p(G) \cap \boldsymbol{CV_p}(\mathcal{G})$ for this space, which is endowed with the sum of the two natural norms, the norm in $\boldsymbol{L}^p(G)$ and the operator norm of the convolution operator:

$$\|f\|_{\boldsymbol{L}^t_p} := \|f\|_p + |||C_f|||_{\boldsymbol{L}^p}.$$

Multipliers from $\left(L^p(G), \|\cdot\|_p\right)$

There are some interesting results describing the space of all "multipliers" (bounded linear operators commuting with translations on $\left(\mathbf{L}^p(G), \|\cdot\|_p\right)$) for $1 < p < \infty$.
In a recent paper a new derivation has been obtained using the so-called Herz – Figa-Talamanca spaces (in fact pointwise algebras) $(\mathbf{A}_p(G), \|\cdot\|_{\mathbf{A}_p(G)})$.
We denote by $\mathbf{PM}_p(G)$ (p-pseudomeasures) the dual space of $(\mathbf{A}_p(G), \|\cdot\|_{\mathbf{A}_p(G)})$. Then it the following chain of equality for multiplier spaces (homomorphisms commuting with group action):

$$\mathbf{H}_{\mathcal{G}}(\mathbf{L}^p(G)) = \mathbf{H}_{\mathcal{G}}(\mathbf{L}^t_p(G)) = \mathbf{H}_{\mathcal{G}}(\mathbf{A}_p(G)) \equiv \mathbf{PM}_p(G).$$

Pointwise relationship

One of the (long-standing) open problems in this area, often mentioned in various discussions is the following:
CONJECTURE: *Given $p > 2$, is there any tempered element $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ (or $\boldsymbol{L}^p(G)$) such that the pointwise definition of the convolution product fails, e.g. such that for some $g \in \boldsymbol{L}^p(\mathbb{R}^d)$ there is a problem with the existence of the (pointwise) convolution product, for a set of strictly positive measure?*
It is even conjectured that the more restrictive assumption which would require the existence of the Lebesgue-integral almost everywhere for *every* $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ would not even make a Banach algebra "with convolution".

Convolution via Regularization

Here we point to the work of Michael Oberguggenberger concerning the multiplication of distributions.

Problems with individual definitions

There are a few attempts to define the (convolution or pointwise) product of "generalized functions" for individual pairs of distributions. While such a definition may be useful in a few particular cases one has to ask whether these definitions are natural and whether the expected formulas still hold true, among them the *commutativity* and *associativity law*.
In fact, there are examples in the literature showing that there may convolvable distributions creating problems:

Convolution of distributions is not associative!

Why should we discuss convolution at all?

The answer according to **H. Reiter's book**: Because we have to analyze the properties of the closed ideal structure of $(\boldsymbol{L}^1(G), \|\cdot\|_1)$. But why should be do so: Because we want to do *spectral analysis* of functions in $\boldsymbol{L}^\infty(\mathcal{G})$.
DOWNSIDE: Connection to the original questions has been lost!

The answer of **engineers** will be: Because we want to understand and make efficient use of time-invariant linear systems.
DOWNSIDE: Mathematical rigor is mostly lost, symbolic manipulations and heuristic descriptions prevail!

A rigourous approach to TILS

Taking the applied situation as a model and deriving corresponding mathematical (mostly functional analytic) tools, appears as a viable way to go.

Definition

Let us call a bounded linear operator T on $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$ which commutes with translation, a BIBOS, because it creates bounded input from bounded output, since

$$\|TF\|_\infty \leq |||T|||_{\boldsymbol{C}_0(G)} \|f\|_\infty, \quad f \in \boldsymbol{C}_0(G).$$

Theorem

Any $T \in \mathcal{H}_G(\boldsymbol{C}_0(G))$ can be represented as moving average, by some $\nu \in \boldsymbol{M}_b(G) = \boldsymbol{C}_0'(G)$, namely

$$Tf(x) = [T_x\nu](f), \quad x \in \mathcal{G}.$$

Rewriting convolution

The above relationship between $\nu(f) = Tf(0)$ and the *moving average operator* $T = A_\nu$ is isometric, i.e.

$$\|\nu\|_{\boldsymbol{M}_b(G)} = |||T|||_{\boldsymbol{C}_0(G)} \, |||A_\nu|||_{\boldsymbol{C}_0(G)} \, .$$

Given the bounded measure ν we can write A_ν for the corresponding *moving average operator*, and vice versa, every $T \in \mathcal{H}_G(\boldsymbol{C}_0(G))$ corresponds a unique bounded measures ν (with $\nu(f) = Tf(0)$). This identification is in fact *isometric*!
We can thus transfer the composition law from the Banach algebra of operators to the functionals and call the unique measure which corresponds to the concatenation $A_{\nu_1} \circ A_{\nu_2}$ simply $\nu := \nu_1 \star \nu_2$.

$$A_{\nu_1 \star \nu_2}(f) = A_{\nu_1}(A_{\nu_2}(f)), \quad f \in \boldsymbol{C}_0(G).$$

Convolution

It is clear that any translation operator commutes with all the other translations, thus $T_x \in \mathcal{H}_G(\mathbf{C}_0(G))$ for any $x \in \mathcal{G}$. The corresponding linear functional is $\delta_{-x} : f \mapsto f(-x)$ on $(\mathbf{C}_0(G), \|\cdot\|_\infty)$, obviously with $|||T_x|||_{\mathbf{C}_0(G)} = 1 = \|\delta_{-x}\|$.
In order to establish the *more convenient* relationship between a translation operator and the corresponding Dirac-measure one has the add the inversion mapping (on the group), i.e. $x \mapsto -x$, or the isometric flip-operator on $(\mathbf{C}_0(G), \|\cdot\|_\infty)$: $f \mapsto f^\checkmark$, with $f^\checkmark(x) = f(-x)$.
Thus instead the *more popular* relationship between elements $T \in \mathcal{H}_G(\mathbf{C}_0(G))$ and bounded linear functionals μ on $(\mathbf{C}_0(G), \|\cdot\|_\infty)$ is the choice $\mu = \nu^\checkmark$, with $\mu(f) = \nu(f^\checkmark)$, and obviously $\|\mu\|_{\mathbf{M}_b} = \|\nu\|_{\mathbf{M}_b}$.

TILS via Convolution

Altogether we arrive at the an isometric identification:

Theorem

There is an isometric isomorphism between the closed subalgebra of $\mathcal{H}_G(\boldsymbol{C}_0(G))$ (within the Banach algebra of all bounded linear operators on $(\boldsymbol{C}_0(G), \|\cdot\|_\infty)$) and the dual space $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b}) := (\boldsymbol{C}_0'(G), \|\cdot\|_{\boldsymbol{C}_0'})$, the space of all bounded measures on $\mathcal{G}$ (!by definition!), which is a Banach algebra with respect to convolution.
Given $\mu \in \boldsymbol{M}_b(G)$ we define the convolution operator $C_\mu : f \mapsto C_\mu(f) = \mu * f$ given pointwise by

$$(\mu * f)(x) = \mu(T_x f^\checkmark) = [T_{-x}\mu](f^\checkmark) = T_x\nu(f).$$

$$\text{with} \quad |||C_\mu|||_{\boldsymbol{C}_0(G)} = \|\mu\|_{\boldsymbol{M}_b} = \|\nu\|_{\boldsymbol{M}_b}.$$

Discretization

An important ingredient in the proof of this identification is the observation that $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$ is not only a Banach algebra with respect to *pointwise multiplication*, and thus obviously $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ is Banach module by the adjoint action, with

$$(\mu \cdot h)(f) = \mu(h \cdot f), \quad f, h \in \boldsymbol{C}_0(G), \mu \in \boldsymbol{M}_b(G),$$

combined with the fact that it is an essential Banach module over $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$, or in fact, for any BUPU $\Psi = (\psi_i)_{i\in I}$ one has

$$\sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}_b} = \|\mu\|_{\boldsymbol{M}_b} = \sup_{f\in \boldsymbol{C}_0(G), \|f\|_\infty \le 1} |\mu(f)|.$$

Consequently compactly supported measures are (norm) dense in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$. This is not true for the discrete measures.

Discrete Measures

Discrete measures are the closed linear span of the Dirac measures inside of $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$. They can be described as absolutely convergent sums of Dirac measures, thus

$$\boldsymbol{M}_d(\mathcal{G}) = \{\nu = \sum_{i\in I} c_i \delta_{x_i}, \text{ with } \|\nu\|_{\boldsymbol{M}_b} = \sum_{i\in I} |c_i| < \infty\}.$$

In fact, $\boldsymbol{M}_d(\mathcal{G})$ is a closed subalgebra of $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$.
On the other hand it is true that $\boldsymbol{M}_d(\mathcal{G})$ is w^*–dense in $\boldsymbol{M}_b(G)$, i.e. for any $\mu \in \boldsymbol{M}_b(G)$ and $f_1, ..., f_n \in \boldsymbol{C}_0(G)$ and $\varepsilon > 0$ there exists a discrete measure $\nu \in \boldsymbol{M}_d(\mathcal{G})$ such that

$$|\mu(f_k) - \nu(f_k)| \leq \varepsilon, \quad k = 1, ..., n.$$

A constructive way of obtaining such discrete measures is

$$\nu := \sum_{i\in I} \mu(\psi_i)\delta_{x_i}, \quad \text{with } \operatorname{supp}(\psi_i) \subseteq x_i + U\}.$$

Homogeneous Banach Spaces

According to Y. Katznelson a Banach space of locally integrable functions is called a *homogeneous Banach space* if ones has:

1. for any compact set $K \subset \mathcal{G}$ there exists $C_K > 0$ such that $\int_K |f(x)| dx \leq C_k \|f\|_{\boldsymbol{B}}, \; f \in \boldsymbol{B}$.
2. Translation is isometric: $\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, x \in \mathcal{G}$;
3. $lim_{x \to e} \|T_x f - f\|_{\boldsymbol{B}} = 0, \quad f \in \boldsymbol{B}$.

Typical examples are the spaces $(\boldsymbol{L}^p(G), \|\cdot\|_p)$, for $1 \leq p < \infty$, or any reflexive Lorentz or Orlicz spaces and many others!
The first condition boils down to the assumption that $f \psi_i \in \boldsymbol{M}_b(G)$ for any $i \in I$ (for some BUPU).

Homogeneous Banach Spaces

An old result (published in 1977) can be used as a motivation:

Theorem

Given a tight and w^*-convergent net $(\mu_\alpha)_{\alpha \in I}$ in $\boldsymbol{M}_b(G)$, with limit $\mu_0 \in \boldsymbol{M}_b(G)$, and a homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ on $\mathcal{G}$, then

$$\mu_\alpha * f \to \mu_0 * f \text{ in } (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}),$$

can be taken as a motivation to DEFINE the action of $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ after verifying that the family, indexed by Ψ:

$$\mathrm{D}_\Psi \mu * f = \sum_{i \in I} \mu(\psi_i) T_{x_i} f$$

is a Cauchy net in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, for $|\Psi| \to 0$.

Associativity and Commutativity

While the associativity law, which can be written as

$$(\mu_1 \star_{\mathbf{M}_b(G)} \mu_2) *_{conv} f = \mu_1 *_{conv} (\mu_2 *_{conv} f) \quad f \in \mathbf{C}_0(G),$$

follows DIRECTLY from the associativity in the algebra of operators[1] we have to take care of the commutativity:

$$\mu_1 \star \mu_2 = \mu_2 \star \mu_1, \quad \mu_1, \mu_2 \in \mathbf{M}_b(G).$$

This is true for discrete measures (since $\delta_x \star \delta_y = \delta_{x+y}$), and hence for series of such measures. But $|\Psi| \to 0$ implies

$$\|[\mathrm{D}_\Psi(\mu_1) \star \mathrm{D}_\Psi(\mu_2)] * f - [\mu_1 \star mu_2] * f\|_\infty \to 0$$

[1] comparable to the associativity of matrix multiplication, arising from the composition law for linear mappings.

Action of bounded measrues on homogeneous Bsps

in the recent paper (2022) it has been demonstrated that convolution of the form $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^p(\mathbb{R}^d) \subseteq \boldsymbol{L}^p(\mathbb{R}^d)$ can be obtained without duality considerations (using explicit knowledge of the dual space, or Bochner integrals) by demonstrating that $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{L}^p(\mathbb{R}^d) \subseteq \boldsymbol{L}^p(\mathbb{R}^d)$ with corresponding norm estimates, for general LCA groups, using the fact that for any homogeneous Banach space (defined properly) on has:

1. The existence of arbitrary fine BUPUs (without using the Haar integral!), we write $|\Psi| \to 0$ for such a net;
2. The observation that the discrete convolutions of the form

$$\mathrm{D}_\Psi \mu * f = \sum_{i \in I} \mu(\psi_i) T_{x_i} f$$

form a Cauchy net of elements in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, for any $f \in \boldsymbol{B}$.

NOTE: "**There is just ONE convolution!**".

Invoking the Banach Gelfand Triple

In order to bring the Fourier transform let us quickly recall the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$, with

$$(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^2(G), \|\cdot\|_2) \hookrightarrow (\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}),$$

where $\boldsymbol{S}_0(G) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathcal{G})$, the subspace of function in $\mathcal{F}\boldsymbol{L}^1(\mathcal{G}) \subset \boldsymbol{C}_0(G)$ with $\sum_{\mathbf{i}\in\mathbf{I}} \|\mathbf{f}\psi_{\mathbf{i}}\|_{\mathcal{F}\boldsymbol{L}^1} < \infty$. for some BUPU which is bounded in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathcal{G})$.
Since both $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ and $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (as above) are contained in $\boldsymbol{S}_0'(G)$ one can transfer questions about convolutions to questions of pointwise multiplication on the Fourier
side (and vice versa).

Multipliers

For me the associativity question is usually a (simple) consequence of the connection to operators, specifically convolution operators or so-called *multipliers*, as **opposed to the question of individual composition of two distributions via some special form of extended convolution**.

If these convolution operators or of an elementary nature, be it
(1) convolution by Dirac or discrete measures;
(2) convolution of test functions (e.g. in $\boldsymbol{S}_0(G)$);
neither associativity (Fubini) nor commutativity are a problem (even based on Riemann integrals!).

Theorem

For any $T \in \boldsymbol{H}_{\mathcal{G}}(\boldsymbol{S}_0(G), \boldsymbol{S}_0'(G))$ *there exists a unique* $\sigma \in \boldsymbol{S}_0'(G)$ *such that* $Tf = \sigma * f$, *with* $(\sigma * f)(x) = \sigma(T_x f^{\checkmark})$. *In addition there is norm equivalence between* $\|\sigma\|_{\boldsymbol{S}_0'(G)}$ *and* $|||T|||_{\boldsymbol{S}_0(G)\to\boldsymbol{S}_0'(G)}$.

Regularization

In a way similar to the Schwartz setting, where tempered distributions can be regularized an thus approximated (in the distributional sense) by test functions from $\mathcal{S}(\mathbb{R}^d)$ one can regularize "mild distributions", i.e. elements from $\boldsymbol{S}_0'(G)$ by first smoothing it (by convolution with elements from $\boldsymbol{S}_0(G)$) and then localized the result (by pointwise multiplication by other elements from $\boldsymbol{S}_0(G)$), and using

$$\boldsymbol{S}_0(G) \cdot (\boldsymbol{S}_0(G) * \boldsymbol{S}_0'(G)) \subset \boldsymbol{S}_0(G)$$

one can expect that also convolution operators can be approximated by "decent convolution operators" (which arise from elements of $\boldsymbol{S}_0(G)$), typically in the so-called *ultra-weak sense*, and thus associativity and commutativity can be granted in such cases.

Back to the Chirp

Coming back to the problem of *pointwise convolution* of chirps, with the SINC function or with each other, while this is quite easy on the Fourier transform side:
In such a situation one just has to apply a bounded approximate unit in the Fourier algebra $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$, i.e. some *summability kernel*, in order to get inside of $\boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$, and since $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d)$ everything (the regularization etc.) is well under control on both the time and the frequency side and all the possible interpretations of the convolution product coincide properly.
Final hint: it is similar to the usual re-interpretation of duality pairs in functional analysis, of both sides belong to different Banach spaces!

THANKS

Thanks for your attention

A course description of an approach to Fourier Analysis over $\mathbb{R}^d$, without Lebesgue integrals, is found at

www.nuhag.eu/ETH20

and the links given there.