NuHAG Numerical Harmonic Analysis Group

On the Foundations of Computational Time-Frequency Analysis

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OEMG Meeting Graz 2023, 19.09.2023

In the analysis courses we teach our students that there are fields such as rational, real and complex numbers, which allow to do the usual *arithmetics*.

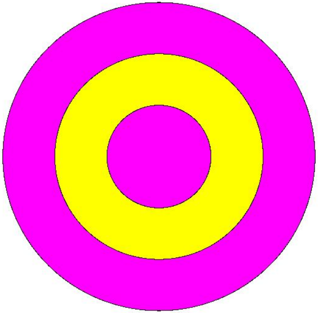


Abbildung: Our GOAL: COMPARISON with $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Number systems

The field of rational numbers is convenient for exact computations, for example, for any non-zero ration number $p/q \neq 0$ it is clear that it has a multiplicative inverse which is simply q/p.
The advantage of the real numbers is the *completeness*, which implies the existence of expressions such as $\sqrt{2}$ or π. Note that expression such as $1/\sqrt{2}$ or $1/\pi$ are to be taken as *SYMBOLS*.
The advantage of complex number is the existence of $i = \sqrt{-1}$ and still having completeness, thus the exponential function is well defined over $\mathbb{C}$, AND we have Euler's formula

$$e^{2\pi ix} = \cos(2\pi x) + i \sin(2\pi x), \quad x \in \mathbb{R},$$

and the **exponential law**:

$$exp(z_1) \cdot \exp(z_2) = \exp(z_1 + z_2), \quad z_1, z_2 \in \mathbb{C}.$$

Of course the *cyclic group* $\mathbb{U}_N$ of unit roots of order N, of the (natural) form $u_k = exp(2\pi ik/N) = \omega_N^k$,
with $k = 0, ..., N-1$, isomorphic to the additive group $\mathbb{Z}_N := \mathbb{Z}($ mod $N\mathbb{Z})$.
We can also do linear algebra on $\mathbb{C}^N$, but for the purpose of (discrete) Fourier Analysis (via DFT/FFT) we consider vectors as functions on $\mathbb{U}_N$, or equivalently as *periodic and discrete* signals.
Endowed with the usual Euclidean scalar-product we view $\mathbb{C}^N = \ell^2(\mathbb{U}_N)$ as *Hilbert space*.
The DFT/FFT can be viewed as a mapping from coefficients of a polynomial

$$p(z) = a_0 + a_1 z + ... a_N z^{N-1}$$

to the values

$$[p(u_0), p(u_1), ... p(u_{N_1})]$$

.

It turns out that (up to the scaling factor $\sqrt(N)$ this is a unitary mapping (realize by a Vandermode matrix).
The FFT (Fast Fourier transform) allows to perform this linear mapping with $O(N)log(N)$ multiplications (using that U_{2^N} consists of just two copies of $U_{2^{N-1}}$), i.e. by *recursion*. Since it is unitary the inverse can be done in the same way (up to a flip).
This allows to replace the cumbersome *convolution* (computation of the Cauchy-product at coefficient level) by pointwise multiplication (convolution theorem).
This is one of the main reasons why the FFT is an important tool for digital signal processing.

Number theory

Going deeper in the analysis, also helping us to understand why the (original) FFT algorithms works better for certain integers N which are rich in divisors (like $360, 480, 640, 720$, or look up formats of popular screens, or take the HiFi convention of $44100 = (2 \cdot 3 \cdot 5 \cdot 6)^2$) we have to see that there is a natural *nested* structure for the family of these unit groups:

$$\mathbb{U}_N \lhd \mathbb{U}_M \Leftrightarrow N|M \quad \text{(divisor)}.$$

For us the trivial case $M = 4N$ will be of particular interest, which allows us to build chains of the form

$$\mathbb{U}_N \lhd \mathbb{U}_{4H} \lhd \mathbb{U}_{16N} \lhd \mathbb{U}_{64N} \lhd \ldots$$

Identifying $[0, 1)$ with $\mathbb{U}$ via the mapping $x \mapsto \exp(2\pi ix)$ we can identify (starting with $N = 1$) the functions on $\mathbb{U}_{2^k N}$ with step functions on $[0, 1)$ which are constant on the partitions obtained from $[0, 1)$ by continued bisection.

Abstract Harmonic Analysis

The DFT/FFT can be explained in simple term of linear algebra, as a change of basis from the basis of ordinary unit vectors $\delta_k(j) = \delta_{k,j}$ (Kronecker delta) to the basis of pure frequencies. These pure frequencies (the columns or rows of the symmetric DFT matrix!) are the joint eigenvectors of the cyclic shift operators (and hence of all linear combinations, i.e. exactly of the cyclic convolution operators). This makes them so useful. However the historical approach was a different one! Starting from the theory of Fourier series (1822) important branches of mathematical analysis have been developed, the Riemann and Lebesgue integral, $\boldsymbol{L}^p$-spaces, Functional Analysis and finally *distribution theory* ($>>$ PDEs, Hörmander).

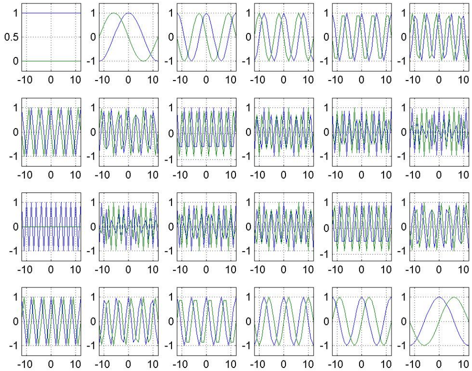


Abbildung: Illustration of pure frequencies

Abstract Harmonic Analysis: AHA

Given a LCA (locally compact Abelian) group G we consider the dual group, i.e. all the *characters* of this group, i.e. the group homomorphism from G into the (multiplicative) group $\mathbb{U} = \{z \,|\, |z| = 1\}$ (the torus group). They form the *dual group* (with respect to pointwise multiplication).

The classical cases are the following *elementary groups*:

1. The torus group itself $\mathbb{U}$, with characters of the form $z \mapsto z^k$, for a uniquely determined $k \in \mathbb{Z}$, or $\widehat{\mathbb{U}} \simeq \mathbb{Z}$.
2. The integer group $(\mathbb{Z}, +)$ with $k \mapsto \exp(2\pi isk)$ for a uniquely determined $s \in [0,1)$ (in fact: $\widehat{\mathbb{Z}} \simeq \mathbb{U}$).
3. The real line $(\mathbb{R}, +)$, which is self-dual (! $\widehat{\mathbb{R}} \simeq \mathbb{R}$), via

$$\chi_s(t) = \exp(2\pi is \cdot t), \quad s \in \mathbb{R}.$$

Elementary LCA groups

Using the structure theory for LCA groups one can extend concepts of AHA to general LCA, using the existence of an invariant *Haar measure*, the Banach convolution algebra $(\boldsymbol{L}^1(G), \|\cdot\|_1)$, find a Fourier transform, satisfying the Riemann-Lebesgue Lemma, i.e. with

$$\mathcal{F}(\boldsymbol{L}^1(G)) \hookrightarrow \boldsymbol{C}_0(\widehat{G}), \quad \text{with} \quad \|\widehat{f}\|_\infty \leq \|f\|_1.$$

For elementary groups one can use product constructions:

1. The dual group of a direct product is the product of dual groups (thus characters of $\mathbb{R}^2$ are *plane waves* etc.;
2. Any finite Abelian group is a product of finite cyclic groups;
3. Elementary LCA groups are of the form

$$G = \mathbb{R}^d \times \mathbb{U}^k \times \mathbb{Z}^m \times D.$$

Engineering terminology

In the world of engineering these settings are described by the classifications of "signals" as

- DISCRETE or CONTINUOUS
- PERIODIC or NON-PERIODIC
- FINITE: periodic and discrete

See for example [ca11]

G. Cariolaro. *Unified Signal Theory.*
Springer, London, 2011. He refers to books of Abstract Harmonic Analysis (Loomis, Rudin) in content and presentation.
Connections are discussed e.g. by Jens Fischer in [fi18]

J. Fischer. Four particular cases of the Fourier transform.
Mathematics, 12(6):335, 2018.

Technical difficulties of Classical Fourier Analysis

A first step towards a FA view-point was the introduction of the Lebesgue integral, thus e.g. $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, later $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for $1 \leq p \leq \infty$. We do not have functions anymore, but *equivalence classes of measurable functions*. Moreover, even in the classical setting we may see divergence of the partial sums of the Fourier series at every point (by Kolmogorov) for some functions in $\boldsymbol{L}^1(\mathbb{U})$. On the other hand summability theory (Fejer, Riesz etc.) was developed in order to overcome such problems with (what was thought to be "natural") *pointwise convergence almost everywhere*.

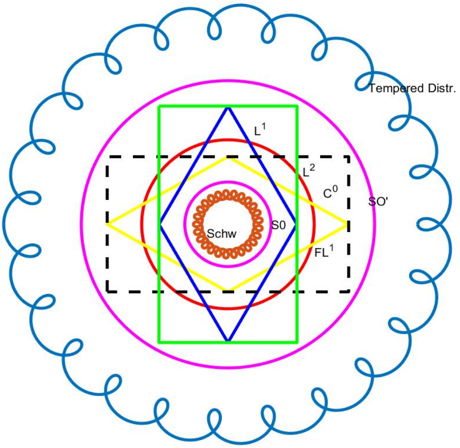
Moreover, the building blocks, the characters χ_s do NOT belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, thus it is hard to view them as eigenvectors for the (still unitary!) FT (Plancherel's Theorem).

Practical Issues

Although there is a “natural Fourier transform” for each such group there are many practical shortcomings:

- Given two periodic functions with incompatible periods, their some is not periodic anymore (such cases are still covered by the theory of almost periodic functions);
- Often signals are only locally periodic (vowels) or have varying periodicity (like heart-beat);
- In the theory of translation invariant systems one cannot describe such a TILS as a convolution operator (with impulse response) or ans a Fourier multiplier (with transfer function) in an equivalent way.

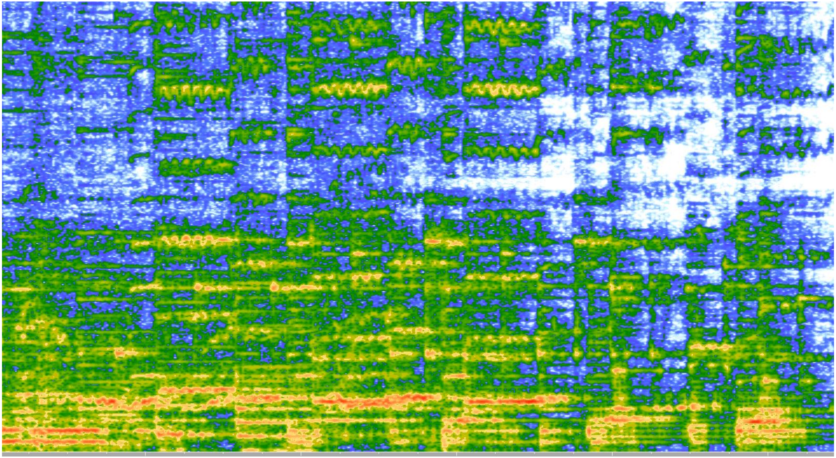
A Zoo of Banach Spaces for Fourier Analysis



Time-Frequency Analysis

To some extent time-frequency analysis tries to overcome some of these difficulties, e.g. by taking Fourier transforms only *locally*, via the so-called Sliding-Window (or Short-Time) Fourier transform.
Very much like a musician analyzes the change of the sound and the energy distributions of a given instrument as something which changes over time. So to say a "mathematical score", given the recorded sound.
The procedure consists in a sliding window (which has smooth and vanishing at the boundaries of its support) which localizes the given signal (function, or distribution) and then a Fourier transform is taken (usually plotted in the vertical direction), thus producing a function of time (x-axis) and frequency (y-axis). The natural parameters are in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, the so-called *phase space*.

A Musical STFT: Brahms, Cello



Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

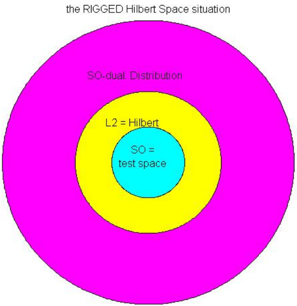


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$,

known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (**lo83-1 :[10]**).
For $\mathbb{R}^d$ the most elegant way (which is describe in **gr01 :[7]** or **ja18 :[8]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}\,dy$$

and the corresponding norm

$$\|f\|_{\mathbf{S}_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|\,dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $\mathbf{S}_0(\mathbb{R}^d)$.

Theorem

1. $S_0(\mathbb{R}^d) \hookrightarrow \left(W(C_0,\ell^1)(\mathbb{R}^d), \|\cdot\|_W\right) \hookrightarrow L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$;
2. $\mathcal{F}(S_0(\mathbb{R}^d)) = S_0(\mathbb{R}^d)$ *(isometrically);*
3. *Isometrically invariant under TF-shifts*

$$\|\pi(\lambda)(f)\|_{S_0} = \|M_s T_t f\|_{S_0} = \|f\|_{S_0}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

4. $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ *is an essential double module (convolution and multiplication)*

$$L^1(\mathbb{R}^d) * S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d) \quad \mathcal{F}L^1(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d),$$

in fact a Banach ideal and hence a double Banach algebra.

5. *Tensor product property* $S_0(\mathbb{R}^d)\widehat{\otimes} S_0(\mathbb{R}^d) \approx S_0(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(S_0(G)) = S_0(H)$.

1. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ has various equivalent descriptions, e.g.
   - as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$;
   - via *atomic decompositions* of the form

$$f = \sum_{i \in I} c_i \pi(\lambda_i) g \text{ with } (c_i)_{i \in I} \in \ell^1(I).$$

2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under group automorphism;
3. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under the *metaplectic group*, and thus under the *Fractional Fourier transform* as well as the multiplication with *chirp signals*: $t \mapsto exp(-i\alpha t^2)$, for $\alpha \geq 0$.

In addition $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is quite universally useful in Classical Fourier Analysis and of course for *Time-Frequency Analysis* and *Gabor Analysis*, and as I am going to show also for QHA: Quantum Harmonic Analysis. In short, it is easier to handle than the Schwartz-Bruhat space or even the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, and since $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ it is (much) bigger.

Mild distributions are the "signals" we care!

By definition a mild distribution is a tempered distribution which has a bounded STFT. The norm on $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ as a dual space is equivalent to the sup-norm of $V_g\sigma$ (over phase space) (for any fixed, non-zero window g from $\boldsymbol{S}_0(\mathbb{R}^d)$ (e.g. from the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$).
We also need w^*-convergence of mild distributions, which corresponds to uniform convergence over compact subsets of phase space (or just pointwise convergence, for bounded families).

A CD-record provides such a finite-dimensional approximation of a piece of music, for the duration of the song and up to $20kHz$, up to minor quantization errors.
Note that there is also a *sequential* (Lighthill style) approach to mild distributions!

Members of the dual space

- First of all ordinary functions (e.g. from $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$) define such mild distributions, because $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$, so called *regular distributions*:
$$\sigma_k(f) = \int_{\mathbb{R}^d} f(x)k(x)dx, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$
- Dirac measures (point evaluations) belong to $\boldsymbol{S}_0'(\mathbb{R}^d)$, with $\delta_x(f) = f(x)$, also *Dirac combs* over lattices:
$$⧢_\Lambda(f) := \sum_{\lambda \in \Lambda} \delta_\lambda(f) = f(\lambda).$$
- In fact Λ could be any discrete set with minimal distance (or finite unions of such sets), appearing in sampling theory (as *relatively separated sets* Λ.
- The Haar measure of a subgroup H, applied to the restriction of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ to H.

Theorem

1. $\mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$ *via* $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0'$.
2. *Identification of TLIS:* $\boldsymbol{H}_G(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$, $(\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \to \boldsymbol{S}_0(H) \hookrightarrow \boldsymbol{S}_0(G)$ *via* $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \boldsymbol{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \boldsymbol{S}_0(G) \mid \mathrm{supp}(\tau) \subset H\}$.

Theorem

1. $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}) = (\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$, *with* $V_g(\sigma)$ *and* $\|\sigma\|_{\boldsymbol{S}_0'} = \|V_g(\sigma)\|_\infty$, *hence norm convergence corresponds to uniform convergence on pahse space. Also* w^*-*convergence is uniform convergence over compact subsets of phase space.*
2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, *with density for* $1 \leq p < \infty$, *and* w^*-*density in* $\boldsymbol{S}_0'$. *Hence, facts valid for* $\boldsymbol{S}_0$ *can be extended to* $\boldsymbol{S}_0'$ *via* w^*-*limits.*
3. *Periodic elements* $(T_h\sigma = \sigma, h \in H)$ *correspond exactly to those with* $\tau = \mathcal{F}(\sigma)$ *having* $\mathrm{supp}(\tau) \subseteq H^\perp$.
4. *The (unique) spreading representation* $T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)d\lambda$, $F \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *for* $T \in \mathcal{B}$ *extends to the isomorphism* $T \leftrightarrow \eta(T)$ $\eta : \mathcal{B} \approx \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, *uniquely determined by the correspondence with*
$$\eta(\pi(\lambda)) = \delta_\lambda, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Some conventions

Scalarproduct in $\mathcal{HS}$:

$$\langle T, S \rangle_{\mathcal{HS}} = \operatorname{trace}(T * S^*)$$

In **feko98 :[5]** the notation

$$\alpha(\lambda)(T) = [\pi \otimes \pi^*(\lambda)](T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

and the covariance of the KNS-symbol is decisive:

$$\sigma(\pi \otimes \pi^*(\lambda)(T)) = T_\lambda(\sigma(T)), \quad T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Periodization and sampling

The context of *mild distributions* allows to discuss periodic functions of (!) different periods within one setting (without normalizing the period), and the concept of w^*-convergence allows to give the *heuristic* derivation of the continuous Fourier transform and its inverse, from the theory of Fourier series. Thus providing the classical integral formulas

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i s \cdot t} \, dt, \quad t, s \in \mathbb{R}^d \tag{1}$$

The inverse Fourier transform then has the form

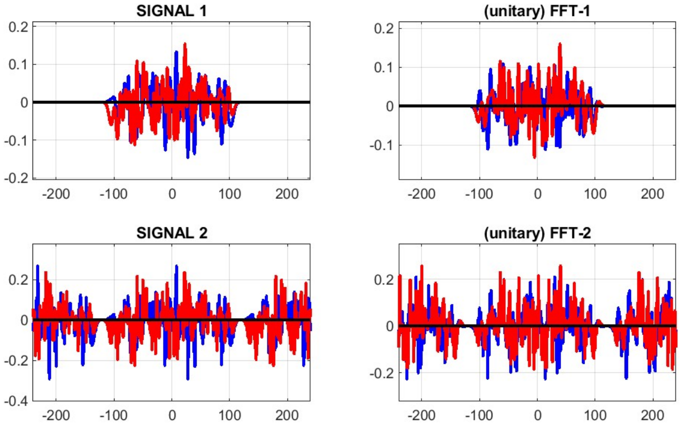
$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega. \tag{2}$$

Sampling and Periodization

The setting does not only provide the possibility of defining the Fourier transform of a Dirac comb, in fact, the integer Dirac comb $⊔⊔_{\mathbb{Z}^d} = \sum_{k\in\mathbb{Z}^d} \delta_k$ is *invariant under the Fourier Transform*, as a consequence (in fact equivalent reformulation) of

$$\text{Poisson's Formula}: \quad \sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \quad (3)$$

Combined with the Convolution Theorem, claiming that multiplication on the "time side" corresponds to convolution at the frequency side (and vice versa!) we come to a mathematical justification of the claim that *sampling on the time side* corresponds to periodization of the Fourier spectrum.



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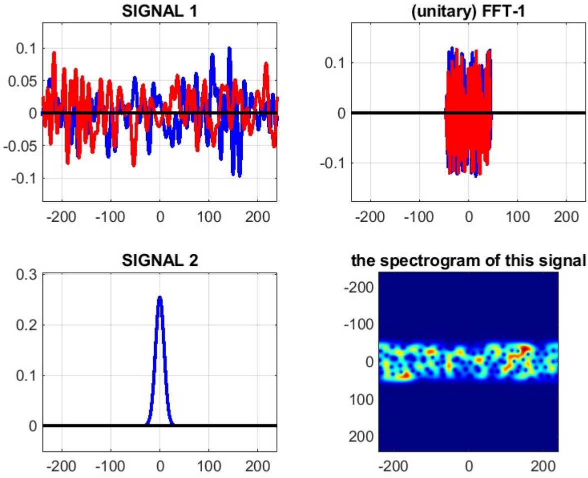


Abbildung: A low pass signal, with spectrogram

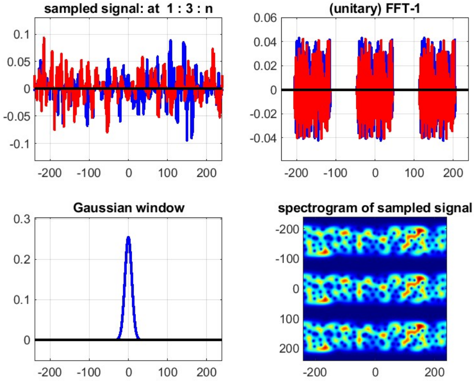


Abbildung: Effect of sampling in the spectrogram

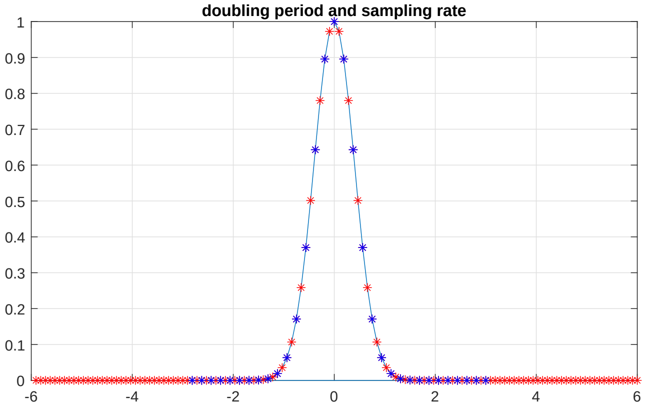


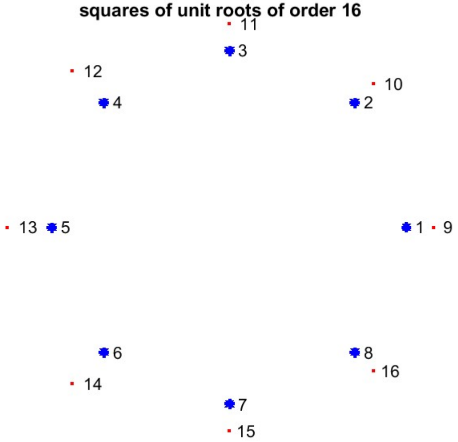
Abbildung: Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ I

It is a harmless but important observation that the squares of the elements of $\mathbb{Z}_n$ (rotation by multiples of $2\pi/n$) are just the elements of $\mathbb{Z}_{n/2}$ (only for n is even!), repeated twice.
Thus for us the operator which replaces a given function (or matrix) by its 2-periodic and 2-sampled version will be of big relevance. Also, since all the information comes twice (for matrices in both the world of column AND the world of rows) we have to understand how to extract properly the subsequence of indices "most representative" for such a reduction (turning vectors of length n into vectors of length $n/2$) or just of length $2n$ into vectors of length n and matrices of size $2n \times 2n$ into matrices of size n, in a compatible way.
We will illustrate this by some plots and also verify that this procedure is well compatible with many of the representations of functions of operators.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ II

As a basic example let us take a function with small support, then produce its p-periodic version, and then sample at the rate of $1/p$, $p \in \mathbb{N}$. Then you will find that the "representing sequence" of the Fourier version of such a function, treated in the same way, will be just (suitable normalized) the FFT of the finite vector (of length p^2, of course) of the vector in $\mathbb{C}^{p^2}$ representing the discrete and periodic signal on $\mathbb{R}$.



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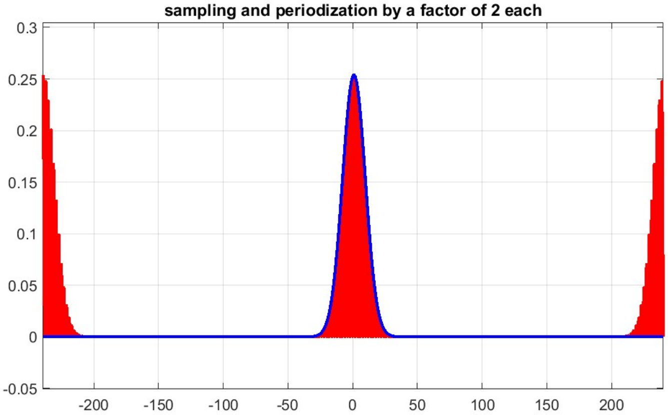

Abbildung: The reduction from the original curves (in blue) to the red curve is by sampling. Since every second value is zero the graph looks filles, plus periodic repetition.

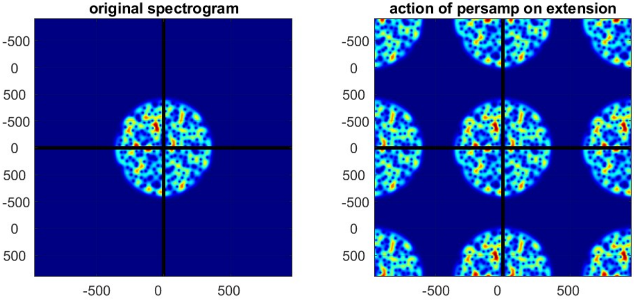

Abbildung: The picture in the STFT domain. Actually, it is the (twisted) convolution with a 2D Dirac comb with 4 points, so it is a 2D-periodization of the spectrogram.

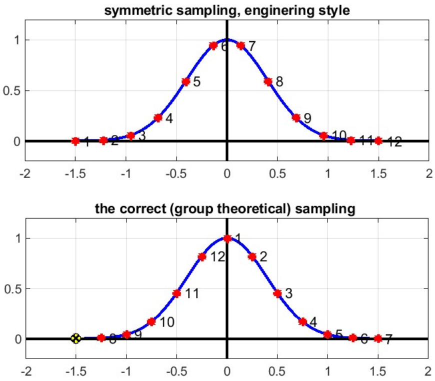

Abbildung: Naive versus correct (group theoretical) sampling. There is natural behavior with respect to refinement of the sampling and taking multiples of the period

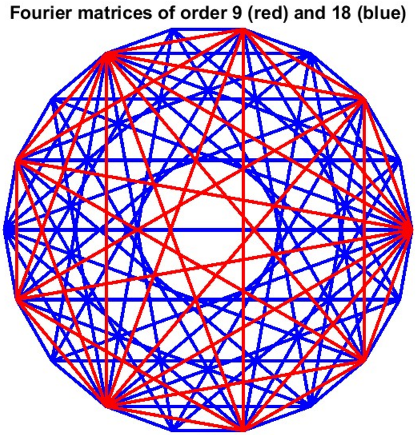

Abbildung: Overlay: fft(eye(9)) (red) over fft(eye(18)) (blue)'

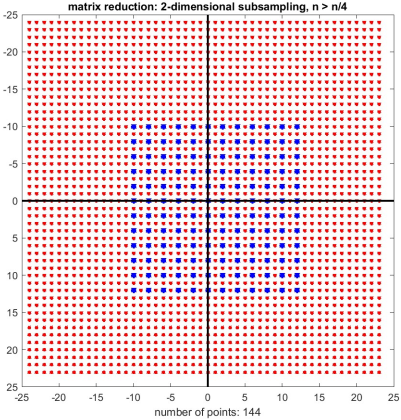

Abbildung: Subsampling from a matrix, using a QUARTER of each row and column, reducing the number of entries by the factor 16.

Structure perservig operations

We summarize the situation in the finite/discrete case:

- Many of the relevant operation on functions on phase space, or $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ or $\mathbb{Z}_n \times \mathbb{Z}_n$ are highly compatible with the reduction steps announced;
- In particular the DFT/FFT of a periodized and sampled version can be obtained via the corresponding FFT;
- The mapping from functions on $\mathbb{Z}_{2n}$ to functions on $\mathbb{Z}_n$ is a homomorphism of algebras, for both the pointwise (obvious) and thanks to the FFT-observation also (circular) convolution (up to rescaling)!
- It is also complatible with the shear-operator (first step towards Kohn-Nirenberg), and the spreading representaton;
- As a special case (!flat tori) the STFT is compatible with the reduction step.

Next we start to discuss the approximation of functions from the samples of a peridiozed and sampled version of a given function in $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$. The established results use quasi-interpolation operators arising from certain BUPUs (bounded in the Fourier algebra, like the sequence of B-splines of order 2 and higher).
These are *qualitative results* and may not provide optimal speed of recovery, BUT they apply to all functions in $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$.
Let us first give an explanation for the case $d = 1$, i.e. for the real line, where all the discrete (cocompact) subgroups $\Lambda \lhd \mathbb{R}$ are of the form $\Lambda = \alpha\mathbb{Z}$, for some $\alpha > 0$.

Definition

We call a sequence of pairs $(\alpha_k, \beta_k)_{k\geq 1}$ with $d_k = \beta_k/\alpha_k \in \mathbb{N}$ *exhausting* if they satisfy

$$\lim_{k\to\infty} \alpha_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \beta_k = \infty$$

1. A natural, bounded sequence of operators SP_k from $\big(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\big)$ to $(\boldsymbol{S}_0'(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0'})$ is then given by **s**ampling combined with **p**eriodization:

$$(f) := \alpha_k \sqcup\!\sqcup_{\alpha_k} \cdot (f * \sqcup\!\sqcup_{\beta_k}) = [\alpha_k \sqcup\!\sqcup_{\alpha_k} \cdot f] * \sqcup\!\sqcup_{\beta_k}$$

2. There exists a sequence of operators R_k with

$$\lim_{k\to\infty} \|((f)) - f\|_{\boldsymbol{S}_0} = 0.$$

Note: it is plausible that the sequence (f) contains all the required information about f because one has for $k \to \infty$:

$$w^*\text{-}\lim_{k\to\infty} \alpha_k \sqcup\!\sqcup_{\alpha_k} = \mathbf{1} \quad \text{and} \quad w^*\text{-}\lim_{k\to\infty} f \sqcup\!\sqcup_{\beta_k} = \delta_0,$$

so that one has in fact for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$:

$$f = (f \cdot \mathbf{1}) * \delta_0 = w^*\text{-}\lim_{k\to\infty}(f) = (f * \delta_0) \cdot \mathbf{1}.$$

Lemma

It is clear that each of the periodic discrete signals which are in the range of can be viewed as an element of the cyclic group $G_k = \mathbb{Z}_{d_k}, k \geq 1$ of order d_k. Moreover the Fourier transform in the $\boldsymbol{S}_0'$-sense of these discrete periodic functions corresponds to the DFT/FFT on the corresponding finite group G_k.

Written in formulas (and ignoring the explicit formulation of the isomorphism, even if one has to be careful in practice) this means, let us assume for simplicity that $\alpha_k = 1/\beta_k$, with $\beta_k \in \mathbb{N}$ tending to infinity, like $\beta_k = 2^k$:

$$(\widehat{f}) = ((f)). \tag{4}$$

The approximation result of a joint paper with N. Kaiblinger (**feka07 :[4]**, then gives:

Theorem

$$\lim_{k\to\infty} \|\widehat{f} - ((f))\|_{S_0} = 0, \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

For the computation of dual FTs and dual Gabor atoms this has been used by N. Kaiblinger in **ka05 :[9]**.

Recipes for transfer

The question addressed here is the transfer between insight (e.g. MATLAB simulations) in the finite-discrete case and related continuous problems:

1. By taking the AHA view-point one can expect that replacing sums by integrals and the FFT by the continuous Fourier transform will give an heuristic starting point.
2. At the level of $\mathcal{B}$ (inner kernel theorem) this is actually true for all the cases discussed in this paper
3. Showing unitarity one can extend to the $\mathcal{HS}$ level
4. By duality or a sequential approach to mild distributions one can extend it to the outer layer.
5. NEW: one can expect to get for the "inner case" good numerical evidence by applying the above SP-principles.

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A non-factorization theorem I

It is a simple consequence of the Hilbert-Schmidt kernel theorem and the characterization of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ as $\boldsymbol{L}^2(\mathbb{R}^d) * \boldsymbol{L}^2(\mathbb{R}^d)$ which gives the following factorization theorem:

Theorem

$HS \star_{OP} \mathcal{H}S = \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^{2d})$.

A non-factorization theorem II

Theorem

For any pair of operators $T1, T2$ *in* $\mathcal{L}(w^*\boldsymbol{S}_0', \boldsymbol{S}_0)$ *we have*

$$F = T_1 *_{OP} T_2 \in \boldsymbol{S}_0(\mathbb{R}^{2d}), \ \text{with } \|F\|_{\boldsymbol{S}_0} \leq C\|T_1^n\|_{\boldsymbol{\mathcal{B}}} \|T_2^n\|_{\boldsymbol{\mathcal{B}}}$$

but the finite linear combinations of such functions do not exhaust all of $\boldsymbol{S}_0(\mathbb{R}^{2d})$*. On the other hand there exists a constant* $C_1 > 0$ *(depending in the choices of norms on the different spaces) such that one can find for every* $F \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ *two sequences of operators* $(T_1^n)_{n\geq 1}$ *and* $(T_2^n)_{n\geq 1}$ *such that*

$$\sum_{n=-\infty}^{\infty} \|T_1^n\|_{\boldsymbol{\mathcal{B}}} \|T_2^n\|_{\boldsymbol{\mathcal{B}}} < C_1\|F\|_{\boldsymbol{S}_0}, \quad \text{and} \quad F = \sum_{n=-\infty}^{\infty} T_1^n *_{OP} T_2^n.$$

Periodization of operators I

In the discussion of Gabor frame operators one has to consider the Gabor frame-operator

$$S_{g,\Lambda} := \sum_{\lambda\in\Lambda} P_{g_\lambda} = \sum_{\lambda\in\Lambda} \pi\otimes\pi^*(\lambda)P_g.$$

In the QHA context this can be written as convolution of the mild distribution $⊔⊔_\Lambda$ with the rank-one operator P_g: $S_{g,\Lambda} = ⊔⊔_\Lambda \star P_g$. According to the FT-rules (symplectic Fourier transform for us) this means (see **feko98 :[5]**!) that

$$\mathcal{F}_W(S_{g,\Lambda}) = (C_\Lambda ⊔⊔_{\Lambda^\circ} \cdot \eta(P_g) = C_\Lambda \sum_{\lambda^\circ\in\Lambda^\circ} V_g(g)(\lambda^\circ)\delta_{\lambda^\circ} \;,$$

or taking the (inverse) symplectic FT we get

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ\in\Lambda^\circ} V_g(g)(\lambda^\circ)\pi(\lambda^\circ).$$ Janssen representation.

Periodization of operators II

Since $V_g(g)(0,0) = \langle, g, \pi(0,0)g \rangle_{\boldsymbol{L}^2} = \|g\|_2{}^2 = 1$ for normalized windows this gives access to the invertibility $S_{g,\Lambda}$ (Gabor frame property) if $\sum_{\lambda^\circ \in \Lambda^\circ, \lambda^\circ \neq 0} |V_g(g)(\lambda^\circ)| < 1$.
Double preconditioning provides methods to reach this status within the algebra of Λ-invariant operators.
For the case that $\Lambda^\circ \lhd \Lambda$ this lattice is a commutative lattice and Zak transform (Gelfand transform) methods apply. For $d = 1$ this corresponds to the case of integer redundancy (see Gestur).
Since $V_g(g) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ this corresponds to the well-known principle that periodization of a function corresponds to sampling on the Fourier side. E.g. every periodic function in $h \in \left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$ is the $\mathbb{Z}$ periodization of some function in $f \in \boldsymbol{S}_0(\mathbb{R})$ and the Fourier coefficients of h are just the samples $(\widehat{f}(n))_{n \in \mathbb{Z}}$, or $h(t) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n t}$.

The overall landscape I

CLAIM 1: At the level of functions, signals, distributions and even for the analysis of operators (via their kernels) the world of

mild distributions, i.e. members of $\mathbf{S}_0'(\mathbb{R}^d)$ or $\mathbf{M}^\infty(\mathbb{R}^d)$

appears to be the suitable framework, not only for *time-frequency analysis*, but also for many questions of classical Fourier analysis and engineering applications. Mild distributions are so-to-say the signals which can be measured, e.g. the sound recorded by a microphone. There are no point values of such signals (e.g. the Dirac measure), but they all have a (bounded) spectrogram, obtained by measuring the impact on a translated and modulated Gauss-function, providing the "average" TF-content of the signal at time t and frequency ω, displayed as $|V_g\sigma(t,\omega)|$ as a function over phase phase $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The allowed profiles are the members of the Feichtinger algebra $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$.

The overall landscape II

CLAIM 2: The environment of $\boldsymbol{S}_0'(\mathbb{R}^d)$ (or more generally $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$, for LCA groups) allows to deal not only with functions which decay (like $\boldsymbol{L}^1(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d)$), but also with periodic or discrete ones (measures, weighted Dirac combs). All of them have a Fourier transform and the concept of w^*-convergence (locally uniform convergence of STFTs over compact subsets of phase space) is the appropriate concept of convergence. It includes Fourier transforms for *almost periodic functions* or *translation-bounded measures* as they arise in quasi-crystallography. It can be used to turn heuristic derivations of one of the Fourier transforms to another one (e.g. discrete FT from continuous, or even continuous FT from FFT) into mathematically correct arguments. As Jens Fischer has formulated it (and of course this is known since Laurent Schwartz):

There is only one Fourier Transform!

The overall landscape III

Sampling, periodization, Shannon's sampling Theorem (at the basis of modern mobile communication) and many other engineering questions can be formulated properly. Finally the "sifting property" of the Dirac Delta, or the view on $(\delta_x)_{x\in\mathbb{R}^r}$ as a family of "unit vectors" (as done in physics) can be given a precise meaning.

CLAIM 3: The so-called Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which consists of the three spaces

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$$

can be used very much in the spirit of the triple of numbers

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

The overall landscape IV

At the level of $\boldsymbol{S}_0(\mathbb{R}^d)$ all the integrals exists, Fourier (!) inversion works well (just using Riemann integrals), Poisson's formula is valid and much more. As in linear algebra one has a representation of linear operators, now from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, by means of a "continuous matrix" $K \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$. The outer level gives the most freedom and explanations, e.g. telling us that the FT maps *pure frequencies* into corresponding Dirac Deltas. There are also rules how to move within the BGT, e.g. by regularizing a mild distribution. By analogy (integrals correspond to sums, etc.) we have a very natural transfer from one setting to the other (and computational implementation).

CLAIM 4:

Structure preserving approximation allows numerical work!

Finally we can go to the finite, or better discrete and periodic setting (isomorphic!!), where actual computations take place.

The overall landscape V

The FFT implements the (generalized) Fourier transform of periodic and discrete signals (linear combinations of Dirac combs of lattices in $\mathbb{R}^d$) exactly!
The key result here is the fact, that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ has the property, that its members $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be reconstructed from the samples of a periodized version of f, e.g. by local reconstruction using piecewise linear interpolation (not step function!) or quasi-interpolation using cubic B-splines (say).
In the TF-picture this corresponds to the availability of more and more information so that larger and larger pieces of the spectrogram can be well recovered (in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$).
A *nested structure of finite Abelian groups arising as quotients of fine over coarse lattices* can then be used to form *structure preserving approximations* of many problems in Fourier analysis.

The overall landscape VI

For real functions this means essentially that one has finer and finer sampling over longer and longer intervals. The situation is particular easy for sampling at rate $1/p$, where $p \in \mathbb{N}$ is the period. In such a case the operator of periodization and sampling commute with the Fourier transform. The transition from p to $4p$ has a very natural interpretation (double the period and the sampling rate!).

There is a LONG LIST of expressions which can be treated in this way: Starting from a unitary Banach Gelfand Triple isomorphism (such as the FT) one comes up with a discrete analogue and a clear concept of *validated transition between the two worlds*: STFT, KNS-symbol, spreading function, Wigner distribution, Gabor expansion, dual Gabor atoms and so on.

THANKS to the audience

THANKS you for your attention

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