Conceptual Harmonic Analysis: Tools and Goals
The ubiquitous role of BUPUs

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OVERVIEW

As I tried to indicate in the (lengthly) abstract which was sent out this talk is another attempt to promote a kind of new thinking about Fourier Analysis. The buzzword is Conceptual Harmonic Analysis (CoHaA), the attempt to integrate Abstract and Computational Harmonic Analysis, but also the mathematical and the engineers' and physicists' way of dealing with Fourier Analysis.

Most of the tools visible so far are related to questions of time-frequency and Gabor Analysis. Notions arising in this context are Banach frames, Gabor expansions of distributions, Wiener amalgam spaces, modulation spaces, and Banach Gelfand Triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$, .

The focus of today's presentation is on the ubiquitous usability of BUPUs, so-called Bounded Uniform Partitions of Unity.

The ideal requests on the concept of CHA

- From the point of view of mathematics (AHA) the approach should allow to deal with Fourier Analysis over LCA (locally compact Abelian) groups G;
- For engineers it should provide a unified framework for *discrete and continuous signals*, but also for periodic and non-periodic signals (all defined over $\mathbb{R}$ and $\mathbb{R}^d$, e.g.);
- For physicists it should provide a justification for the use of continuous "bases" such as $(\delta_x)_{x\in\mathbb{R}^d}$ or $(\chi_s)_{s\in\mathbb{R}^d}$ with $\chi_s(t) = exp(2\pi i\langle s,t\rangle)$ (and the intuitive interpretation of the *Fourier transform* as a change of bases).

In addition, as a mind-set put up by Hans Reiter: it should work without the use of structure theory for LCA groups!

What do we have so far? I

- MATH: First one has to establish the existence of the Haar measure, then define $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ and convolution and the Fourier transform, mapping $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ into $\left(\boldsymbol{C}_0(\widehat{G}), \|\cdot\|_\infty\right)$ (Riemann-Lebesgue Lemma);
- FT as *integral transform*, and *convolution* understood pointwise (a.e.), all based on Lebesgue's integral;
- ENGINEERS: Start from the *sifting property* of the Dirac:

$$f = \int_{-\infty}^{\infty} f(x)\delta(x-t)dt,$$

  in order to derive that any TILS is a convolution operator!
- The FFT is the backbone of *digital signal processing*, is just a fast implementation of the DFT, with little immediate connection to the integral transform $\mathcal{F}$!

What do we have so far? II

After ca. 200 years of Fourier analysis we have a lot of tools:

1. We can do Fourier Analysis using $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, a Banach algebra with convolution and a well defined Fourier transform via Lebesgue integration;
2. We can do similar things over general LCA groups
3. We have fast algorithms, thanks to the FFT (basis for digital signal processing!);
4. Thanks to L. Schwartz we have the wonderful theory of tempered distributions, which is based on the properties of the *nuclear Frechet space* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, called the Schwartz space.
5. We have a large variety of function space, in particular the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ which is required to prove Plancherel's Theorem and then the Hausdorff-Young inequality for the Fourier transform.

Some references I

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Work to be done I

The ambitious claim and demands on CoHaA requires a lot of work, as it is supposed to provide a more elementary approach towards Fourier and Time-Frequency Analysis with a wider range of applicability and mathematical precision compared to existing methods.

The universal role of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$ (especially for $G = \mathbb{R}^d$) has been in the focus of the courses at TU Muenich and ETH (2015-2020). Teaching the material to engineers also has forced the speaker to distill the (modest) functional-analytic background needed in order to formulate the general principles (essentially up to the use of the w^*-convergence).

Although the concept of **CoHaH** was introduced already ca. 14 years ago there is still a lot to be done!

Work to be done II

The work done so far has

- Establish a relatively simple approach to the kind of Fourier and Gabor Analysis over $\mathbb{R}^d$ and its subgroups, making use of Gaussians and dilations;
- Provide alternative approaches to the ingredients needed (e.g. the details of a sequential approach to "mild distributions", the elements of $\boldsymbol{S}_0'(G)$);
- Clarify the tools needed for different communities (engineers, pure mathematicians, physicists);
- Demonstrate that this view-point also allows to support studies connected with NHA (Numerical Harmonic Analysis), i.e. the "discrete to continuous" problem!
- Check how far the construction of the Haar measure can be postponed in such an approach;

BUPUs I

The technical part of this talk will focus on a somewhat technical aspect of Fourier Analysis, which however plays a crucial role in various places. It is the notation and the use of BUPUs, i.e. bounded uniform partitions of unity. They turn out to be very useful and helpful in many places:

1. a new approach to convolution of bounded measures;
2. the definition of the Fourier-Stieltjes transform;
3. the approximation of functions by spline-type functions;
4. the w^*-approximation of measures by discrete ones;
5. the definition of Wiener amalgams and modulation spaces;
6. the development of *coorbit theory*;
7. recovery of band-limited functions from samples;
8. computation of Fourier transforms with the help of FFT.

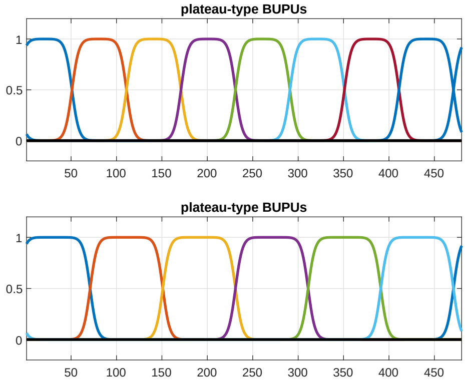
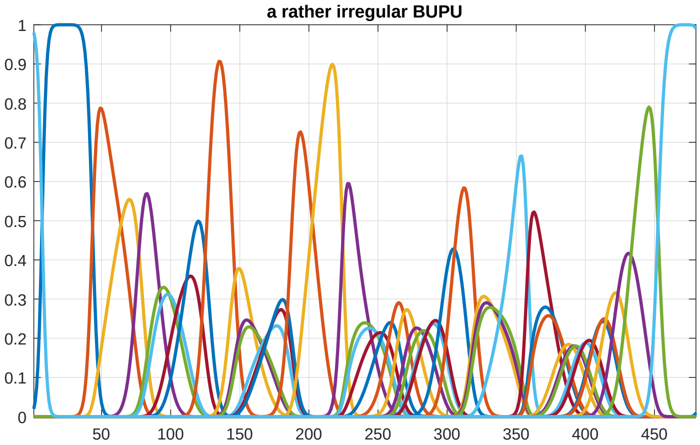


Abbildung: BUPUplat01.eps



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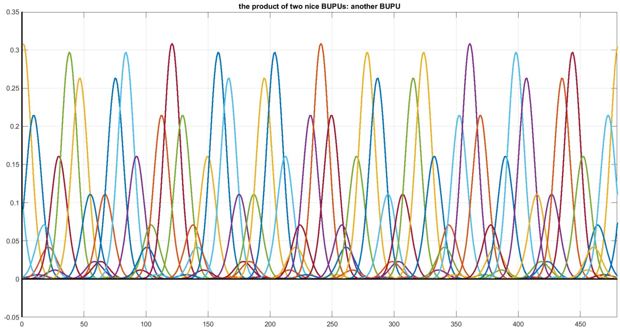

Abbildung: BUP1BUP2A: Pointwise product of 2 regular BUPUs

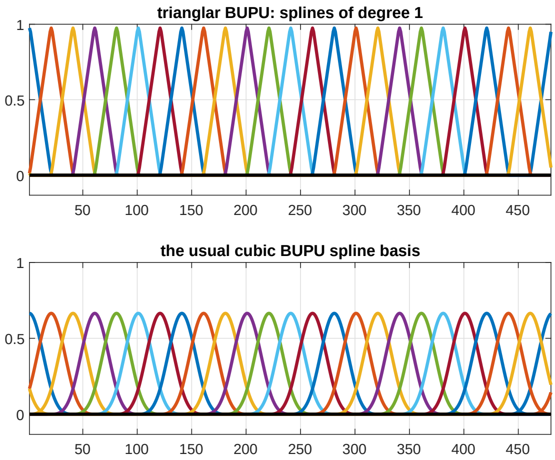

Abbildung: BUPUdem03.eps

General Λ-BUPUs

In a more general context we have the following definition for BUPUs over general LC groups G (regular A-BUPUs):

Definition

A *coherent family* $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} = (T_\lambda\psi)_{\lambda\in\Lambda}$ in a Banach algebra $(\mathbf{A}, \|\cdot\|_A)$ is forms a **regular A-Bounded Uniform Partition of Unity** generated by the pair (ψ, Λ) if ψ has compact support, if the family is bounded in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ and

$$\sum_{\lambda\in\Lambda} T_\lambda\psi(x) = \sum_{\lambda\in\Lambda} \psi(x-\lambda) \equiv 1 \quad \text{for all} \quad x \in G$$

For many examples Λ can be a lattice (discrete subgroup, e.g. $\Lambda = A * \mathbb{Z}^d$ in $\mathbb{R}^d$), but this is not a requirement.

Defining general BUPUs

Definition

A bounded family $\Psi = (\psi_i)_{i \in I}$ in some normed algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a called an $\boldsymbol{A}$**- Uniform Partition of Unity of size** $\delta > 0$ if:

1. $\sup_{i \in I} \|\psi_i\|_{\boldsymbol{A}} = C_A < \infty$;
2. there is a family of points $(x_i)_{i \in I}$ such that
$$\operatorname{supp}(\psi_i) \subseteq B_\delta(x_i) \quad \forall i \in I;$$
3. there is limited overlap of the supports, i.e.
$$\sup_{i \in I} \#\{j \mid B_\delta(x_j) \cap B_\delta(x_i) \neq \Phi\} = C_X < \infty;$$
4. $$\sum_{i \in I} \psi_i(x) \equiv 1 \quad \text{over } \mathbb{R}^d$$

Bounded Measures I

Given any LCA group G, we identify $\left(\mathbf{C}_0(G), \|\cdot\|_\infty\right)$ as the closed subspace (ideal) of $\left(\mathbf{C}_b(G), \|\cdot\|_\infty\right)$ (with respect to sup-norm) arising as the closure of $\mathbf{C}_c(G)$. In fact it is a Banach algebra with respect to pointwise multiplication, and also translation invariant under

$$T_x f(y) = f(y - x), \quad x, y \in G.$$

From a functional-analytic view-point it is then natural to make use of the dual space. Based on the Riesz representation it is justified to *call* use the symbol $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$ for dual space, and call the elements of $\mu \in \mathbf{M}_b(G)$ the *bounded measures* !

Convolution of bounded measures I

It is well known that $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ is a Banach algebra with respect to convolution, with $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ being a closed subspace. In fact, the study of the Banach algebra $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ is often seen as the key problem of Fourier analysis (due to its connection to the spectral analysis problem).
The key to convolution between functions and bounded measures is the standard formula

$$\mu * f(x) = \mu(T_x f^{\checkmark}), \;\; x \in G, f \in \boldsymbol{C}_0(G),$$

with $f^{\checkmark}(x) = f(-x)$. Obviously

$$\delta_x * f = T_x f,$$

But how to convolve measures (internal convolution)?

There are different uses of BUPUS

Using BUPUs $\Psi = (\psi_i)_{i \in I}$ we define operators on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and their adjoints $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$:

Definition

$\mathrm{Sp}_\Psi f := \sum_{i \in I} f(x_i)\psi_i, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d)$. QUASI-INTERPOLATION

Definition

$\mathrm{D}_\Psi \mu := \sum_{i \in I} \mu(\psi_i)\delta_{x_i}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. DISCRETIZATION

In fact, a simple operation like "piecewise linear interpolation" for functions in $\left(\boldsymbol{C}_b(\mathbb{R}), \|\cdot\|_\infty\right)$ can be seen as an operator of the form Sp_Ψ, using the triangular BUPU, and many numerical integration methods are just providing closed formulas for the integrals of such approximations.

The Spline Quasi-interpolation operators I

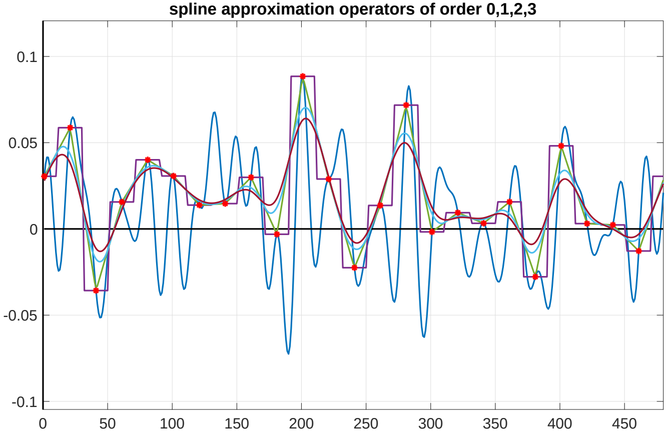

Abbildung: Approximation by spline functions of order 1, 2, 3, 4

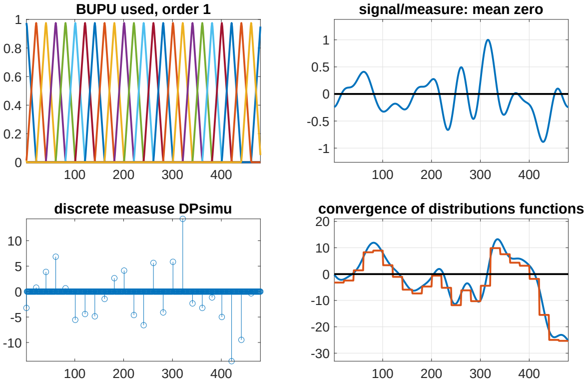

Abbildung: The (triangular) BUPU, the smooth signal/measure, the discretized measure illustrate by the STEM-command, and the corresponding distribution function, which has a jump of sice c_k at the position of the corresponding Dirac measure.

How to Define Convolution I

In order to define convolution of bounded measures we may keep in mind that in the case of finite groups we simply have to take the group multiplication law $z = x + y$ and turn it into a convolution relation $\delta_x * \delta_y = \delta_z$.
An elegant approach is this identification theorem:

Theorem

There is an isometric identification between the translation invariant operators $T : \boldsymbol{C}_0(G) \to \boldsymbol{C}_0(G)$ commuting with translation and the bounded measures, via

$$C_\mu f(x) = \mu(T_x f^\checkmark), \quad \textit{with} \quad f^\checkmark(x) = f(-x),$$

with $|||T|||_{\boldsymbol{C}_0} = \|\mu\|_{\boldsymbol{M}_b(G)}$. Converse: $\mu(f) = T(f^\checkmark)(0)$.

How to Define Convolution II

Engineers would call such systems BIBOS-TLIS!

The definition of *convolutions* for bounded measures on G is then realized via the transfer of structure from the obvious algebra properties of the space of operators to their "representatives". Moreover we have the convolution theorem

$$\widehat{\mu \star \nu} = \widehat{\mu} \cdot \widehat{\nu}, \quad \mu, \nu \in \boldsymbol{M}_b(G).$$

The non-trivial step is the verification of

$$\boldsymbol{M}_b(G) * \boldsymbol{C}_0(G) \subset \boldsymbol{C}_0(G),$$

based on the fact that in both $\boldsymbol{M}_b(G)$ and $\boldsymbol{C}_0(G)$ the compactly supported elements are dense in both spaces.
Convolution by δ_x corresponds to the translation operator, hence convolution with discrete measures creates linear combinations of shifted copies of the input function.

The following observation will be crucial:

Lemma

For any non-negative BUPU $\Psi = (\psi_i)_{i \in I}$ one has

$$\|\mu\|_{\boldsymbol{M}} = \sum_{i \in I} \|\mu\psi_i\|_{\boldsymbol{M}}, \tag{1}$$

hence $\mu = \sum_{i \in I} \mu\psi_i$ is absolutely convergent for $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. Consequently the action of $\mu \in \boldsymbol{C}_0'(G)$ extends naturally to all of $\left(\boldsymbol{C}_b(G), \|\cdot\|_\infty\right)$.
This allows us to set

$$\widehat{\mu}(\chi) := \mu(\overline{\chi}), \quad \chi \in \widehat{G}.$$

Existence of fine BUPUs on LC groups

Formulating the principles of CoHaA over LCA groups seems to require the use of arbitrary fine BUPUs over such groups. Since such a result could not be found in the literature this basis step had to be taken (without relying on the existence of a Haar measure):

Theorem

Let G be any locally compact group and $U \in \mathcal{U}(e)$ be any neighborhood of the neutral element $e \in G$. Then there exist (plenty of) BUPUs $\Psi = (\psi_i)_{i \in I}$ of non-negative functions $\Psi = (\psi_i)_{i \in I}$ which form a BUPU, a partition of unity of size $\leq U$, meaning with

$$\mathrm{supp}(\psi_i) \subset x_i U, \quad \forall i \in I, \tag{2}$$

for a suitable discrete (in fact uniformly separated) family $X = (x_i)_{i \in I}$ in G

Convergence of $D_\Psi \mu$ in the w^*–sense

Given $\mu \in \boldsymbol{M}_b(G)$ we note that the bounded net $(D_\Psi \mu)_{|\Psi| \to 0}$ is w^*-convergent, i.e.

$$D_\Psi \mu(f) \to \mu(f), \quad \forall f \in \boldsymbol{C}_0, |\Psi| \to 0. \tag{3}$$

In fact, we have for any BUPU Ψ:

$$\|D_\Psi \mu\|_{\boldsymbol{M}_b} \leq \|\mu\|_{\boldsymbol{M}_b}, \quad \mu \in \boldsymbol{M}_b(G). \tag{4}$$

Moreover, the family $(D_\Psi \mu)_{|\Psi| \leq 1}$ is *tight* in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$[1]

[1] A bounded set $S \subset \boldsymbol{M}_b(G)$ is called tight if for every $\varepsilon > 0$ there exists $p \in \boldsymbol{C}_c(G)$ such that $\|p\mu - \mu\|_{\boldsymbol{M}_b} \leq \varepsilon, \forall \mu \in S$.

Towards integrated group representations I

Theorem

Any abstract homogeneous Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ *with respect to a given, strongly continuous and isometric representation* ρ *of a locally compact group* G *is also a Banach module over the Banach algebra* $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$ *(with respect to convolution). The mapping* $(\mu, f) \mapsto \mu \bullet_\rho f$ *is the natural extension of the action of discrete measure given by* $\delta_x \bullet_\rho f = \rho(x) f$*, and satisfies the norm estimate*

$$\|\mu \bullet_\rho f\|_{\mathbf{B}} \leq \|\mu\|_{\mathbf{M}} \|f\|_{\mathbf{B}}, \quad \text{for all } f \in \mathbf{B}. \tag{5}$$

We thus can summarize our findings so far in the following theorem:

Towards integrated group representations II

Theorem

Let $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ be a homogeneous Banach space of an LCA group G. Then $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach module over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution. In fact, the action of μ on $f \in \boldsymbol{B}$ is defined as the limit of expressions of the form $\mathrm{D}_\Psi \mu * f$, in the norm of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

Towards integrated group representations III

Theorem

Given a HBSG and a bounded and tight net $(\mu_\alpha)_{\alpha \in I}$ *in* $(\boldsymbol{M}_b(G), \| \cdot \|_{\boldsymbol{M}_b})$ *with*

$$\mu_0 = w^*\text{-}\lim_{\alpha \to \infty} \mu_\alpha$$

then one has norm convergence

$$\lim_{\alpha \to \infty} \|\mu_\alpha * f - \mu_0 * f\|_{\boldsymbol{B}} = 0, \quad f \in \boldsymbol{B}.$$

Towards integrated group representations IV

Theorem

Let ρ be a strongly continuous, isometric representation of the locally compact group G on the Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $(\mu_\alpha)_{\alpha\in I}$ a bounded and tight net in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with $\mu_0 = w^*\text{-}\lim_{\alpha\to\infty}\mu_\alpha$. Then one has

$$\lim_{\alpha\to\infty} \|\mu_\alpha \bullet_\rho f - \mu_0 \bullet_\rho f\|_{\boldsymbol{B}} = 0, \quad \forall f \in \boldsymbol{B}. \tag{6}$$

How to extend from discrete to general bd. measures

The usual consideration related to a definition of $\mu * f$ for $f \in (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (some Banach space) is via vector-valued integrals. However, using BUPUs (replacing the concept of Riemannian sums) we only have to make use of the following simple fact:

Lemma

A normed space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach space if and only if one of the following equivalent statements is valid

1. *Every Cauchy-sequence is convergent;*
2. *Every absolutely convergent series is norm convergent;*
3. *Every Cauchy-net is convergent.*

Thus, finally it is enough to check that $(\mathrm{D}_\Psi \mu \bullet f)_{\mathrm{diam}(\Psi)\to 0}$ is a Cauchy-net!

Literature on BUPUs

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Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Ccolloq. Mmath. Sco. Jnos Bolea*, pages 509–524. North-Hfolland, AAmsterdam, Edz. B. Sz.-Ngay and J. Sbarbados. edition, 1983.

[lemu91] H. Leptin and D. Müller.
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Adv. Math., 90(1):1–14, 1991.

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Banach spaces related to integrable group representations and their atomic decompositions, I.
J. Funct. Anal., 86(2):307–340, 1989.

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A novel mathematical approach to the theory of translation invariant linear systems.
In I. Pesenson, Q. Le Gia, A. Mayeli, H. Mhaskar, and D. Zhour, editors, *Revent Aapplications of Harsonic Aanalysis to Function Spaces, Ddifferential Eequations, and Ddata Sscience.*, Aplied and Nnumerical Harsonic Aanalysis., pages 483–516. Birkhäuser, Cham, 2017.

UPUs exist on general LC groups!

The reference [lemu91] above is of particular interest and has received very little attention in the past: Here Leptin and Müller establish the fact that for each LC group one can define UPUs, i.e. partitions of unit obtained as translates of a given function $\varphi \in \boldsymbol{C}_c(G)$, i.e. with the property that

$$\sum_{i \in I} \varphi(x_i^{-1}x) \equiv 1 \text{ on } G. \qquad (7)$$

Using their result it is also easy to derive that the family $(x_i)_{i \in I}$ is well-spread and thus $(T_{x_i}\varphi)_{i \in I}$ constitutes a BUPU of size supp(φ). The proof requires structure theory of LC groups and it is not clear whether one can modify the proof in order to obtain arbitrary fine UPUs in this way (of course one can obtain fine BUPUs by just splitting φ into small pieces).

Nice inclusion results

The classical norms for Wiener amalgams decompose $\mathbb{R}^d$ into cubes, obtained by shifting the *fundamental domain* of $\mathbb{R}^d$ with respect to the standard lattice $\mathbb{Z}^d$, and thus takes the $\boldsymbol{L}^p$-norm over all these cubes and then takes a global ℓ^q-sum.
The extreme cases are to take a local sup-norm and a global ℓ^1-norm. The space (of continuous functions inside of) $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)(\mathbb{R})$ consists of all continuous functions with finite upper Riemannian sum (we write $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R})$).
Correspondingly the largest of these classical spaces is $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)(\mathbb{R})$ which consists of all locally integrable functions of "uniform density" in the sense of boundedness of the local integrals.
This space is a closed subspace of the dual of $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$, which is the space of *translation bounded* Radon measures, we write $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R})$.

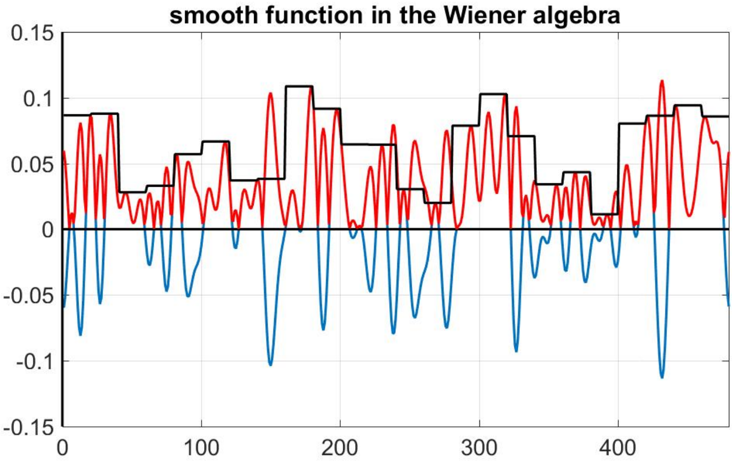

Abbildung: Interpreting the norm in Wiener's algebra $W(C_0, \ell^1)$

The relevance of Wiener Amalgam norms for Sampling

The Wiener amalgam spaces, especially those of the form $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)(\mathbb{R}^d)$, are of particular importance for the theory of irregular sampling. Convergence proofs for iterative methods to reconstruct from irregular samples of a band-limited functions rely heavily on the use of such spaces.

The other ingredient is of course that a smooth function will not deviate too much from e.g. a piecewise interpolation obtained from the given irregular samples. A key step is the norm equivalence between the usual $\boldsymbol{L}^p$-norm and the $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$-norm, for functions in $\boldsymbol{L}^p(\mathbb{R}^d)$ with given compact spectrum Ω.

Such results have been already extended to the LCA setting long ago!

Wiener amalgam spaces I

One of the motivations for the development and study of the concept of BUPUS was to wish to have spaces which allow to distinguish between local norms and global behaviour (of a given local property), in the framework which is now known as *Wiener amalgam spaces*. Applying such principles on the Fourier transform side was then possible (Using BUPUs in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$) and led to the concept of modulation spaces (all between 1980 and 1984).

Let us just remind the reader that the *Segal algebra* $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ has been introduced as amalgam space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(G)$, which shows some minimality property similar to the *Wiener algebra* $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(G)$.

Wiener amalgam spaces II

Given a BUPU in $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ we have respectively

$$\boldsymbol{W}(\boldsymbol{C}_0, \ell^1) = \{f \in \mathcal{F}\boldsymbol{L}^1, \|f\| := \sum_{i \in I} \|f\psi_i\|_{\boldsymbol{C}_0} < \infty\},$$

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \{f \in \mathcal{F}\boldsymbol{L}^1, \|f\| := \sum_{i \in I} \|f\psi_i\|_{\mathcal{F}\boldsymbol{L}^1} < \infty\}.$$

Different BUPUs define the same space and equivalent norms.

Given the fact that locally $(\mathcal{F}\boldsymbol{L}^1)_{loc} = \mathcal{F}(\boldsymbol{M}_b)_{loc}$ we can thus define $\boldsymbol{S}_0(G)$ as $\boldsymbol{W}(\mathcal{F}\boldsymbol{M}_b, \ell^1)$ on general LCA groups and derive

$$\mathcal{F}_G(\boldsymbol{S}_0(G)) = \boldsymbol{S}_0(\widehat{G}).$$

TILS on SO I

Equipped with these tools we can already answer a number of classical questions. The Fourier transform (defined first for $\boldsymbol{M}_b(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$) extends to all of $\boldsymbol{S}_0'(\mathbb{R}^d)$ via

$$\widehat{\sigma}(f) = \sigma(\widehat{f}), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

Both periodic functions (in the classical sense) or discrete signals, meaning weighted Dirac combs with bounded coefficients

$$\sigma = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad \text{with } \mathbf{c} = (c_\lambda)_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$$

belong to $\boldsymbol{S}_0'(\mathbb{R}^d)$, they have a Fourier transform in this sense. *Poisson's formula* appears as

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp}$$

and the DFT/FFT can be derived as the "natural FT" for periodic and discrete signals.
The *spectrum* of $h \in \mathbf{L}^\infty(G) \hookrightarrow \mathbf{S}_0'(G)$ is nothing else but $\mathrm{supp}(\hat{h})$.
There are many heuristic considerations in the books (mostly on applied Fourier analysis) which appear to motivate e.g. the shape of the forward and the inverse Fourier transform on $\mathbf{L}^2(\mathbb{R}^d)$, by approximating a given function by periodic version (which are expanded in a Fourier series). This can be made *rigorous* in the context of $\mathbf{S}_0'(\mathbb{R}^d)$, using w^*-convergence.

We can also characterize the translation invariant operators from $\mathbf{S}_0(G)$ to $\mathbf{S}_0'(G)$, without referring to the more complicated theory of quasi-measures (which coincide with $(\mathcal{F}\mathbf{L}^\infty)_{loc}$):

TILS on SO III

Theorem

[TILS are convolution operators]
Any bounded linear operator from $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *to* $\left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right)$ *which commutes with translations can be described as a convolution operator by a uniquely determined* $\sigma \in \boldsymbol{S}_0'$*, i.e.*

$$Tf(x) = \sigma(T_x f^{\checkmark}), \quad f \in \boldsymbol{S}_0(G), x \in G.$$

Moreover, the operator norm of T *and the norm of the representing functional* $\sigma \in \boldsymbol{S}_0'$ *are equivalent. Alternatively, we have a characterization of a TILS via the transfer "function"*

$$\mathcal{F}(T(f)) = \widehat{\sigma} \cdot \widehat{f}, \quad f \in \boldsymbol{S}_0(G). \tag{8}$$

Piecewise linear interpolation I

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ has many beneficial properties. Among others one can show that for $\mathbf{S}_0(G) \hookrightarrow \mathbf{C}_0(G) \cap \mathbf{L}^1(G)$ the restriction mapping is well defined for any subgroup $H \lhd G$ and that

$$R_H(\mathbf{S}_0(G)) = \mathbf{S}_0(H),$$

with norm equivalence for the quotient norm.
Correspondingly (by duality and the use of spectral synthesis) there is a natural identification of the subspace of $SOPG$, given by

$$\{\nu \in \mathbf{S}_0'(G) \text{ with } \operatorname{supp}(\nu) \subset H\}$$

and those elements in $\mathbf{S}_0'(G)$ which arise from some $\sigma \in \mathbf{S}_0'(H)$ via

$$\nu(f) = \sigma(R_H(f)), \quad f \in \mathbf{S}_0(G).$$

Kernel Theorem and Consequences I

One of the most effective tools for the discussion of operators between function spaces relevant for harmonic analysis is the so-called kernel theorem, which allows to identify the Banach space $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ of all bounded linear operators from $\boldsymbol{S}_0(G)$ to $\boldsymbol{S}_0'(G)$ with the space of functionals $\boldsymbol{S}_0'(G \times G)$.
This is the analogue of matrix representation of linear mappings between finite dimensional space.
It allows to even extend it to a *unitary Banach Gelfand Triple isomorphism* between the operator space $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{H}S, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and the corresponding operator kernels (or spreading functions, or Kohn-Nirenberg symbols).
The behaviour of the spreading representation of operators can be compared with the properties of the Fourier transform with respect to periodization.

Kernel Theorem and Consequences II

We all know that we have for functions: Periodization corresponds to sampling on the Fourier transform side. A typical relation is

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda * f) = \widehat{f} \cdot \widehat{\sqcup\!\sqcup_\Lambda} = C_\Lambda(\widehat{f} \cdot \sqcup\!\sqcup_{\Lambda^\perp}), \quad f \in \boldsymbol{S}_0(G).$$

At the operator level we can define for every $\lambda = (x, \chi) \in G \times \widehat{G}$ the (unitary) "time-frequency shift operator" $\pi \otimes \pi^*(\lambda) = M_\chi T_x$ and raise it to the operator level via conjugation:

$$\pi \otimes \pi^*(\lambda)(T) = \pi(\lambda) \circ T \circ \pi(\lambda)^{-1}.$$

While $\lambda \mapsto \pi(\lambda)$ is only a *projective representation* of $G \times \widehat{G}$ one can check that $\lambda \to \pi \otimes \pi^*$ establishes a strongly continuous, isometric group representation of $G \times \widehat{G}$ on e.g. $\mathcal{HS}(G)$.

Kernel Theorem and Consequences III

Consequently we can apply the general principles from the beginning and obtain the integrated group representation, hence in particular for $f \in \boldsymbol{L}^1(\mathbb{R}^{2d}) \subset \boldsymbol{M}_b(\mathbb{R}^{2d})$ the expression

$$f \bullet_{\pi \otimes \pi^*} T$$

is well defined and turns the Banach Gelfand Triple $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ into a Banach module over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (with convolution).
This is one of the key ingredients of QHA (Quantum Harmonic Analysis) as described in the work of Werner (1984) and Luef/Skrettingland (2017).

Kernel Theorem and Consequences IV

On the other hand the standard MATLAB code used nowadays in order to *efficiently form the matrix defining a Gabor multiplier* is based on the fact. In fact, it is done via a convolution operator at the KNS level which is (by the *symplectic Fourier transform*) a pointwise multiplication operator at the level of spreading functions. The key relationship is the fact that the KNS-symbol intertwines $\pi \otimes \pi^*$ with ordinary translation, i.e. that

$$\kappa[\pi \otimes \pi^*(\lambda)(T)] = T_\lambda\left(\kappa(T)\right), T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0').$$

Fourier Standard Spaces I

The attentive listener/reader may have observed that so far $\boldsymbol{L}^p$-spaces have not played a role in our description of Fourier Analysis. But of course one can make use of the Haar integral (once it is established) and use it to embed $(\boldsymbol{C}_c(G), \|\cdot\|_1)$ isometrically into $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ and the go on the define $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$ (as a subspace of $\left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right)$) or the other $\boldsymbol{L}^p$-spaces. One may ask the classical questions:

- What are the bounded linear operators from $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ to $\left(\boldsymbol{L}^q(G), \|\cdot\|_q\right)$?
- How can one describe Fourier multipliers, resp. pointwise multipliers from $\mathcal{F}\boldsymbol{L}^p(G)$ into itself?
- Are test functions dense in a space of Fourier multipliers?
- Maybe similar question for the Wiener amalgams $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(G)$ or modulation space $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$?

Fourier Standard Spaces II

The following setting allows to properly discuss such questions:

Definition

A Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is called a Fourier Standard Space (earlier called a *(restricted) standard space*) if

1. $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}) \hookrightarrow (\mathbf{B}, \|\cdot\|_{\mathbf{B}}) \hookrightarrow (\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ (continuous embeddings);
2. $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d) \cdot \mathbf{B} \subseteq \mathbf{B}$, with $\|h \cdot f\|_{\mathbf{B}} \leq \|h\|_{\mathcal{F}\mathbf{L}^1} \|f\|_{\mathbf{B}}$ for $h \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d), f \in \mathbf{B}$;
3. $\mathbf{L}^1(\mathbb{R}^d) * \mathbf{B} \subseteq \mathbf{B}$ with $\|g * f\|_{\mathbf{B}} \leq \|g\|_{\mathbf{L}^1} \|f\|_{\mathbf{B}}$; for $g \in \mathbf{L}^1(\mathbb{R}^d), f \in \mathbf{B}$.

Fourier Standard Spaces III

Lemma

Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is a Banach space of locally integrable functions on* $\mathbb{R}^d$ *such that* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *is contained in* $\boldsymbol{B}$ *as a dense subspace and that* $\|M_\omega T_t f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ *for all* $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. *Then it is a Fourier Standard Space.*

The notion of a *Fourier Standard Space* offers great flexibility and allows a unified treatment of a huge collection of spaces which play a role in Fourier Analysis. It has been the topic of another talk, just recall that the notation is invariant under the (even !fractional) Fourier transform, it includes spaces of Fourier multipliers or convolution kernels, Wiener Amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$ as well as the (unweighted) Modulation Spaces $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$.

QHA: Quantum Harmonic Analysis papers I

[we84] R. F. Werner.
Quantum harmonic analysis on phase space.
J. Math. Phys., 25(5):1404–1411, 1984.

[feko98] H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

[lusk18-3] F. Luef and E. Skrettingland.
Convolutions for localization operators.
J. Math. Pures Appl. (9), 118:288–316, 2018.

[feja18] H. G. Feichtinger and M. S. Jakobsen.
The inner kernel theorem for a certain Segal algebra.
Monh. f. Math., to appear.

[de21] M. A. de Gosson.
Introduction to Quantum Harmonic Analysis.
De Gruyter, 2021.

Sampling and approximate recovery

One of the possible further applications of BUPUs (specifically the regular BUPUs arising from the family of B-splines) is the recovery of a function in $\boldsymbol{S}_0(\mathbb{R}^d)$ form its regular samples at a lattice. There are general results concerning the approximate recovery of functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ from sufficiently fine regular samples:

[feka07] H. G. Feichtinger and N. Kaiblinger.
Quasi-interpolation in the Fourier algebra.
J. Approx. Theory, 144(1):103–118, 2007.

Guaranteed rates of convergence in various function spaces for various tasks (e.g. computation of the FT based on the FFT of the sampling sequences) are among the agenda in the context of CoHaA. This means, that one should derive rates of convergence for functions belonging to a subspace of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ which is compactly embedded into $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, such as the Shubin classes $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$ for $s > d$.

TILS: Lüke/Ohm

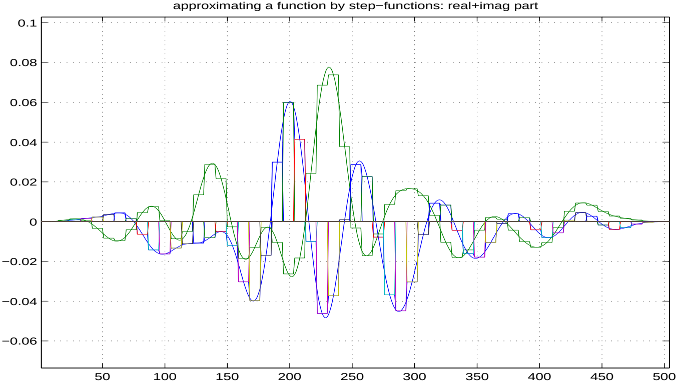

Abbildung: A typical illustration of an approximation to the input of a TILS T, preparing for the use of the Dirac impulse.

The explanation of this plot I

1. The picture shows (real and imaginary part of a) function, which is approximated by step functions;
2. These step functions are obtained by compressing a boxcar function $\mathbf{1}_{[-1/2,1/2]}$ and shifting it to the correct position;
3. Obviously the step functions get closer to the original function as the spacing gets more and more narrow ($h \to 0$);
4. On the other had the compressed rectangular function, all assumed to satisfy $\int_{-\infty}^{\infty} b(x)dx = 1$ tend, in the limit, to δ_0, the Dirac "function", thus justifying the rule

$$\int_{-\infty}^{\infty} \delta(y)dy = 1.$$

One must say, that an attempt to make the statement found in this context mathematically solid claims is a challenging task!

Questions arising from these pictures

- In which sense does the limit of the rectangular functions exist? Maybe the symbol δ or δ_0 is just a *phantom*?
- What kind of argument is given for the transition to integrals? Do we collect (as I learned in the physics course) uncountably many infinitely small terms in order to get the integral?
- In which sense are these step function convergent to the input signal f, e.g. uniformly, or in the $\boldsymbol{L}^1$-sense, and how are the steps determined (samples, local averages)?
- What has to be assumed about the boundedness properties of the operator T? In other words, which kind of convergence of signals in the domain will guarantee corresponding (or different) convergence in the target domain?

Starting from systems

Let us first look at BIBOS systems, i.e. at systems which convert bounded input in bounded output. If one wants to avoid problems with integration technology and sets of measure zero it is reasonable to assume that the operator has $\left(C_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ as a domain and as target space.

Recall the **scandal in systems theory** observed by I. W. Sandberg.

I. W. Sandberg. A note on the convolution scandal.
Signal Processing Letters, IEEE, 8(7) (2001) p.210–211.

I. W. Sandberg The superposition scandal.
Circuits Syst. Signal Process., 17/6, (1998) p.733-735.

Piecewise Linear Interpolation in $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$

The situation can be save, if one just applied *piecewise linear interpolation* at the input level.
It is obvious that $\mathrm{Sp}_\Psi(f) \to f$ for $|\Psi| \to 0$ in $\left(\boldsymbol{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$, for any $f \in \boldsymbol{C}_0(\mathbb{R})$.
But in a paper with N. Kaiblinger the same has been shown to be true for $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$, using the fact that the triangular BUPU is bounded in $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.
Details can be found in my course notes at
`www.nuhag.eu/ETH20` *with YouTube links.

Piecewise linear interpolation is better

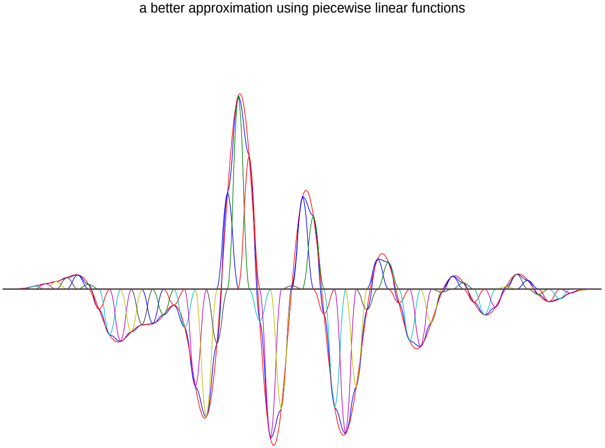

Abbildung: The piecewise linear functions $\mathrm{Sp}_\Psi(f)$ converge to f in the uniform norm.

Wiener Amalgam Spaces

Spaces like $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ can be defined via BUPUs, measuring the "pieces" $f\psi_i$ in the $\boldsymbol{L}^p$-norm and summing with a (weighted or unweighted) ℓ^q-norm.
Subsequently one can replace the local and global norms using BUPUs, allowing local components which measure local smoothness, or membership in some Fourier-Beurling algebra.
For band-limited functions ($f \in \boldsymbol{L}^p(\mathbb{R}^d)$, $\mathrm{supp}(\widehat{f}) \subseteq \Omega$) the $\boldsymbol{L}^p(\mathbb{R}^d)$-norms and the norms of $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)(\mathbb{R}^d)$ are *equivalent*!

THANKS for your ATTENTION!

Material can be found at: www.nuhag.eu,
in particular at www.nuhag.eu/talks

Access via ''visitor'' and ''nuhagtalks''

and papers at www.nuhag.eu/bibtex
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