NuHAG Numerical Harmonic Analysis Group

The Bounded Approximation Property in the Context of FOURIER STANDARD SPACES

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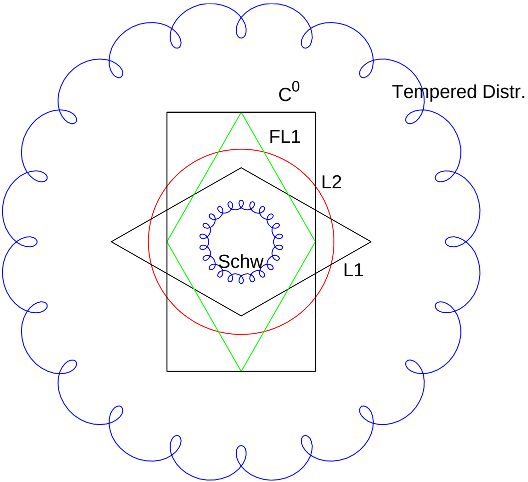
A quick outline of the talk

- The Metric Approximation Property for Banach spaces;
- In order to prove this property for large family of Banach spaces of distributions or functions, we have to introduce some concepts from time-frequency analysis;
- It helps to recall the so-called integrated group action;
- and the so-called Banach Gelfand Triple, consisting of the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and the dual space (of mild distributions) $\boldsymbol{S}_0'(\mathbb{R}^d)$;
- Some terminology concerning Banach modules;
- Finally Fourier Standard Spaces,
- Characterization of compact sets.

The Schwartz Gelfand triple



Classical spaces and Fourier

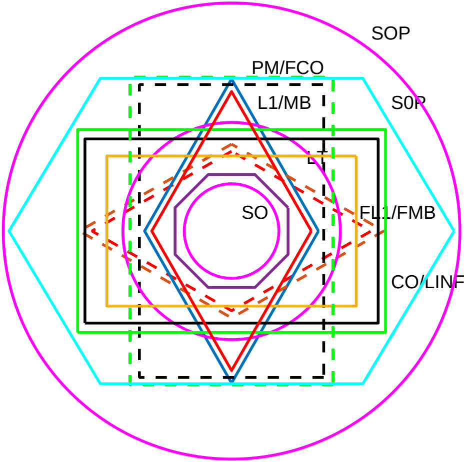


Figure: Classical spaces and Fourier

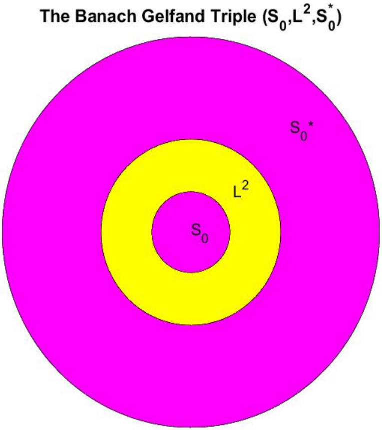


Figure: THE Banach Gelfand Triple

$L^1(\mathbb{R}^d)$ and the Fourier Algebra $\mathcal{F}L^1(\mathbb{R}^d)$

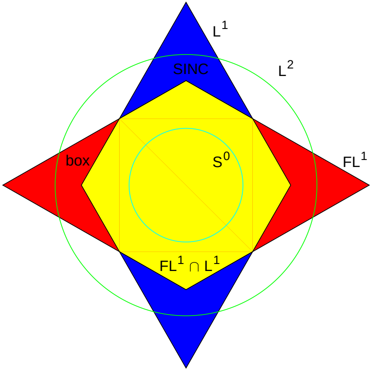


Figure: Yellow area is the domain for the Fourier Inversion Theorem, $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1$, and $\boldsymbol{S}_0$ is a natural domain for Poisson's formula, or a reservoir for summability kernels!

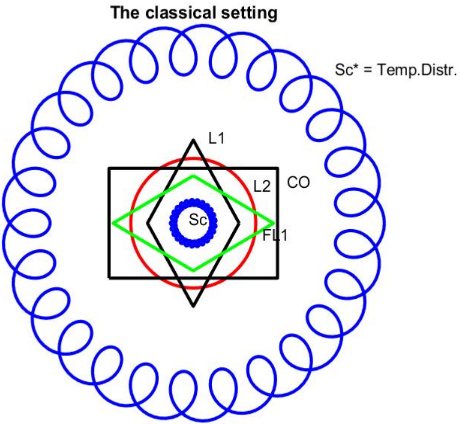


Figure: Classical spaces and tempered distributions

The Bounded Approximation Property

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ has the **Bounded Approximation Property** if there exists some constant $C_{\boldsymbol{B}} > 0$ such that for every compact subset $K \subset \boldsymbol{B}$ and $\varepsilon > 0$ there exists some finite rank operator T on $\boldsymbol{B}$ with operator norm $|||T|||_{\boldsymbol{B}} \leq C_{\boldsymbol{B}}$ and

$$\|T\mathbf{x} - \mathbf{x}\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall \mathbf{x} \in K. \tag{1}$$

The case $C_{\boldsymbol{B}} = 1$ describes the *Metric Approximation Property*. Equivalently on can describe the situation by asking for a bounded net $(T_\alpha)_{\alpha \in I}$ of finite rank operators on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ converging strongly to the identity operator.

Psychological/Technical Background

The chosen topic is part of an extensive program of **reshaping Fourier Analysis** in the light of the experiences of the last decades, in particular influenced by the needs (and functional analytic methods) required in order to handle the questions arising in *Time-Frequency Analysis* and *Gabor Analysis*. At the same time it is influenced by the need of teaching the principles of Fourier Analysis in a mathematically (yet rather straightforward) way to *engineers or physicists*. So this talk is in a way a demonstration of the usefulness of such concepts for questions of approximation theory.

First of all let us recall convolution!

For mathematicians convolution is a (commutative) multiplication in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, defined pointwise a.e. via

$$f * g(x) = \int_{\mathbb{R}^d} g(x-y)f(y)dy, \quad x \in \mathbb{R}^d.$$

For engineers any Translation Invariant System (TILS) T (i.e. operators commuting with translations) can be described as a convolution operator, i.e.

$$T(f) = \mu * f$$

for some measure μ, called the *impulse response function* of T, or alternatively via a *transfer function* h on the Fourier side:

$$\mathcal{F}(T(f)) = h \cdot \mathcal{F}(f).$$

What are the domains?

Usually such descriptions are given in an intuitive way, or rather technically without reference to the applications. What can be said, is the claim that Lebesgue integration theory seems to provide the natural domain for the Fourier transform, normalized here as

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s t} dt, \quad s \in \mathbb{R}^d,$$

and also in order to derive the *convolution theorem*:

$$\mathcal{F}(f * g) = \widehat{f} \cdot \widehat{g}, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}^d).$$

The Riemann Lebesgue Theorem and Fourier Inversion

We can formulate the Riemann Lebesgue Lemma as follows:

Theorem

The Fourier transform $\mathcal{F} : f \mapsto \widehat{f}$ defines an injective and nonexpansive Banach algebra homomorphism from $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (with convolution) into $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (with pointwise multiplication).

Unfortunately neither *Plancherel's Theorem*, establishing a unitary automorphism for $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ nor the Fourier inversion formula can make us of the integrability for all the functions in their domain (hence: *summability methods* are required! Functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ are in fact good summability kernels!)

Rieffel's use of Banach Modules

In his paper on Induced Representations (JFA, 1967) M. Rieffel has provided a very powerful, but abstract view on the subject (mostly referring to E. Hewitt)
It is based on the theory of Banach modules over *Banach algebras*, in our case Banach modules over the Banach convolution algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, such as $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$.
Aside from other (mostly algebraic) properties (such as associativity of convolution in the given context) we are viewing $\boldsymbol{L}^p$-spaces as Banach modules over $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, satisfying

$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1}\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{2}$$

Families of Banach spaces of (locally integrable) functions with this property appear in the book of Katznelson by the name of Homogeneous Function Spaces and in the work of H. Reiter as Segal algebras.

Homogeneous Banach Spaces

In his book on Fourier Analysis (which first appeared in 1968) Y. Katznelson describes homogeneous Banach spaces as one possible generalization of ordinary $\boldsymbol{L}^p(G)$-spaces.

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of locally integrable functions is called a *homogeneous Banach space* on $\mathbb{R}^d$ if it satisfies

1. $\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in B, x \in \mathbb{R}^d$;
2. $\|T_x f - f\|_{\boldsymbol{B}} \to 0$ for $x \to 0, \quad \forall f \in \boldsymbol{B}$.

EX: $\boldsymbol{B} = \boldsymbol{L}^p(\mathbb{R}^d)$ for $1 \leq p < \infty$, or *reflexive BF spaces*.
Via vector-valued integration one derives

$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}, \quad \forall g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}. \qquad (3)$$

Characterization of BIBOS linear systems

Definition

A BIBOS is a bounded linear operator T on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ which commutes with translations, i.e. which satisfies

$$\exists\, C > 0 \text{ such that } \forall f \in \boldsymbol{C}_0(\mathbb{R}^d): \quad \|T(f)\|_\infty \leq C\|f\|_\infty \tag{4}$$

and

$$T \circ T_z = T_z \circ T, \quad \forall z \in \mathbb{R}^d. \tag{5}$$

TILS as convolution operators

Theorem

[Characterization of TILSs on $\mathbf{C}_0(\mathbb{R}^d)$]
There is a natural isometric isomorphism between the Banach space $\mathcal{H}_{\mathbb{R}^d}(\mathbf{C}_0(\mathbb{R}^d))$, endowed with the operator norm, and $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}})$, the dual of $(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$, by means of the following pair of mappings:

1. *Given a bounded measure $\mu \in \mathbf{M}(\mathbb{R}^d)$ we define the operator C_μ (to be called convolution operator with convolution kernel μ later on) via:*
$$C_\mu f(x) = \mu(T_x f^\checkmark). \tag{6}$$
2. *Conversely we define for $T \in \mathcal{H}_{\mathbb{R}^d}(\mathbf{C}_0(\mathbb{R}^d))$ the linear functional $\mu = \mu_T$ by*
$$\mu_T(f) = [Tf^\checkmark](0). \tag{7}$$

Discretization

Definition

A δ-REPU, i.e. a (non-negative) **re**gular **p**artition of **u**nity in $\boldsymbol{C}_0(\mathbb{R}^d)$ *of diameter* (or size) $\delta > 0$, is a family $\Psi = (\psi_k)_{k\in\mathbb{Z}^d}$ of non-negative functions on $\mathbb{R}^d$, satisfying:

1. There exists some lattice $\Lambda \lhd \mathbb{R}^d$, i.e. a discrete subgroup of the form $\Lambda = \boldsymbol{A}(\mathbb{Z}^d)$, for some non-singular $d \times d$-matrix $\boldsymbol{A}$, such that $\psi_k = T_\lambda \psi_0 = T_{\boldsymbol{A}(k)}\psi_0$;
2. $0 \leq \psi_0(x) \leq 1$, and $\text{supp}(\psi_0) \subseteq B_\delta(0)$;
3. $\sum_{\lambda\in\Lambda} \psi_\lambda(x) \equiv 1$.

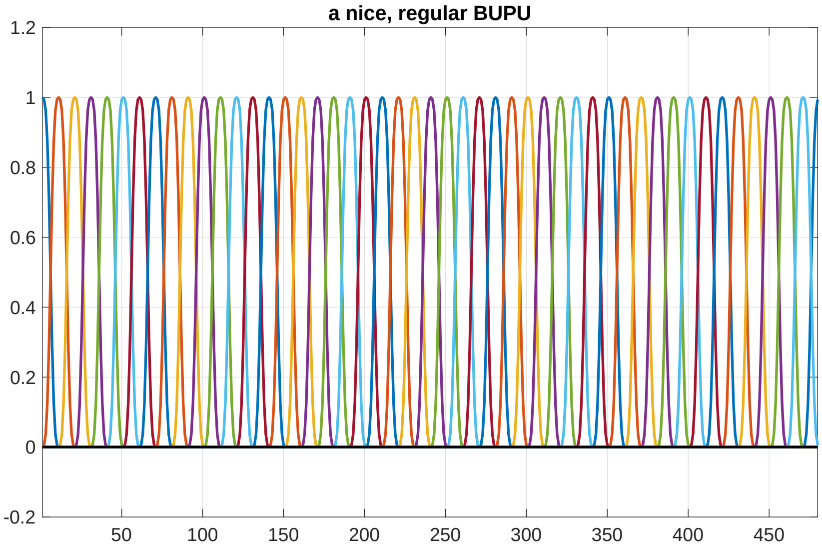


Figure: BUPUETH01.pdf

Quasi-interpolation operators

Definition

For any BUPU Ψ the *spline-type quasi-interpolation operator* Sp_Ψ is defined on the space of continuous functions as follows:

$$f \mapsto \mathrm{Sp}_\Psi(f) := \sum_{i \in I} f(x_i)\psi_i. \quad (9)$$

For the case of a δ-REPU constisting of triangular functions Δ_δ of width $\delta > 0$ the operator corresponds to piecewise linear interpolation of the underlying continuous function at regular positions $(\alpha k))_{k \in \mathbb{Z}}$. This prototype of a *spline approximation operator* can be rewritten as

$$\mathrm{Sp}_{\Delta,\alpha} := \sum_{k \in \mathbb{Z}} f(\alpha k) T_{\alpha k} \Delta_\alpha. \quad (10)$$

Lemma

The operators Sp_Ψ *map continuous functions into continuous functions, and form a uniformly bounded family of operators on* $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ *and* $(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$ *respectively, with* $|||\mathrm{Sp}_\Psi|||_\infty = 1$[a], *because* $\|\mathrm{Sp}_\Psi(f)\|_\infty \le \|f\|_\infty$. *Moreover*

$$\|\mathrm{Sp}_\Psi(f) - f\|_\infty \to 0 \quad \text{for} \quad \mathrm{diam}(\Psi) \to 0$$

whenever f *is uniformly continuous, i.e.* $f \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$. *For the case of regular BUPUs the converse is true as well.*

[a] This means that the operator is non-expansive, and maybe isometric for some functions f. It does not claim that it is an isometric mapping in general!

The adjoint Discretization Operator I

Such a property entails that the corresponding dual operators, acting boundedly on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, will converge in the w^*-sense to the identity operator on $\boldsymbol{M}(\mathbb{R}^d)$. The adjoint operators, which we will call *discretization* operators will be denoted by the symbol D_Ψ. The concrete form of these operators can be obtained as follows:

$$\mathrm{Sp}'_\Psi(\mu)(f) = \mu(\mathrm{Sp}_\Psi(f)) = \mu\left(\sum_{i\in I} f(x_i)\psi_i\right) = \left[\sum_{i\in I} \mu(\psi_i)\delta_{x_i}\right](f). \tag{11}$$

The adjoint Discretization Operator II

Definition

For a BUPU $\Psi = (\psi_i)_{i \in I}$ we define

$$D_\Psi \mu := \sum_{i \in I} \mu(\psi_i) \delta_{x_i}.$$

From this approach it is clear that we have

$$\|D_\Psi \mu\|_{\boldsymbol{M}_b} \leq \|\mu\|_{\boldsymbol{M}_b}, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$$

and also the *strong convergence* relation (w^*−convergence):

$$\lim_{|\Psi| \to 0} D_\Psi \mu(f) = \mu(\mathrm{Sp}_\Psi f) \to \mu(f), f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

The adjoint Discretization Operator III

In fact, for any homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one can demonstrate that for any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ the net $(\mathrm{D}_\Psi \mu * f)_{|\Psi| \to 0}$ is a *Cauchy net* in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and thus the definition

$$\mu * f := \lim_{|\Psi| \to 0} \mathrm{D}_\Psi \mu * f$$

makes sense and gives rise to the fact that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach module over the (commutative) Banach algebra $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (with convolution, DEFINED via composition of the corresponding TILS!).
Note that these discrete (bounded) measures form a bounded and tight (i.e. uniformly concentrated) family in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, vaguely convergent to μ. Note that discrete measures as such form a closed subalgebra (generated by the Dirac measures).

Action of $M_b(\mathbb{R}^d)$ on $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$

It turns out that these discretizations are an ideal tool to define even $\mu * f$ for $\mu \in M_b(\mathbb{R}^d)$ and f $in\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for $1 \leq p < \infty$ or $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ as limit of the convolutions by discrete measures, so essentially be defining it as limit of the expressions of the form

$$D_\Psi \mu * f = \sum_{i \in I} \mu(\psi_i)\delta_{x_i} * f = \sum_{i \in I} \mu(\psi_i) T_{x_i} f.$$

Another important step is the verification (starting from the properties of $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and avoiding measure-theoretical arguments and the Riesz Representation Theorem) of the fact that the sum describing $D_\Psi \mu$ in $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$ is in fact in the form of an absolutely convergent sum.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

Behavior under Fourier transform

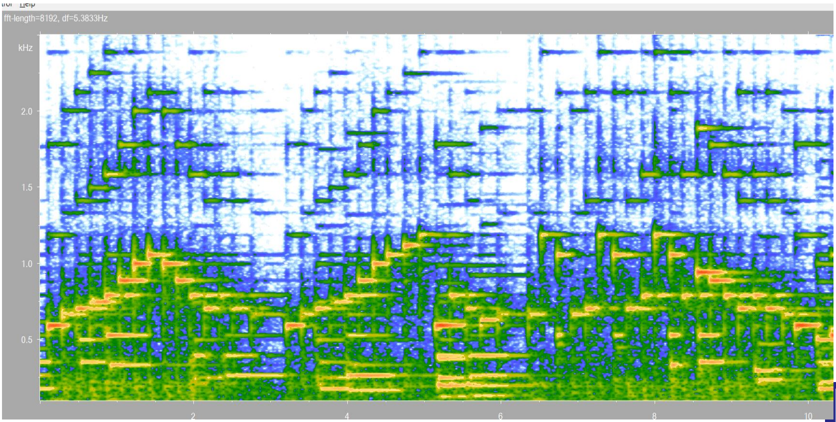
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

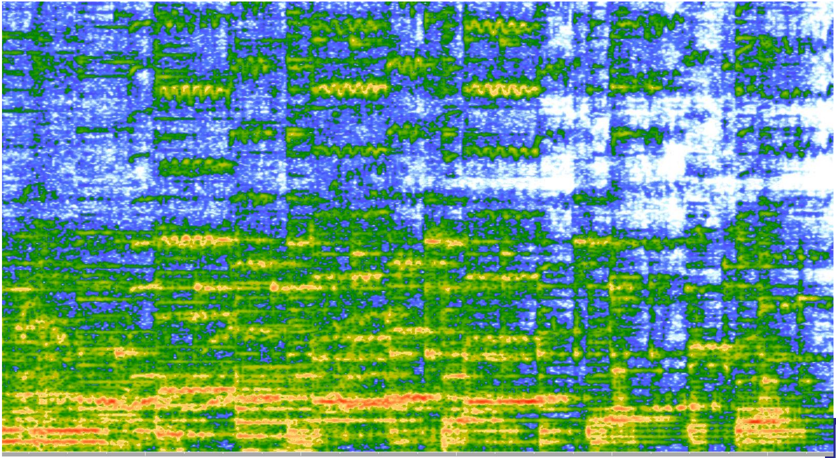
$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

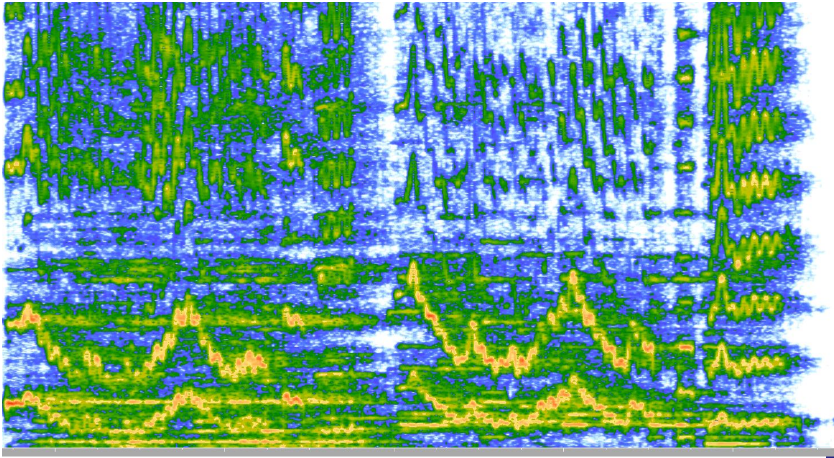
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello

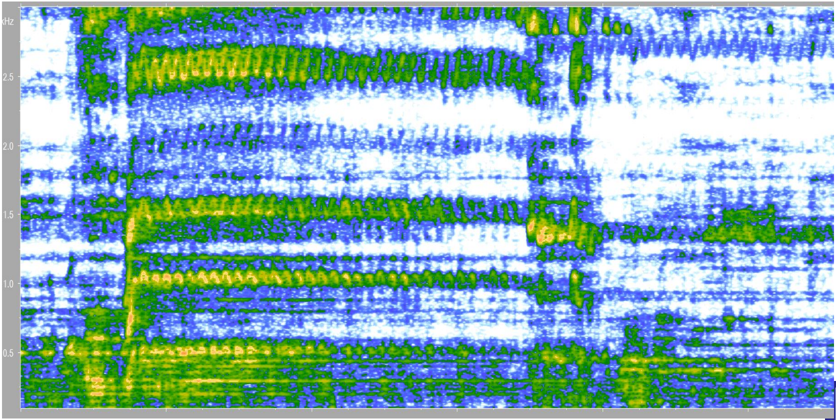


A Musical STFT: Maria Callas



A Musical STFT: Tenor: VINCERA!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$x contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of V_g is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$.
In addition it gives a nice reconstruction formula

$$f = \frac{1}{\|g\|_2{}^2} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \pi(\lambda) g.$$

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

The main result of my paper concerning the validity of the Metric Approximation Property for MINSTAs contains as a special case the validity of this property for *minimal Fourier Standard Spaces*, i.e. for those Fourier standard spaces which contain $\mathbf{S}_0(\mathbb{R}^d)$ as a dense subspace, or equivalently those Banach spaces $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ sandwiched between $\mathbf{S}_0(\mathbb{R}^d)$ and $\mathbf{S}_0'(\mathbb{R}^d)$ which allow a (strongly) continuous (projective) isometric representation of phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ by TF-shifts (see [3]).
makes use of the characterization of compact subsets in spaces with a double module structure as described in [2]. We will rephrase the result given there in a form which is more suitable for the current setting. It just makes use of the fact that approximate units can always be chosen from a dense subset of the Banach algebra acting on $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ (by convolution resp. pointwise multiplication):

Theorem

A closed and bounded subset M of a MINSTA is compact in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only if it is tight and equicontinuous, which means, that for any $\varepsilon > 0$ there exist a band-limited function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and a compactly supported function $h \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ such that $f \mapsto g * f$ and $f \mapsto h \cdot f$ are bounded operators on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and satisfy

$$\|h \cdot f - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M, \tag{12}$$

and

$$\|g * f - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M. \tag{13}$$

A crucial step for our main results is constituted by the approximation of the identity operator via compact product-convolution operators, which make fully use of the assumed double module structure of a MINSTA.

Let us first recall the Kolmogorov-type compactness criterion established in fe84 and also used effectively in [4]:

Theorem

Let M be a bounded subset of a minimal Fourier Standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then M is relatively compact in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only if for any any $\varepsilon > 0$ there exists some $g, h \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, both with compact support (or alternatively band-limited), such that

$$\|g * (h \cdot f) - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M, \tag{14}$$

and (respectively alternatively)

$$\|h \cdot (g * f) - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M. \tag{15}$$

Note that the functions g, h above can be chosen from any dense subset of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, in particular, one may assume that they have compact support on the time or on the frequency side.

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Fourier Standard Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of (tempered) distributions is called a **Fourier standard space** if it satisfies the following conditions:

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$,
2. $\boldsymbol{B}$ is translation and modulation isometrically invariant, i.e.
$$\|M_\omega T_x f\|_{\boldsymbol{B}} = \|\pi(\lambda) f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \lambda = (x, \omega).$$
3. The Fourier algebra $\boldsymbol{A}$ defines pointwise multipliers on $\boldsymbol{B}$:
$$\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\boldsymbol{A}} \|f\|_{\boldsymbol{B}}, \quad h \in \boldsymbol{A} := \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \, f \in \boldsymbol{B}.$$
4. $\boldsymbol{B}$ is a Banach convolution module over $\boldsymbol{L}^1(\mathbb{R}^d)$, with
$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_1 \|f\|_{\boldsymbol{B}}, \quad g \in \boldsymbol{L}^1(\mathbb{R}^d), \, f \in \boldsymbol{B}.$$

Some Remarks

Assuming (3) and (4) one can start equivalently from the situation

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

If $\mathcal{S}(\mathbb{R}^d)$ (hence $\boldsymbol{S}_0(\mathbb{R}^d) \supset \mathcal{S}(\mathbb{R}^d)$) is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ then (2) implies automatically (3). In fact, from the continuity of translation (and modulation) one could obtain the "integrated group action" by standard vector-valued integration methods. Combining the two actions into the group action of the *reduced Heisenberg group* (via the so-called Schrödinger representation) one has in fact *Heisenberg modules*.

Operations within the family I

The setting of Fourier standard spaces allows to treat a large number of "derived spaces" in a unified viewpoint. In fact, most of the operations which play a role in Fourier Analysis, but also in Gabor or Time-Frequency Analysis can be applied to Fourier Standard Spaces.
Hence one can treat those questions in a more systematic way, avoiding the purely technical questions of integrability and concentrate on various interesting, and sometimes completely overlooked questions.
OBVIOUSLY Banach function spaces, such as $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, but also Lorentz or Orlicz spaces satisfy all these conditions.
If such a space is reflexive it is a minimal FouSSt.

Constructions within the FSS Family

1. Taking **Fourier transforms**;
2. Conditional dual spaces, i.e. the **dual space** of the closure of $\boldsymbol{S}_0(G)$ within $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. only for minimal spaces;
3. With two spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$: take **intersection or sum**
4. forming **amalgam spaces** $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$; e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$;
5. forming **modulation spaces** $\boldsymbol{M}^{p,q} = \mathcal{F}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q))$;
6. defining pointwise or convolution **(Fourier) multipliers**;
7. using complex (or real) **interpolation methods**, so that we get the spaces $\boldsymbol{M}^{p,p} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)$ (all Fourier invariant);
8. Applying **automorphism** such as dilations, rotations;
9. any **metaplectic** image of such a space, e.g. the **fractional Fourier transform**.

PROOF and Arguments I

Theorem

Let $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ be a Banach space of tempered distributions, which is an essential Banach module over some Beurling algebra $(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w})$ with respect to convolution. Then one has for any $g \in \boldsymbol{B}$ and $f \in \boldsymbol{L}^1_w(\mathbb{R}^d)$:

$$\|g * f - g * \mathrm{D}_\Psi f\|_{\boldsymbol{B}} \to 0 \quad \textit{for } |\Psi| \to 0. \tag{16}$$

The convergence is uniform for g from a fixed bounded, equicontinuous subsets $M \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and f from a fixed bounded, tight subset $M_1 \subset \boldsymbol{L}^1_w(\mathbb{R}^d)$.

PROOF and Arguments II

The main statement of [3] is the following, which can be specialized to minimal Fourier standard spaces:

Theorem

Any MINSTA, in particular any minimal Fourier Standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ satisfies the bounded approximation property.

PROOF and Arguments III

The claim can be verified if there is some constant $C = C_B > 0$ such that for any given compact set $M \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $\varepsilon > 0$ we can find a finite rank operator F on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with $|||F|||_{\boldsymbol{B}} \leq C_B$, such that

$$\|F(f) - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M. \tag{17}$$

First we apply Theorem 12 and choose $\varphi, \tau \in \boldsymbol{\mathcal{D}}(\mathbb{R}^d)$ such that

$$\|f - \tau \cdot (\varphi * f)\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M. \tag{18}$$

$$\boldsymbol{\mathcal{D}}(\mathbb{R}^d) \cdot (\boldsymbol{\mathcal{S}}(\mathbb{R}^d) * \boldsymbol{B}) \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^d) \cdot (\boldsymbol{\mathcal{S}} * \boldsymbol{\mathcal{S}}') \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^d) \subset \boldsymbol{L}^1_w(\mathbb{R}^d) \cap \boldsymbol{B},$$

implies that the (*regularizing*) convolution-product operator

$$R : f \mapsto \tau \cdot (\varphi * f)$$

PROOF and Arguments IV

is a continuous from $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ into $\boldsymbol{L}^1_w(\mathbb{R}^d) \cap \boldsymbol{B}$. Thus it maps the compact set $M \subset \boldsymbol{B}$ into a compact subset $R(M)$ of $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right) \cap (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. In particular

$$\sup_{f \in M} \|\tau \cdot (\varphi * f)\|_{\boldsymbol{L}^1_w} \leq C_2 < \infty. \tag{19}$$

Of course it is also a bounded as an operator on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, with a different norm. The estimate for $|||R|||_{\boldsymbol{B}\to\boldsymbol{B}}$ depends only on the bounds of the approximate units in $\boldsymbol{L}^1_w(\mathbb{R}^d)$ resp. the corresponding Fourier-Beurling algebra acting on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ by pointwise multiplication, typically $|||R|||_{\boldsymbol{B}\to\boldsymbol{B}} \leq 1$.
We also have compactness of $R(M)$ in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ which implies that equicontinuity in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Thus for any bounded approximate identity (g_ρ) in $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$ one has uniform

PROOF and Arguments V

convergence on $R(M)$. Given $\varepsilon > 0$ there exists $\rho_0 > 0$ such that $\rho \in (0, \rho_0]$ implies

$$\|R(f) - R(f) * g_\rho\|_{\boldsymbol{B}} \leq \varepsilon/4, \quad \forall f \in M. \tag{20}$$

For our purpose it is advantageous to assume that the chosen function g_ρ is taken from the dense subspace of band-limited functions in $\boldsymbol{L}^1_w(\mathbb{R}^d)$. This is possible due to the (BD)-condition (see rest00). Fixing one such parameter ρ (we can use $\rho = \rho_0$, and will write g for the chosen element, for notational convenience). We continue our argument for this fixed $g \in \boldsymbol{L}^1_w(\mathbb{R}^d)$.
Next we invoke the crucial *discretization step* as described in Theorem 14 and define F via

$$F : f \mapsto \mathrm{D}_\Psi(R(f)) * g = \mathrm{D}_\Psi(\tau \cdot (\varphi * f)) * g. \tag{21}$$

PROOF and Arguments VI

Due to the compactness of the support of τ F can be rewritten as a finite sum

$$F(f) = \sum_{k=1}^{K} c_k T_{x_k} g, \quad \text{with} \quad c_k = \tau(x_k)(\varphi * f)(x_k), \tag{22}$$

hence F is a *finite rank operator*, whose range is a set of finite linear combinations of shifts of g.
Given the preparatory steps the key estimate will be the one obtained with the help of Theorem 14:

$$\|F(f) - R(f) * \rho\|_{\boldsymbol{B}} \leq \varepsilon/4, \quad f \in M. \tag{23}$$

PROOF and Arguments VII

The final estimate (17) is then obtained by combining the estimates (23), (20) and (18). In fact, we have established the following estimate for *any* $f \in M$:

$$\|F(f) - f\|_{\boldsymbol{B}} \leq \|F(f) - R(f) * g\|_{\boldsymbol{B}} + \|R(f) * g - R(f)\|_{\boldsymbol{B}} + \|R(f) - f\|_{\boldsymbol{B}}$$

In order to verify the uniform boundedness of the family of finite rank operators on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ established in this way we proceed as follows. First we recall that Beurling algebras $\boldsymbol{L}^1_w(\mathbb{R}^d)$ always have bounded approximate units (for convolutions), because the usual Dirac sequences with joint compact support are also serving as approximate units. Due to the local boundedness of submultiplicative, so-called *Beurling weights* any such family is also bounded in $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$, see Reiter's book [5] respectively

PROOF and Arguments VIII

[6]. Consequently also the Fourier- Beurling algebra $\mathcal{F}\boldsymbol{L}^1_v(\mathbb{R}^d)$ arising from the action of modulation operators on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (according to the double module assumption) will have bounded approximate units as well. These bounded approximate units can be chosen to have compact support due to the (BD) (Beurling-Domar) condition (see [1] respectively [6]). Hence the regularizing operators

$$f \mapsto R(f) = \tau \cdot (\varphi * f)$$

are uniformly bounded (typically by 1) on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Thus also $T_a : f \mapsto R(f) * g$ is bounded.
The (finite rank) operator T is bounded (also by $1 + \varepsilon$, for any ε) as well. For this purpose we recall that the operators R can be viewed as a compact operators from $\boldsymbol{B}$ into $\boldsymbol{L}^1_w(\mathbb{R}^d) \cap \boldsymbol{B}$, but also

PROOF and Arguments IX

as a bounded (in fact compact) operator on $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$. In fact, it maps $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \supset \boldsymbol{B}$ into $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \subset \boldsymbol{L}^1_w(\mathbb{R}^d)$ and is bounded (by the closed graph theorem). The elements of the form $R(f), f \in \boldsymbol{B}$ have joint support (for each fixed operators R):

$$\operatorname{supp}(R(f)) \subseteq \operatorname{supp}(\tau), \quad \forall f \in \boldsymbol{B}.$$

Thus $M_1 := R(\boldsymbol{B})$ is a bounded and tight subset of $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$.
This means that we can control the operator norm $|||T_a - T|||_{\boldsymbol{B}\to\boldsymbol{B}}$ in the following way, for $b \in \boldsymbol{B}$

$$\|T_a(b) - F(b)\|_{\boldsymbol{B}} = \|R(g) * g - \mathrm{D}_\Psi(R(b) * g)\|_{\boldsymbol{B}} < \varepsilon', \qquad (24)$$

for any given value of $\varepsilon' > 0$, because $M_1 := \{R(b), \|b\|_{\boldsymbol{B}} \leq 1\}$ is a bounded and tight subset of $\left(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}\right)$, by applying

PROOF and Arguments X

Theorem 14. Thus by limiting the consideration to sufficiently fine BUPUs with $|\Psi| \leq \delta' > 0$ we can assure that

$$|||\, T_a - F |||_{\boldsymbol{B}\to\boldsymbol{B}} \leq \varepsilon'. \tag{25}$$

Altogether this implies in the general case that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ satisfies the *bounded approximation property*, but for all relevant cases, where we have seen that we can have $|||\, T_a |||_{\boldsymbol{B}\to\boldsymbol{B}} = 1$, hence

$$|||\, F |||_{\boldsymbol{B}\to\boldsymbol{B}} \leq |||\, R |||_{\boldsymbol{B}\to\boldsymbol{B}} + |||\, R - T_a |||_{\boldsymbol{B}\to\boldsymbol{B}} \leq C_2 + \varepsilon', \tag{26}$$

and in particular for the case of approximate units of norm one: $|||\, F |||_{\boldsymbol{B}\to\boldsymbol{B}} \leq 1 + \varepsilon'$, for any given $\varepsilon' > 0$. In such a case we can of course renormalize the sequence of finite rank operators to have norm exactly one.

Modulation spaces and Gabor expansions

The spaces $\boldsymbol{S}_0(\mathbb{R}^d)$, $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$, corresponding to the spaces of tempered distributions with STFT in $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ respectively are members of the more comprehensive family of *modulation spaces*. The by now classical modulation space $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$ are modelled after the more well-known *Besov spaces* using uniform partitions of unity (BUPUs) instead of the dyadic decompositions on the Fourier transform sight. They are characterized by the membership of their STFTs (for e.g., Gaussian windows) in weighted mixed-norm spaces of *phase space* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Gabor analysis is then concerned with the reconstruction of the function $f \in \boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ from a sampled version of the STFT, while Gabor analysis (mathematically equivalent) is asking for series expansions of f as a double-series using TF-shifted copies of some Gabor atoms, typically the Gaussian, or any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Summary: CONCEPTUAL HARMONIC ANALYSIS

- Conceptual Harmonic Analysis is a blend of abstract, computational and applied harmonic analysis, including the more flexible world of time-frequency analysis and thus also pseudo-differential operators;
- THE Banach Gelfand Triple (and other version which allow to include tempered or even ultra-distributions) allows to provide the right setting for Fourier Analysis (even over LCA groups)
- Wiener amalgams and modulation spaces as function spaces.
- Fourier Standard Spaces are a subclass of translation and modulation invariant function spaces.
- Basis for treatment of the finite-discrete versus continuous variables case (!approximation theory).

THANKS to the audience

THANKS you for your attention

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