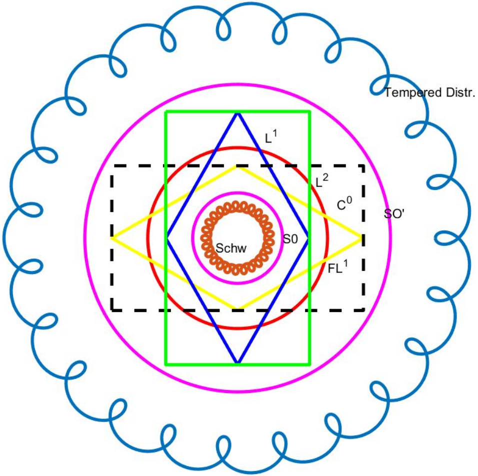
Classical and Modern Fourier Analysis in the context of Mild Distributions

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at
www.nuhag.eu TALKS at www.nuhag.eu/talks

Invited Talk at ICAIM2023
via ZOOM, March 25th, 2023

A Zoo of Banach Spaces for Fourier Analysis



Abstract submitted to ICAIM23 I

Fourier Analysis is an important branch of mathematical analysis, but also of our daily life, if we think of applications in signal processing or mobile communication. We transmit sound and images in a digital format, and compute Fourier transforms (at least approximately) using the FFT (Fast Fourier Transform). In my article Ingredients for Applied Fourier Analysis published in **Sharda Conference Feb. 2018**, Taylor and Francis, (2020) I described the setting of the Banach Gelfand Triple, consisting of the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ (a Banach algebra of continuous and integrable functions), the Hilbert space $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space, the space $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ of so-called "mild distributions, as a possible setting for the derivation of a modern course in Fourier Analysis.

In the last 5 years this topic has been further developed, and thus a summary of the new perspectives concerning this approach can

Abstract submitted to ICAIM23 II

be given. Originally this setting turned out to be useful in the context of time-frequency or Gabor analysis, because it allows to describe well the boundedness of operators involved in this theory of local Fourier analysis for non-periodic functions. Meanwhile the speaker has given courses on this subject e.g. at ETH Zuerich (which are found on YouTube, the material can be found at www.nuhag.eu/ETH20) and can thus report on different concrete building blocks for this new approach, even to questions of classical Fourier Analysis (where typically summability methods play a big role). In the talk we will address such results, e.g. in the context of the theory of Fourier multipliers, or concerning a simplified (sequential) approach to mild distributions, inspired by the Lighthill (and Temple) approach to the theory of tempered distributions in the sense of Laurent Schwartz.

Global Orientation I

This talk is (another) PERSPECTIVE talk of mine, trying to contribute to a timely interpretation of mathematical tasks related to Fourier Analysis in the modern world.

Classical (or later Abstract) Harmonic Analysis have been dealing with purely mathematical questions, such as Fourier Analysis over LCA (locally compact Abelian) groups which provides an good, qualitative framework.
One of the key-person (with his book L'integration dans les Groupes Topologiques et ses Applications. Hermann and Cie, (1940), Paris) was **Andre Weil**, who actually was a chair of Aligarh Muslim University from 1931 to 1932.
Unfortunately Lebesgue integration, or almost everywhere convergence of Fourier series (Carleson, 1972) so not play a role for engineering applications.

Abbildung: Andre Weil, Aligarh Muslim University, 1931-1932

Lack of Connection

Very unfortunately the classical tools as such are insufficient in order to deal with the problems that have to be addressed in the world of applications. Of course they form crucial building blocks for an introduction to a modern view on harmonic analysis, e.g. for *wavelet theory* of *time frequency analysis* and *Gabor Analysis*, for non-periodic and not decaying signals, like a piece of music.

I think it is in the very spirit of this conference to indicate that modern mathematical concepts are needed and provide important opportunities for relevant research work.

In my paper `Ingredients for Applied Fourier Analysis` published in the Proceedings of the Sharda Conference of Feb. 2018 , published with Taylor and Francis in 2020 p.1-22, I outline an alternative approach to Fourier Analysis, which does NOT require to first learn about Lebesgue integration or topological vector spaces leading to *tempered distributions*.

Modern Applications

There is a large variety of real-world applications of Fourier Analysis, and in fact the FFT (the Fast Fourier transform, implementing the DFT in an efficient way) is one of the backbones of the modern digital world.

In everyday life we make (mostly unconsciously) use of the FFT:

- making phone calls, exchanging messages;
- streaming music (MP3) or movies;
- taking pictures, face recognition;
- editing and filtering images;
- Scanners and MR-imaging in medicine;
- bar-codes and QR-codes, communication,...
- online conferences such as this one!

Official Abstract (for later reading)

It is the purpose of this presentation to explain certain aspects of Classical Fourier Analysis from the point of view of *distribution theory*. The setting of the so-called *Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ starts from a particular Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ of continuous and Riemann integrable functions. It is Fourier invariant and thus an extended Fourier transform can be defined for $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ the space of so-called **mild distributions**. Any of the $\boldsymbol{L}^p$-spaces contains $\boldsymbol{S}_0(\mathbb{R}^d)$ and is embedded into $\boldsymbol{S}_0'(\mathbb{R}^d)$, for $p \in [1, \infty]$.

We will show how this setting of *Banach Gelfand triples* resp. *rigged Hilbert spaces* allows to provide a conceptual appealing approach to most classical parts of Fourier analysis. In contrast to the Schwartz theory of tempered distributions it is expected that the mathematical tools can be also explained in more detail to engineers and physicists.

Function space norms

Function spaces are typically infinite-dimensional, therefore we are interested to allow convergent series. In order to check on them we need norms and completeness (in the metric sense), i.e. Banach spaces!

The classical function space norms are

- $\|f\|_\infty := \sup_{t\in\mathbb{R}^d} |f(t)|$;
- $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$;
- $\|f\|_2 := (\int_{\mathbb{R}^d} |f(x)|^2 dx)^{1/2}$;
- $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} := \sup_{\|f\|_\infty \leq 1} |\mu(f)|$, or $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} = \int_{\mathbb{R}^d} 1 d|\mu|$.
- $\|h\|_{\mathcal{F}\boldsymbol{L}^1} = \|f\|_1, \;$ for $h = \hat{f}$.

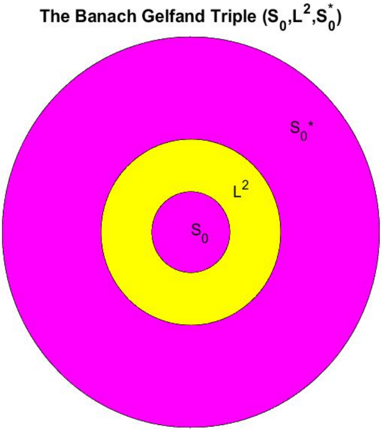


Abbildung: THE Banach Gelfand Triple

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

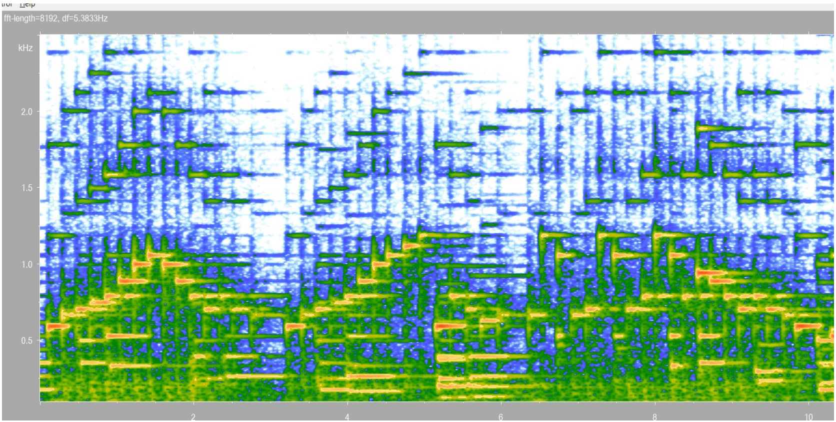
$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

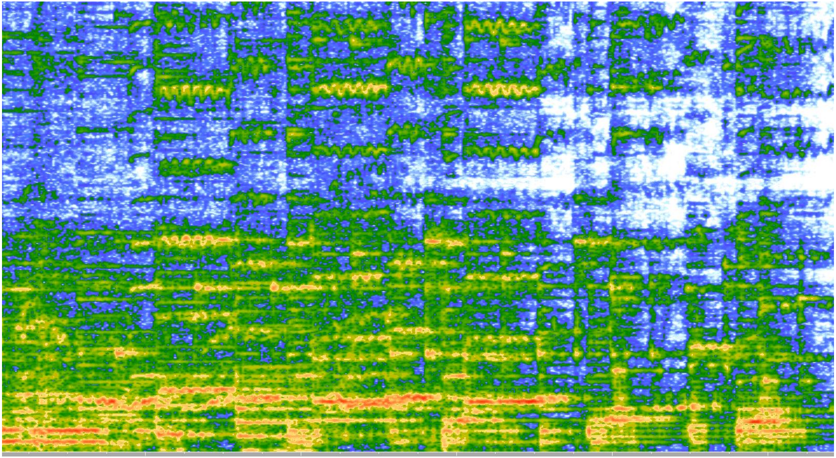
$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

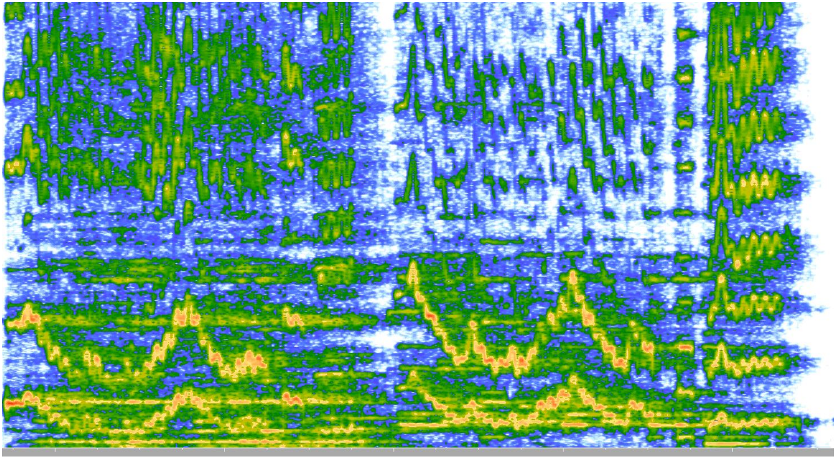
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello

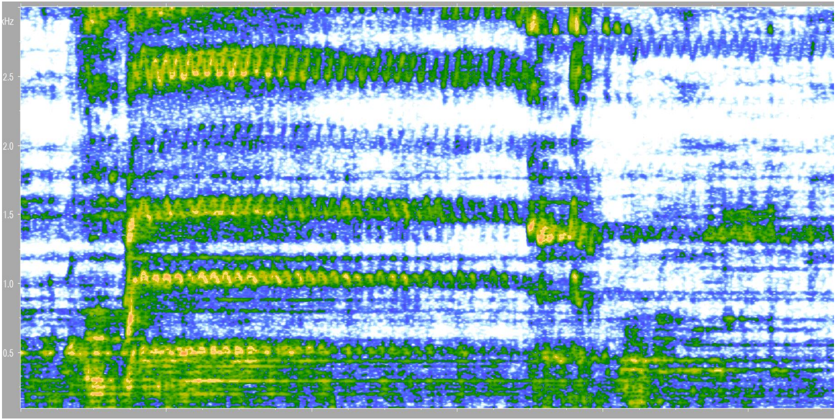


A Musical STFT: Maria Callas



A Musical STFT: Tenor: VINCERO!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$x contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of V_g is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$.
In addition it gives a nice reconstruction formula

$$f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda) g.$$

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ *(dense embedding);*
2. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

Our Aim: Popularizing Banach Gelfand Triples

According to the title I have to first explain what **Banach Gelfand Triples** are, with the specific emphasis on the BGTr $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, arising from the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (as a space of test-functions), alias the *modulation spaces* $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$.

Hence I will describe them, provide a selection of different characterizations (there are *many of them!*) and properties.

Finally I will come to the main part, namely applications or *use of this* (!natural) concept in the framework of **classical analysis**.

Mild Distributions: Convergence I

The abstract definition of a dual space means that one looks at all the bounded, linear mappings from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into the field of complex numbers, i.e., $\sigma \in \boldsymbol{S}_0'$, or σ is a mild distribution means: There exists $C = C(\sigma) > 0$ such that

$$f \mapsto \sigma(f), \quad \text{with} \quad |\sigma(f)| \leq C\|f\|_{\boldsymbol{S}_0}, \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

While this gives the impression that mild distributions are "linear measurements" applied to functions one should better view them as "signals which can be measured", not by taking point values, but arising from profiles found in the vector space $\boldsymbol{S}_0(\mathbb{R}^d)$ to them. So we should look at the (equally) linear mapping induced by $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ on the space of "signals" (called mild distributions):

$$\sigma \mapsto \sigma(f) = M_f(\sigma) \quad \text{and} \quad |M_f(\sigma)| \leq \|f\|_{\boldsymbol{S}_0}\|\sigma\|_{\boldsymbol{S}_0'}.$$

Mild Distributions: Convergence II

Since it is easy to show that any smooth function with compact support, or even any (product of) piecewise linear functions which is continuous belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$, and also that multiplication of $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with a *pure frequency* leaves the norm invariant, we can form for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$
Simple examples of elements in $\boldsymbol{S}_0'(\mathbb{R}^d)$ are the following ones

1. Dirac measures: $\delta_x : f \mapsto f(x)$;
2. regular distributions: $h \in \boldsymbol{C}_b(\mathbb{R}^d), \sigma_h(f) = \int_{\mathbb{R}^d} f(x)h(x)dx$;
3. Dirac combs: $⧢_\alpha = \sum_{k\in\mathbb{Z}^d} f(\alpha k)$; for some $\alpha > 0$; $(⧢ = ⧢_1 = \sum_{k\in\mathbb{Z}^d} \delta_k)$.

The mapping $f \mapsto \sigma_k$ defines an embedding of $\boldsymbol{C}_b(\mathbb{R}^d)$ (hence of $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$) which is continuous, thus ordinary functions can be viewed as "generalized functions" in the above sense.

Mild Distributions: Convergence III

Obviously δ_X or $\sqcup\!\sqcup_\alpha$ is *not a regular distribution*, but can we approximate a mild distribution $\sigma \in \boldsymbol{S}_0'$ by functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ (which are after all Riemann integrable!)?
The answer is given using the concept of w^*-convergence:

Definition

A sequence $(\sigma_n)_{n\geq 1}$ in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is w^*-convergent with limit $\sigma_0 \in \boldsymbol{S}_0'(\mathbb{R}^d)$ if one has for any $f \in$

$$\lim_{n\to\infty} \sigma_n(f) = \sigma_0(f), \quad f \in \boldsymbol{S}_0.$$

The so-called *Banach-Steinhaus Theorem* (from Functional Analysis) implies: it is enough to know that the limit on the left hand side exists for any (!) $f \in \boldsymbol{S}_0$, then it *defines already* a unique mild distribution $\sigma_0 \in \boldsymbol{S}_0'(\mathbb{R}^d)$.

Mild Distributions: Convergence IV

Theorem

$S_0'(\mathbb{R}^d)$ is w^*-dense in $S_0'(\mathbb{R}^d)$, i.e. for any σ_0 there exist sequences $(f_n)_{n\geq 1}$ in $S_0(\mathbb{R}^d)$ such that

$$\sigma_0 = w^*\text{-}\lim_{n\to\infty} \sigma_{f_n}.$$

Rather argument!

One possible proof is obtained by applying a so-called regularization process. Let us just consider the Dirac-comb ⊔⊔, which is neither decaying at infinity nor represented by any continuous or even smooth function. Hence one has to "localize" it (first) and the "smooth it out" (by convolution with some Dirac-like, compressed Gaussian, for example). □

Mild Distributions: Convergence V

This fact opens the way to introduce $\boldsymbol{S}_0'(\mathbb{R}^d)$ in a different (not functional analytic) way. We can show that provides an alternative description of the same vector space of "generalized function" (mild distributions).

Mild Distributions: Convergence VI

Lemma

One can identify the vector space $\boldsymbol{S}_0'(\mathbb{R}^d)$ *(containing* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *as a subspace of regular distributions).*
Conversely, one can form the set of equivalence classes of (equiconvergent) limits of w^*-*convergent sequences from* $\boldsymbol{S}_0(\mathbb{R}^d)$.
Even with the natural (inf-)norm this defines an isomorphism of normed spaces[a]

[a] This can be compared with the idea that the real numbers of the same as the equivalence classes of *Cauchy-sequences* of rational numbers. For details see the paper: "A sequential approach to mild distributions", Axioms, Vol.9 No.1, (2020) p.1-25.

Typical Heuristic Arguments I

Typical examples of w^*-convergence are often related to *heuristic arguments* in the engineering applications (or in physics).

1. We have $\delta_0 = w^*\text{-}\lim_{\alpha\to\infty} \sqcup\!\sqcup_\alpha$.
2. Correspondingly we for any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ one has

$$f = w^*\text{-}\lim_{\alpha\to\infty} \sqcup\!\sqcup_\alpha * f = \sum_{n\in\mathbb{Z}^d} f(x - n\alpha),$$

which is often used to "derive the validity of the Fourier Inversion Theorem for functions in $\boldsymbol{L}^1(\mathbb{R}^d)$ or $\boldsymbol{L}^2(\mathbb{R}^d)$ from the use of Fourier series.

Typical Heuristic Arguments II

3. We also have for any $h \in \mathbf{S}_0(\mathbb{R}^d)$ (or even $f \in \mathbf{C}_b(\mathbb{R}^d)$): the regular distribution σ_h is can be approximated by discrete measures (weighted Dirac combs)

$$\sigma_h(f) = w^*\text{-}\lim_{\alpha\to 0}[\alpha^d \cdot ⊔⊔_\alpha] \cdot h(f) = \lim_{\alpha\to 0} \alpha^d \sum_{k\in\mathbb{Z}^d} h(\alpha k) f(\alpha_k).$$

Checking the technical deatils reveals that this is nothing but the claim that for any $h \in \mathbf{C}_b(\mathbb{R}^d)$ and $f \in \mathbf{S}_0(\mathbb{R}^d)$ the pointwise product is a Riemann integrable function! Specifically for $h(x) \equiv 1$ and writing $\beta := 1/\alpha$ we get

$$\mathbf{1} = w^*\text{-}\lim_{\beta\to 0} \beta^{-d} ⊔⊔_{1/\beta}$$

Another intuitive description of w^*−convergence is to comes.

Why should be work with Banach spaces? I

It is fair to say that the purpose of function spaces is to allow a description of operators or approximation procedures. Living in a world of "signals", more precisely working with a vector space of functions or (mild) distributions over $\mathbb{R}^d$, which is to large to be spanned by a finite collection of *basis vectors*, we often talk in a sloppy way of an *infinite dimensional vector space*. In this situation it is unavoidable to measure the size of vectors (i.e., to introduce norms) in order to define convergence and allow infinite sums (series). Therefor, and for many reasons it is quite useful to ask for completeness of such normed spaces, meaning to work with *Banach spaces* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, sometimes with *Hilbert spaces* endowed with some scalar product.

The generalized Fourier transform I

The setting of THE Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ allows to describe the Fourier transform and other linear mappings of relevance in an elegant, functional analytic way. For example one can DEFINE

$$\widehat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \mathbf{S}_0.$$

This is justified by observing that for any $g \in \mathbf{S}_0(\mathbb{R}^d)$ the *Fundamental Identity for the Fourier Transform* we have

$$\int_{\mathbb{R}^d} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^d} \widehat{g}(x)f(x)dx,$$

(justified by Fubini's Theorem), or in short (!compatibility)

$$\widehat{\sigma_g} = \sigma_{\widehat{g}}, \quad g \in \mathbf{S}_0(\mathbb{R}^d).$$

The generalized Fourier transform II

The sequential approach to mild distributions (in the style of Lighthills treatment of tempered distributions, as described in the work of Laurent Schwartz) allows an alternative description of the extended version of the Fourier transform (as introduced above), avoiding the existence of integrals, and widening the scope of Fourier Analysis (away from spaces of locally Lebesgue-integrable functions and pointwise considerations!).
One can show (not done here) that the standard properties of the ordinary Fourier transform defined as usual (but only requiring Riemann integrals) as

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle s,x \rangle} dx, \quad f \in \boldsymbol{S}_0$$

The generalized Fourier transform III

maps *equivalence classes of Cauchy sequences* into equivalence classes, and since it coincides with the ordinary FT for "constant sequences" (in $\boldsymbol{S}_0(\mathbb{R}^d)$) it is a natural extension of the classical FT! Abstractly speaking, we can say: it is the unique extension of the ordinary FT, which respects w^*-convergence of sequences.

As an application one can show that there are two very important elements in $\boldsymbol{S}_0'(\mathbb{R}^d)$ which are Fourier *invariant*, namely

1. The Dirac comb, noting that $\widehat{\sqcup\!\sqcup} = \sqcup\!\sqcup$ is just the same as the validity of Poisson's formula for all $f \in \boldsymbol{S}_0$;
   ($>$ Shannon Sampling Theorem $>$ CD-player!)
2. The chirp signal $\chi(t) = \exp(-i\pi t^2)$

Both examples describe cases of *double transformable measures*.

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ I

As already indicated, the space $S_0'(\mathbb{R}^d)$ can embedded into $\mathcal{S}'(\mathbb{R}^d)$, the space of *tempered distributions*, because $\mathcal{S}(\mathbb{R}^d) \hookrightarrow (S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, as a dense subspace.
Defining the STFT (Short-Time Fourier Transform) of a distribution, using a *window* $g \in \mathcal{S}(\mathbb{R}^d)$ by

$$V_g(\sigma)(t,s) := \sigma(\overline{M_s T_t g}), (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

One can show: A distribution is a *mild distribution* if and only if $V_g(\sigma)$ is not only continuous (with polynomial growth), but also bounded! In fact, this is then true for any $g \in S_0(\mathbb{R}^d)$.
The abstract norm on $S_0'(\mathbb{R}^d)$ coincides with the supremum of $V_g(\sigma)$ over the TF-plane, and norm convergence for $(\sigma_n)_{n\geq 1}$ in $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is thus just uniform convergence of the corresponding spectrograms $V_g(\sigma_n)$ to $V_g(\sigma_0)$.

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ II

Note that in this context different non-zero functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ define one of the many equivalent norms. Often one takes the Gaussian function $g_0(t) = exp(-\pi|t|^2)$, because it satisfies $\widehat{g_0} = g_0$, and thus allows to argue that $|V_g(\widehat{\sigma})|$ is just a rotated version of $|V_g(\sigma)|$. We fix one such non-zero g for the rest.
It is thus plausible, or a practical description of w^*-convergence for mild distributions to make use of the following simple-minded characterization:

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ III

Lemma

A sequence $(\sigma_n)_{n\geq 1}$ in $S_0'(\mathbb{R}^d)$ is convergent in the w^*-sense if and only if one can observe uniform convergence of $V_g(\sigma_n)$ over compact set, i.e.
Given $R > 0$ and $\varepsilon > 0$ there exists n_0 such that

$$|V_g(\sigma_n)(t,s) - V_g(\sigma_0)(t,s)| < \varepsilon, \quad |t|,|s| < R.$$

Usually I tell the story about what kind of information is stored on a CD (at the rate of 44100 samples per second): It is just a (very reasonable) w^*-approximation of the signal which occurred!
Another more mathematical observation is the following one:
Since we have $S_0(\mathbb{R}^d) \hookrightarrow S_0'(\mathbb{R}^d)$ we can consider finite dimensional vector spaces $V \subset S_0(\mathbb{R}^d) \subset S_0'(\mathbb{R}^d)$ which are

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ IV

generated by a finite collection of (linear independent) elements in $S_0(\mathbb{R}^d)$. We can say, that this are the subspaces that we can really control, which are not too big, and which can be described using methods from linear algebra.

Functional Analysis helps us to demonstrate that for any such space V there exists some projection P_V from $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ onto V. Hence it is nice to realize that one has:

Lemma

A sequence $(\sigma_n)_{n\geq 1}$ in $S_0'(\mathbb{R}^d)$ is convergent in the w^*-sense if and only on has norm convergence in $S_0(\mathbb{R}^d)$ (or equivalently in $(L^2(\mathbb{R}^d), \|\cdot\|_2)$ or in $\big(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\big)$) for any projection, i.e.

$$\lim_{n\to\infty} \|P_V(\sigma_n) - P_V(\sigma_0)\|_{S_0} = 0.$$

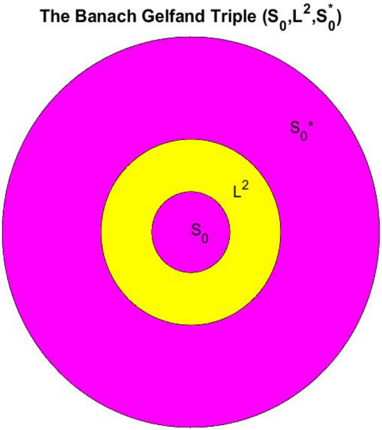


Abbildung: THE Banach Gelfand Triple

The usefulness of $(S_0, L^2, S_0')(\mathbb{R}^d)$ in Gabor Analysis I

Although it is not surprising that $S_0(\mathbb{R}^d)$ and its dual are useful in Time-frequency Analysis, in particular for Gabor Analysis, the number of results where this setting is appropriate, is astonishing. Recall that Gabor Analysis deals with the exact reconstruction of a signal from the spectrogram, resp. the representation of a mild distribution as a (in fact w^*-convergent) double series of TF-shifted copies of some Gabor atom (from $S_0(\mathbb{R}^d)$, of course). Any classical summability kernel belongs to $S_0(\mathbb{R}^d)$!
Given a certain Gabor series expansion (often one requires that it is a tight Gabor frame) of the form

$$f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda.$$

The usefulness of $(S_0, L^2, S_0')(\mathbb{R}^d)$ in Gabor Analysis II

A so-called *Gabor multiplier* is then a linear operator which arises simply by multiplying the coefficients with some numbers $\mathbf{m} = (m_\lambda)_{\lambda \in \Lambda}$, the *upper symbol* of GM_m, given by

$$GM_m(f) = \sum_{\lambda \in \Lambda} m_\lambda \langle f, g_\lambda \rangle g_\lambda.$$

It is not hard to show that any bounded sequence $\mathbf{m} \in \ell^\infty(\Lambda)$ defines a bounded linear operator not only on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but also on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and $\left(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}\right)$, which is also w^*-w^*-continuous. Similar more involved methods allow to describe *pseudo-differential operators*, and in particular so-called Anti-Wick operators (continuous STFT-multipliers) in this setting.

Operating on the audio signal: filter banks

There are many more applications

The list of topics which we could not discuss here is long, we just mention very few of them:

1. A kernel theorem, allowing a kind of "continuous matrix representation" of linear operators from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.
2. Correspondingly description of operators using the spreading representation or equivalently the Kohn-Nirenberg calculus;
3. Using the FFT to compute $\widehat{f}$ from samples of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, followed by e.g., piecewise linear interpolation;
4. more general **modulation spaces**, such as , where $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ arises for $p = 1 = q$, and $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ corresponds to the case of $p = \infty = q$.
5. Wilson bases, coefficients in mixed norm sequence spaces.

Physics Nobel Prize 2017 (Jarnick Lecture in Prague)

Time-Frequency Analysis and Black Holes

Breaking News

Today, Oct. 3rd, 2017, the **Nobel Prize in Physics** was awarded to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!

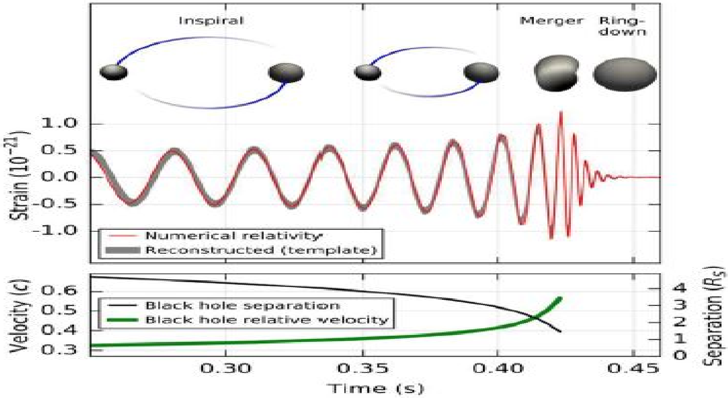
The Prize-Winners are

Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".

There had been two strategies:

- Searching for 2500 explicitly determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.

NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).