-5mm

NuHAG Numerical Harmonic Analysis Group

QHA via the Banach Gelfand Triple

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at
www.nuhag.eu TALKS at www.nuhag.eu/talks

NTNU Trondheim, Quantum Harmonic Analysis
June 7th, 2023

Original Abstract by HGFei I

QHA via the Banach Gelfand Triple

In my talk I would like to give a presentation of QHA from the point of view of "THE" Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$, consisting of the chain

$$\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right) \hookrightarrow \left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right) \hookrightarrow \left(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}\right),$$

i.e. what people call the *Feichtinger algebra* $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$, the Hilbert space $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the ambient dual space $\mathbf{S}_0'(\mathbb{R}^d)$ of *"mild distributions"* on $\mathbb{R}^d$. It contains no only all the spaces $\left(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, but also the translation-bounded measures, hence Dirac measures and Dirac combs, to give an example, and all the (almost) periodic functions.

Original Abstract by HGFei II

The starting point will be a quick review of the concept of an abstract Banach Gelfand Triple, and then (unitary) BGTr isomorphisms (UBGTIs). A classical case is the isomorphism between the triple $(\boldsymbol{A}(\mathbb{T}), \boldsymbol{L}^2(\mathbb{T}), \boldsymbol{PM}(\mathbb{T}))$ and the sequence space $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.
Starting at the function space level any of the usual groups of operators occurring naturally, like translations, modulations, rotations and so on induce such automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
More importantly the Fourier transforms is a unitary Banach Gelfand Triple automorphism (a UBGTA) of "THE" BGT. It is uniquely determined by the fact that it satisfies $\mathcal{F}(\chi_t) = \delta_t$.
Another important ingredient is the Banach Gelfand triple of operators which can be built around the kernel theorem for Hilbert-Schmidt operators. If we restrict the attention to (integral) operators with kernels $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ then we get operators which

Original Abstract by HGFei III

map $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, and turn bounded, w^*-convergent sequences into norm convergent sequences in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (we write $\boldsymbol{B}$ for this operator algebra, and call this the "inner kernel theorem"). Recall that the pairing in $\mathcal{HS}$ is via the Hilbert-Schmidt scalar product $\langle S, T\rangle_{\mathcal{HS}} = trace(S * T')$. One can show that this duality (suitably extended) allows to identify the dual $\boldsymbol{B}'$ with $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, endowed with the operator norm. Thus we can describe the kernel theorem as a UBGTI between the operator BGTr $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$.
There are several other representations of operators which in turn allow to be expressed via connecting UBGTs, above all the spreading representation and the Kohn-Nirenberg representation. Using the notations of [1] we write $\eta(T)$ and $\sigma(T)$, noting that they are identifiable via a *symplectic Fourier transform* on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (another UBGTI). At the "inner level" we describe the operators in

Original Abstract by HGFei IV

$\boldsymbol{B}$ as integral operators with $K(x,y)$ belonging to $\boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, via $K(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y))$, and the spreading symbol $\eta(T)$ allows to describe the operator as simple integral

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(\lambda)\pi(\lambda) d\lambda,$$

with $\eta(T) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
Now the convolutions arising in QHA can be simply explained as convolutions in the Kohn-Nirenberg domain, combined with the identification of operators T with their KNS-symbol $\sigma(T)$. The operator-Fourier-transform coincides correspondingly with spreading representation $T \mapsto \eta(T)$ which provides (at the inner level) an identification of $\boldsymbol{B}$ with $\boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Original Abstract by HGFei V

Many of the corresponding formulas and identities then follow easily from the properties of $(\boldsymbol{S}_0, \|\cdot\|_{\boldsymbol{S}_0})$ and $(\boldsymbol{S}_0', \|\cdot\|_{\boldsymbol{S}_0'})$.
In short we have (with $*$ representing convolution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$), at least for operators

$$S, T \in \boldsymbol{B} := \{T \in \mathcal{L}(\boldsymbol{L}^2, \boldsymbol{L}^2), \sigma(T) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)\}$$

$$= \{T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \text{Int.Op. with kernel in } \boldsymbol{S}_0(\mathbb{R}^{2d})\}.$$

- Given S, T in $\boldsymbol{B}$, we define a function in $\boldsymbol{S}_0(\mathbb{R}^d)$ via $S \star T = \sigma(S) * \sigma(T)$;
- For $T \in \boldsymbol{B}$, $f \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ $f \star T$ is the operator with $\sigma(f \star T) = f * \sigma(T)$;
- $\mathcal{F}_W(T) = \eta(T) = \mathcal{F}_S(\sigma(T)) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$..

Original Abstract by HGFei VI

Some relevant publications on the subject of Banach Gelfand Triples are listed below.
[feko98], **[fezi98]**, **[fe02]** **[cofelu08]**, **[fe09]**, **[fe18-3]**, **[lusk18-3]**.

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

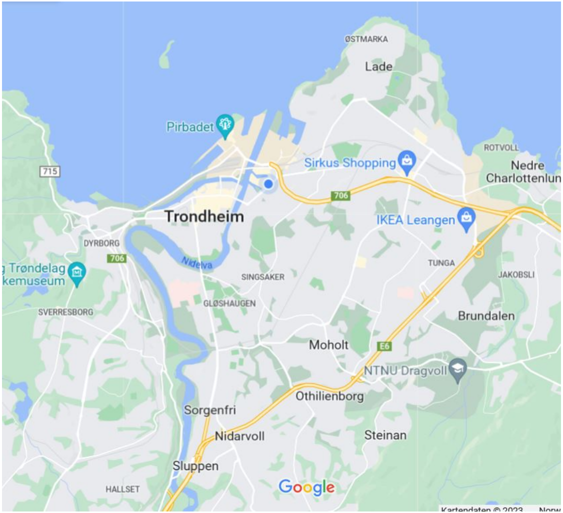


Abbildung: Trondheim23

Conceptual Harmonic Analysis I

Those who know me may expect that also in this talk I will try to promote my ideas about **Conceptual Harmonic Analysis** which is the blend of Abstract Harmonic Analysis (AHA) and Computational Harmonic Analysis.
It raises the question whether the discrete model can be use to approximate the continuous in the right sense.
In fact, I want to point out that this is possible for the case objects and operations arising in QHA, because all of these objects can be defined on any LCA group, hence also on finite Abelian groups.
Consequently one has to find ways to use this information to compute an approximate solution to the "continuous"
(or high resolution) problem.

Benefits from this approach

There are several steps that make the CHA-viewpoint useful.

- First of all the AHA point of view allows to compare the problems that one wants to address on a similar, finite dimensional context (finite LCA groups), like my paper with Werner Kozek and Franz Luef about Gabor analysis on finite Abelian groups (!no converngence problems)
- The algebraic structure is usually visible (strucutre preserving approximations are the goal!) in the finite case;
- Then one deals with continous variable, which requites some functional analysis!
- Finally on has to ask whether there is some good form of approximation of the continuous case by the discrete. E.g. can one approximate $\widehat{f}$ using the FFT only?

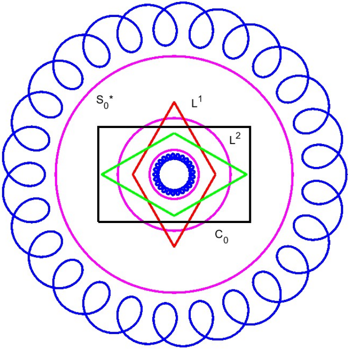


Abbildung: scsoclass23.jpg

Modern Fourier Analysis of the 20^{th} Century I

Following the path of "Modern Analysis" of the 20th century Fourier Analysis can only be understood once we have a good understanding of Lebesgue integration theory, established 100 years after Fourier's ideas concerning Fourier series.
We saw the transition from the treatment of periodic functions via *Fourier series* to the (even more technically demanding) discussion of the *Fourier transform* as an integral transform, with Parseval being replaced by Plancherel's Theorem. *Pontrjagin's Duality Theorem* provides the group theoretical background (the dual group of the dual group is naturally identified with the original, Abelian group!), and the existence of a *Haar measure* allows to transfer the basic facts into the *locally compact Abelian* (LCA) context. Weil's book of 1940 is summarizing these principles (and has inspired Hans Reiter): Abstract Harmonic Analysis.

Modern Fourier Analysis of the 20^{th} Century II

The realization of *Carleson's result* (1972) (almost everywhere convergence of Fourier series arising from $\boldsymbol{L}^2(\mathbb{T}) \subset \boldsymbol{L}^1(\mathbb{T})$) and Malliavin's result on set in $\mathbb{R}^3$ which do not allow *spectral synthesis* are definitely mathematical highlights, but do not influence the daily life of engineers, just look up signal processing books. But, the study of $\boldsymbol{L}^p-$spaces got *Functional Analysis* started.

Cooley/Tukey published their FFT (Fast Fourier Transform) algorithm in 1965. It is the backbone of digital signal processing.

Around the same time Laurent Schwartz was introducing the setting of *tempered distributions*, the elements of the dual $\mathcal{S}'(\mathbb{R}^d)$ of the *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions. It is a (fancy) nuclear Frechet space, thus requiring a theory of topological vector spaces. It is invariant under the (classical) Fourier transform, and an algebra under convolution and pointwise multiplication.

Modern Fourier Analysis of the 20^{th} Century III

It is true (but not obvious) that the *sequential approach to* $\mathcal{S}(\mathbb{R}^d)$ *worked out by Lighthill* is equivalent and more handy for engineers. We also have to point out, that based on the work of Lars Hörmander this setting is nowadays at the basis of a modern theory of PDEs. But is it needed in order to discuss the properties of a Dirac comb in an engineering course??? And how should it be done, technically speaking, using the "sifting property" of the Dirac Delta?

We all know the *engineering definition* (what is 3δ?);

$\delta(x) = 0$ for $x \neq 0$, and $\delta(0) = +\infty$, with the agreement that

$$\int_{\mathbb{R}^d} \delta(x)dx = 1.$$

Shortcomings of the AHA approach

The short-comings of the AHA approach are (at least from my *personal judgement*) the following ones:

- AHA only establishes **analogies** between the Fourier transforms on different groups. It is good in order to determine the "uniqueness" of the Fourier transform (within an isomorphism class of LCA groups);
- It requires quite some technical tools (Haar measure, Lebesgue integral, Schwartz-Bruhat space, etc.);
- The emphasis on the spaces $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ prohibits the discussion of other, perhaps more important aspects (e.g. *local* versus *global* properties of a function). Among others it restricts the consideration of the Fourier transform to $\boldsymbol{L}^p(G)$, for $1 \leq p \leq 2$.

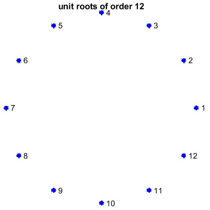
Starting from SCRATCH I

The basis for Fourier Analysis are complex exponentials!

Starting from the chain of fields $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, *polynomials* of the form $p(z) = \sum_{k=1}^{n} a_k z^{n-k}, z \in \mathbb{C}, \mathbf{a} \in \mathbb{C}^n$ can be treated.

```
Hence: polyval([1,2,3],10) == 123
```

We can define unit roots of order n and display them like this (labelling ω_n^k by the number $k+1$):



Starting from SCRATCH II

Finally we have complex exponential functions satisfying the exponential law

$$e^{z_1+z_2} = e^{z_1} + e^{z_2}, \quad z_1, z_2 \in \mathbb{C},$$

and giving us "most beautiful formula" $\exp(2\pi i) = 1$. Hence the unit roots can be obtained as (MATLAB command) `u = exp(2*pi*i*(0:1/N:(N-1)/N));`

Working with numbers is also teaching us a lot about how to "deal with infinity" (meaning actually how to pretend its existence without running into *existential/philisophical troubles*).

$$\text{Verify that} \quad \frac{3\pi^2}{4\pi} = 0.75\,\pi$$

Starting from SCRATCH III

Note: the infinite decimal expression describing $\pi \in \mathbb{R}$ has *never been fully spelled out*, so what is the square of such a number, leave alone the multiplicative inverse (described as $1/\pi$ as a !*symbol only*). In addition the above formula contains a mixture of (simple) *rational numbers* such as $3/4$ and real numbers (such as 0.75 or π). Still it all looks good and works well, justified by approximations and corresponding continuity considerations. Nevertheless certain expressions, better non-sense symbols such as $5/0$, have to be forbidden (for good reasons)!

We will see a similar story in the distributional context!

The BOTTOM UP approach to Fourier Analysis I

Instead of the historical path I think it is time to REBUILD Fourier Analysis from scratch, in a way which is both *mathematically correct* and *useful for the applications in engineering* (signal processing) and *physics* (making heuristic derivations precise, in the sense of giving symbols a clear mathematical meaning!)

After all, discrete Fourier Analysis, i.e. the DFT, realized as FFT, is a piece of Linear Algebra, which is familiar to every student. It can be realized with the help of MATLAB (or other mathematical software programs) and thus also accessible to numerical work. Thus it is not only useful for applications in signal processing, mobile communication or streaming, but also the basis for Numerical (or Computational) Harmonic Analysis.

The BOTTOM UP approach to Fourier Analysis II

Abstract Harmonic Analysis (AHA) is helping us to realize the connections between the different forms of the Fourier transform through the group theoretical glass. Engineers talk about continuous signals or discrete ones. Fourier series are good for periodic, continuous functions, and the DFT/FFT is suitable for the treatment of periodic and discrete signals. Each of those worlds follow similar rules (translation goes to modulation by pure frequencies or characters, and vice versa), or one has a convolution theorem, but again those convolutions depend on the context (e.g. cyclic convolution for the finite case). AHA tells us that the DFT should be read as Fourier Analysis on the cyclic group $\mathbb{Z}_n$ because (easy exercises) the columns (or rows) of the Fourier matrix `fft(eye(n))` are exactly the characters of $\mathbb{Z}_n$.

The BOTTOM UP approach to Fourier Analysis III

There are "magic properties"; here a, b are divisors of n:

1. They form (up to the scaling $\sqrt{n}$) an ONB for $\mathbb{C}^n$;
2. They represent a *Vandermonde Matrix* (see below), for $\mathbb{Z}_n$ taken in the clockwise sense;
3. The subsampling to $\mathbb{Z}_{n/a}$ gives the n/a Fourier matrix.
4. The sequence $\mathbb{Z}_n^b$ corresponds to b copies of unit roots of order n/b, in natural order. Only for p prime $x \mapsto x + x$ is an isomorphism (relevant for Wigner).

```
norm(fliplr(vander(conj(u12)))- fft(eye(12))) ca. 0
compares the Vandermode matrix for the unit roots,
in the clockwise sense, with the DFT matrix. Note
the convention for polynomials as described above!
```

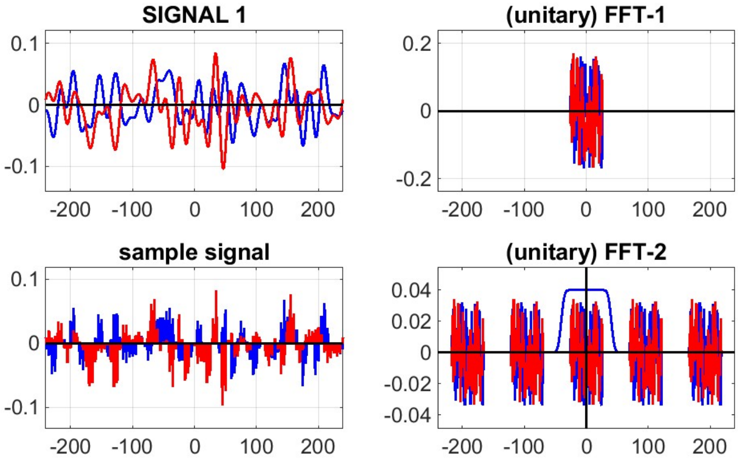


Abbildung: demshannon3.jpg

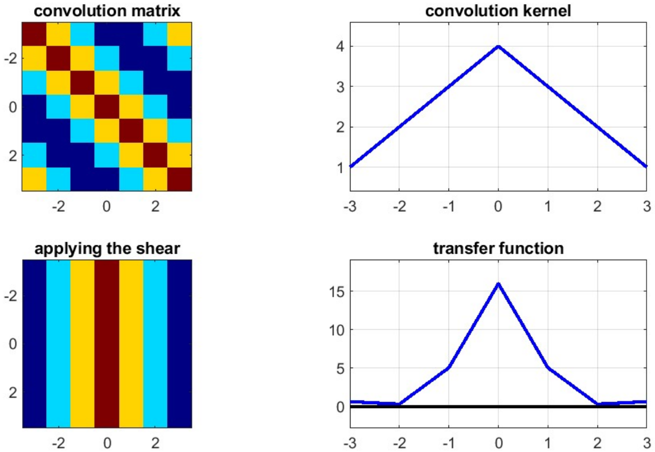


Abbildung: convop4matA.jpg

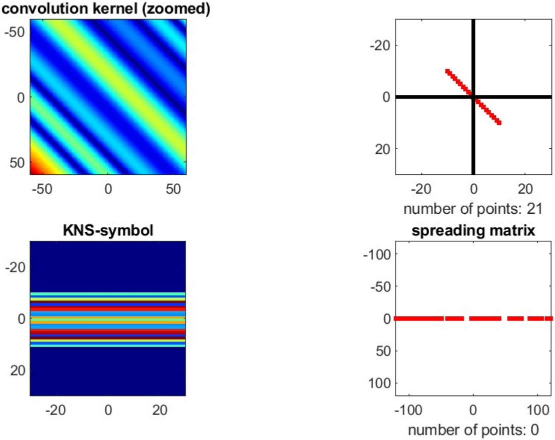


Abbildung: The four different representations of a matrix in the form of an $n \times n$-matrix.

Matrix realization of the four representations I

The idea was to represent an operator by a space variant transfer function, so instead of having a **moving average** with constant profile, one reduces the consideration first to say that it is a variable average, to while the profile is moving it is also changing the shape.

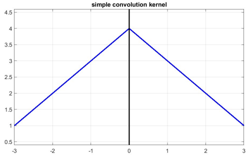


Abbildung: Showing the "convolution kernel".

Realization of KNS for matrices I

Applying first an automorphism of $\mathbb{Z}_n \times \mathbb{Z}_n$:

$$[allowframebreaks] \begin{pmatrix} 1 & 7 & 13 & 19 & 25 & 31 \\ 8 & 14 & 20 & 26 & 32 & 2 \\ 15 & 21 & 27 & 33 & 3 & 9 \\ 22 & 28 & 34 & 4 & 10 & 16 \\ 29 & 35 & 5 & 11 & 17 & 23 \\ 36 & 6 & 12 & 18 & 24 & 30 \end{pmatrix} \qquad (1)$$

Then we take the discrete Fourier transform of the rows (or columns) in order to find the representation of the matrix as in the spirit of a *time-variant transfer function*.

Realization of KNS for matrices II

Alternatively one can start to explain the spreading representation of a a given $n \times n$ matrix $\mathbf{A}$. In fact, we have n cyclic shift matrices and n rows of the DFT matrix, which we can combine, in order to obtain n^2 TF-shifts $\pi(\lambda)$. It is easy to check that these matrices form an ONB in the Hilbert Schmidt sense, because they are supported on different side-diagonals for different shift paramters, and if the shift-parameter is equal then different frequencies or orthogonal to each other (unitarity of Fourier matrix)
Note the the Hilbert-Schmidt structure of matrices with norm

$$\|\mathbf{A}\|_{\mathcal{HS}} := \mathrm{trace}(\mathbf{A} * \mathbf{A}')$$

conincides with the Frobenius norm of a matrix, i.e. with the norm of $\mathbf{A}$ viewed as an element of the *Euclidean space* $\mathbb{C}^{n^2}$!

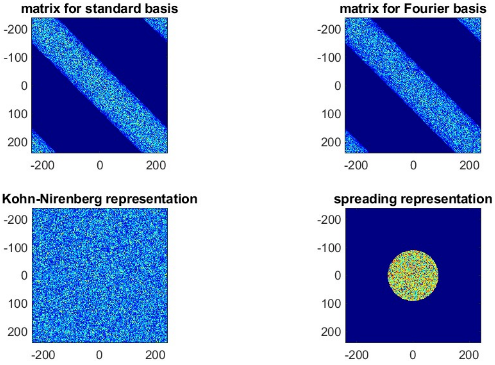


Abbildung: The description of an underspread operator, who is well concentrated in the spreading domain.

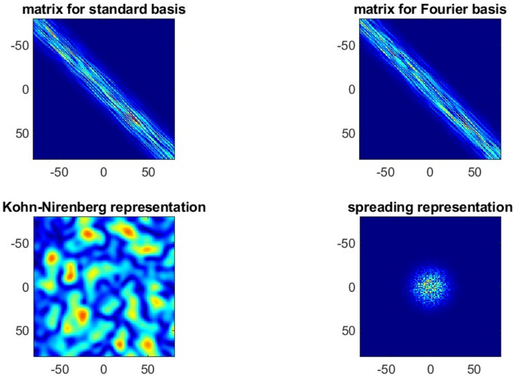


Abbildung: 4 version of the description of a Gabor multiplier, or Anti-Wick operator, with Gaussian window, zoomed version.

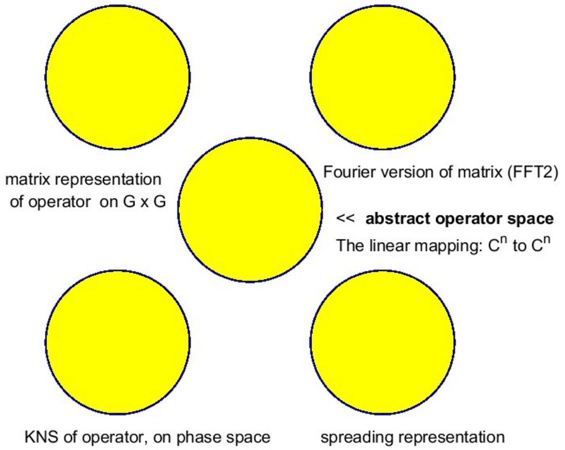


Abbildung: Different Matrix Representations

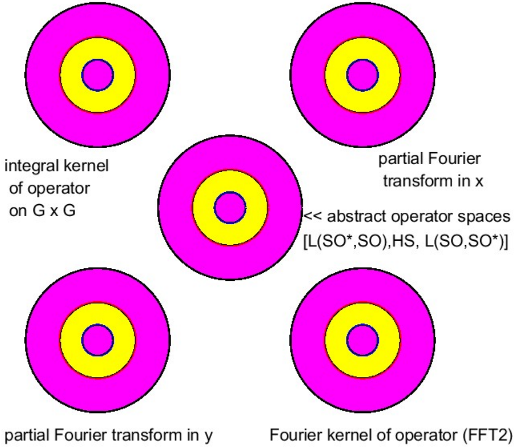


Abbildung: The spaces of operators $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ can be identified with four different representations.

Feichtinger operators I

The operators in $\mathcal{B}$ are exactly those of the form with an integral kernel $K \in \mathbf{S}_0(\mathbb{R}^{2d})$, i.e.

$$T(f) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in \mathbf{S}_0(\mathbb{R}^d),$$

$$\text{with} \quad K(x,y) = T(\delta_y)(\delta_x), \quad x, y \in \mathbb{R}^d.$$

They are trace class operators, in fact dense in $\mathcal{S}^1$, and

$$\text{trace}(T) = \int_{\mathbb{R}^d} K(x,x) dx,$$

as expected. Composition can be written as

$$K_{T_2 \circ T_1} = \int_{\mathbb{R}^d} K_2(z,x) K_1(x,y) dx.$$

Feichtinger operators II

By the classical kernel theorem Hilbert-Schmidt operators with the norm $\|T\|_{\mathcal{HS}} := \sqrt{\mathrm{trace}(HH^*)})$ obtained from the correcponding scalar produce $\langle T, S\rangle_{\mathcal{HS}}$ are isometrically realized as integral operators with $K \in \left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$.
We have the a chain of proper inclusions of operator spaces:

$$\mathcal{B} \hookrightarrow \mathcal{S}^1 \hookrightarrow \mathcal{S}^2 = \mathcal{H}_s \hookrightarrow\hookrightarrow \mathcal{B}'.$$

Typical elements of $\mathcal{B}'$, with kernel K characterized by the equality

$$Tf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)f(y)dy\, g(x)dx = \langle K, g \otimes f\rangle_{\boldsymbol{S}_0', \boldsymbol{S}_0},$$

allow to describe all bounded linear operators from $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ to $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$, for $1 \leq p, q < \infty$.

Spreading representation versus KNS symbol

The Kohn-Nirenberg symbol description of a "time-varying system" is just the description of an operator, given a for each position the current transfer function. Writing σ_T for the KNS-symbol (which is a function on phase space) we can write

$$Tf(x) = \mathcal{F}^{-1}(\widehat{f} \cdot \sigma_T(x,.)) = \int_{\mathbb{R}^d} \widehat{f}(y)\sigma_T(x,y)e^{2\pi ixy}dy$$

It is also known that operators have a *spreadig representation*. The spreading kernel $\eta(T)$ can be defined to be the *symplectic Fourier transform* of the KSN-symbol $\sigma(T)$, and it is characterized by the representation of T as a superposition of TF-shifts:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(\lambda)\pi(\lambda)d\lambda.$$

For Hilbert-Schmidt operators the kernels, KNS-symbols or the spreading symbol are all in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ resp. $\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and Plancherel's Theorem helps.

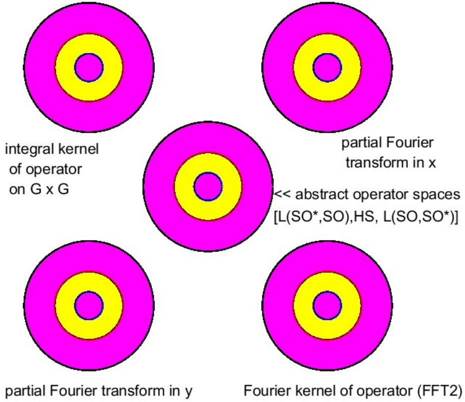


Abbildung: The spaces of operators $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ can be identified with four different representations. WE ADD group automorphisms!

Transfer of Structure

We know from Linear Algebra courses that matrix multiplication is not just a "funny rule". It is compatible with the composition and inversion of linear mappings!
In a similar way I have introduced convolution of measures over LCA groups based on an isometric identification of the operators on $\left(\mathbf{C}_0(G), \|\cdot\|_\infty\right)$ which commute with translations.
Now we can say, that $\mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ can be identified with an algebra of operators and thus we can obviously transfer this structure (composition) to $\mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ by taking the spreading function. We obtain then twisted convolution of functions.

Finally, we can do convolution of functions in $\mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ by viewing them as representations of operators in $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ (or functions in $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$), which turns out to be the starting point for QHA!

Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

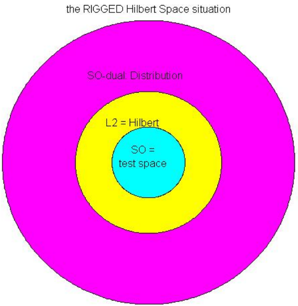


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (**[lo83-1]**).

For $\mathbb{R}^d$ the most elegant way (which is describe in **[gr01]** or **[ja18]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}\,dy$$

and the corresponding norm

$$\|f\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem

1. $\boldsymbol{S_0}(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{W}(\boldsymbol{C_0}, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C_0}(\mathbb{R}^d)$;
2. $\mathcal{F}(\boldsymbol{S_0}(\mathbb{R}^d)) = \boldsymbol{S_0}(\mathbb{R}^d)$ *(isometrically);*
3. *Isometrically invariant under TF-shifts*
$$\|\pi(\lambda)(f)\|_{\boldsymbol{S_0}} = \|M_s T_t f\|_{\boldsymbol{S_0}} = \|f\|_{\boldsymbol{S_0}}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$
4. $\left(\boldsymbol{S_0}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S_0}}\right)$ *is an essential double module (convolution and multiplication)*
$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S_0}(\mathbb{R}^d) \subseteq \boldsymbol{S_0}(\mathbb{R}^d) \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S_0}(\mathbb{R}^d) \subseteq \boldsymbol{S_0}(\mathbb{R}^d),$$
in fact a Banach ideal and hence a double Banach algebra.
5. *Tensor product property* $\boldsymbol{S_0}(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{S_0}(\mathbb{R}^d) \approx \boldsymbol{S_0}(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(\boldsymbol{S_0}(G)) = \boldsymbol{S_0}(H)$.

1. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ has various equivalent descriptions, e.g.
   - as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$;
   - via *atomic decompositions* of the form

$$f = \sum_{i \in I} c_i \pi(\lambda_i) g \text{ with } (c_i)_{i \in I} \in \ell^1(I).$$

2. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is invariant under group automorphism;
3. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is invariant under the *metaplectic group*, and thus under the *Fractional Fourier transform* as well as the multiplication with *chirp signals*: $t \mapsto exp(-i\alpha t^2)$, for $\alpha \geq 0$.

In addition $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is quite universally useful in Classical Fourier Analysis and of course for *Time-Frequency Analysis* and *Gabor Analysis*, and as I am going to show also for QHA: Quantum Harmonic Analysis. In short, it is easier to handle than the Schwartz-Bruhat space or even the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, and since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ it is (much) bigger.

Theorem

1. $\mathcal{F}(\mathbf{S}_0'(\mathbb{R}^d)) = \mathbf{S}_0'(\mathbb{R}^d)$ *via* $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \mathbf{S}_0'$.
2. *Identification of TLIS:* $\mathbf{H}_G(\mathbf{S}_0, \mathbf{S}_0') \approx \mathbf{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0') \approx \mathbf{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0) \approx \mathbf{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\mathbf{S}_0' * \mathbf{S}_0) \cdot \mathbf{S}_0 \subseteq \mathbf{S}_0$, $(\mathbf{S}_0' \cdot \mathbf{S}_0) * \mathbf{S}_0 \subseteq \mathbf{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \to \mathbf{S}_0(H) \hookrightarrow \mathbf{S}_0(G)$ *via* $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \mathbf{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \mathbf{S}_0(G) \mid \mathrm{supp}(\tau) \subset H\}$.

Theorem

1. $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}) = (\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$, *with* $V_g(\sigma)$ *and* $\|\sigma\|_{\boldsymbol{S}_0'} = \|V_g(\sigma)\|_\infty$, *hence norm convergence corresponds to uniform convergence on pahse space. Also* w^*-*convergence is uniform convergence over compact subsets of phase space.*
2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, *with density for* $1 \leq p < \infty$, *and* w^*-*density in* $\boldsymbol{S}_0'$. *Hence, facts valid for* $\boldsymbol{S}_0$ *can be extended to* $\boldsymbol{S}_0'$ *via* w^*-*limits.*
3. *Periodic elements* $(T_h\sigma = \sigma, h \in H)$ *correspond exactly to those with* $\tau = \mathcal{F}(\sigma)$ *having* $\operatorname{supp}(\tau) \subseteq H^\perp$.
4. *The (unique) spreading representation* $T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)d\lambda$, $F \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *for* $T \in \mathcal{B}$ *extends to the isomorphism* $T \leftrightarrow \eta(T)$ $\eta : \mathcal{B} \approx \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, *uniquely determined by the correspondence with*
$$\eta(\pi(\lambda)) = \delta_\lambda, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Some conventions

Scalarproduct in $\mathcal{H}S$:

$$\langle T, S\rangle_{\mathcal{H}S} = \text{trace}(T * S^*)$$

In **[feko98]** the notation

$$\alpha(\lambda)(T) = [\pi \otimes \pi^*(\lambda)](T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

and the covariance of the KNS-symbol is decisive:

$$\sigma(\pi \otimes \pi^*(\lambda)(T)) = T_\lambda(\sigma(T)), \quad T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

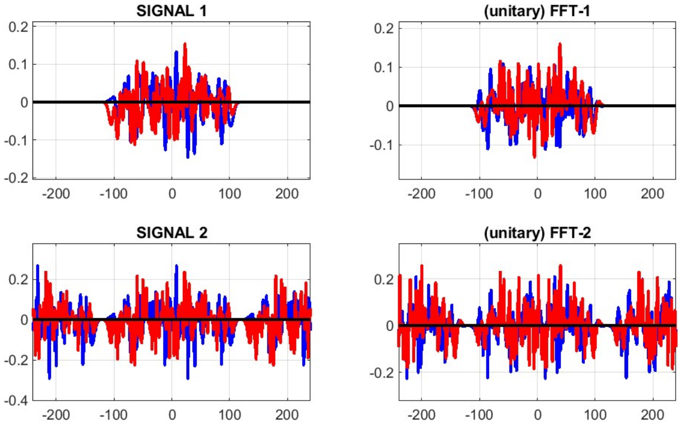


Abbildung: persampf2spA.jpg

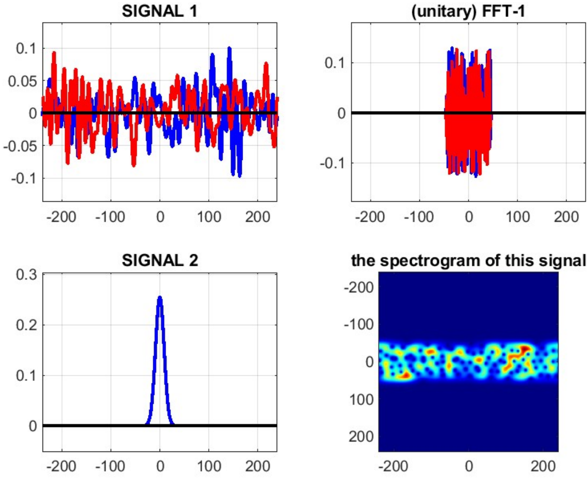


Abbildung: A low pass signal, with spectrogram

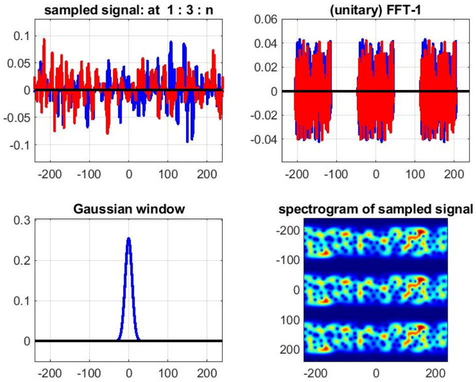


Abbildung: Effect of sampling in the spectrogram

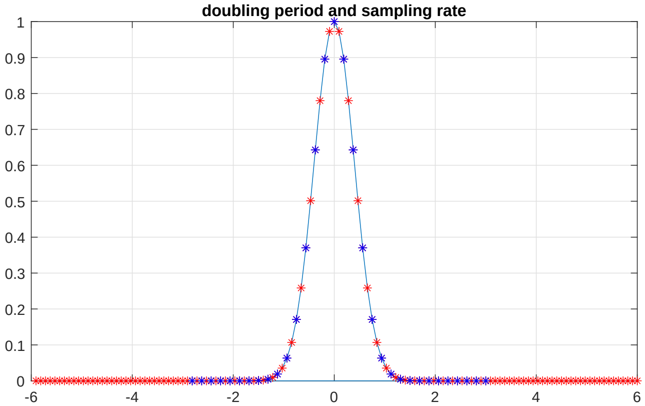


Abbildung: Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ I

It is a harmless but important observation that the squares of the elements of $\mathbb{Z}_n$ (rotation by multiples of $2\pi/n$) are just the elements of $\mathbb{Z}_{n/2}$ (only for n is even!), repeated twice.
Thus for us the operator which replaces a given function (or matrix) by its 2-periodic and 2-sampled version will be of big relevance. Also, since all the information comes twice (for matrices in both the world of column AND the world of rows) we have to understand how to extract properly the subsequence of indices "most representative" for such a reduction (turning vectors of length n into vectors of length $n/2$) or just of length $2n$ into vectors of length n and matrices of size $2n \times 2n$ into matrices of size n, in a compatible way.
We will illustrate this by some plots and also verify that this procedure is well compatible with many of the representations of functions of operators.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ II

As a basic example let us take a function with small support, then produce its p-periodic version, and then sample at the rate of $1/p$, $p \in \mathbb{N}$. Then you will find that the "representing sequence" of the Fourier version of such a function, treated in the same way, will be just (suitable normalized) the FFT of the finite vector (of length p^2, of course) of the vector in $\mathbb{C}^{p^2}$ representing the discrete and periodic signal on $\mathbb{R}$.

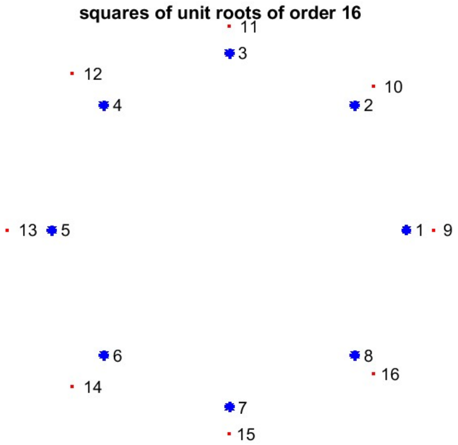


Abbildung: twiceZn16A.jpg

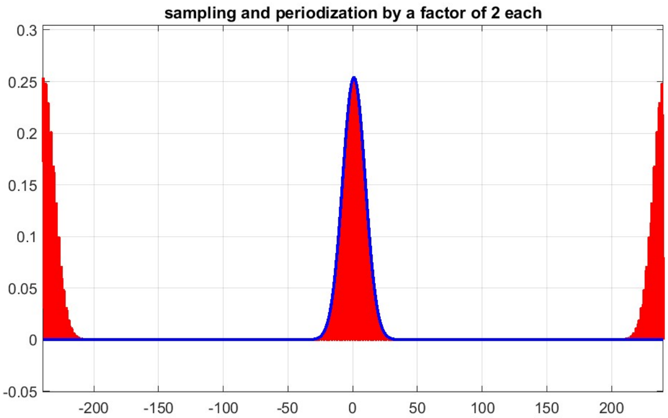


Abbildung: The reduction from the original curves (in blue) to the red curve is by sampling. Since every second value is zero the graph looks filles, plus periodic repetition.

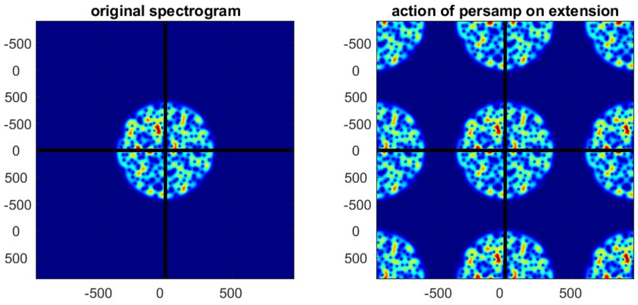


Abbildung: The picture in the STFT domain. Actually, it is the (twisted) convolution with a 2D Dirac comb with 4 points, so it is a 2D-periodization of the spectrogram.

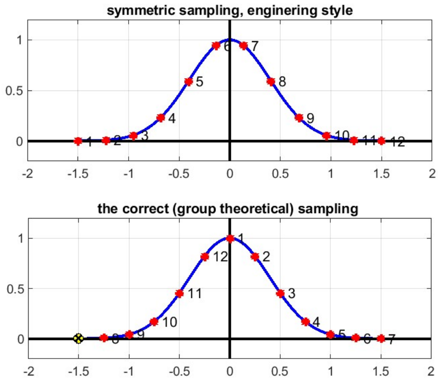


Abbildung: Naive versus correct (group theoretical) sampling. There is natural behavior with respect to refinement of the sampling and taking multiples of the period

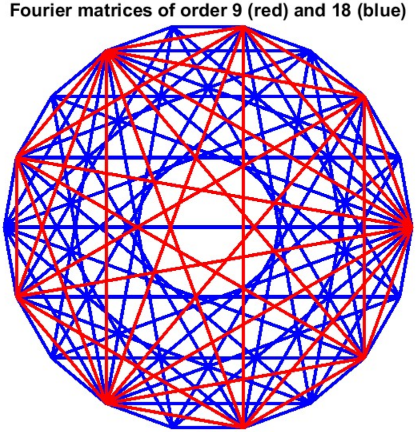


Abbildung: Overlay: fft(eye(9)) (red) over fft(eye(18)) (blue)'

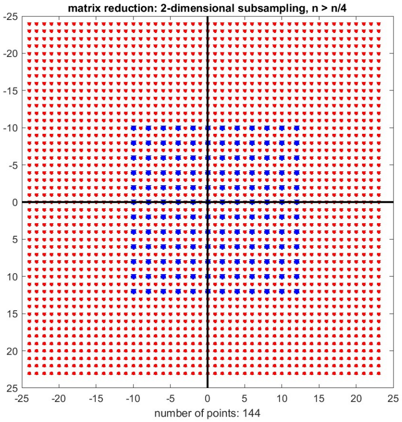


Abbildung: Subsampling from a matrix, using a QUARTER of each row and column, reducing the number of entries by the factor 16.

Structure perservig operations

We summarize the situation in the finite/discrete case:

- Many of the relevant operation on functions on phase space, or $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ or $\mathbb{Z}_n \times \mathbb{Z}_n$ are highly compatible with the reduction steps announced;
- In particular the DFT/FFT of a periodized and sampled version can be obtained via the corresponding FFT;
- The mapping from functions on $\mathbb{Z}_{2n}$ to functions on $\mathbb{Z}_n$ is a homomorphism of algebras, for both the pointwise (obvious) and thanks to the FFT-observation also (circular) convolution (up to rescaling)!
- It is also complatible with the shear-operator (first step towards Kohn-Nirenberg), and the spreading representaton;
- As a special case (!flat tori) the STFT is compatible with the reduction step.

Next we start to discuss the approximation of functions from the samples of a peridiozed and sampled version of a given function in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. The established results use quasi-interpolation operators arising from certain BUPUs (bounded in the Fourier algebra, like the sequence of B-splines of order 2 and higher). These are *qualitative results* and may not provide optimal speed of recovery, BUT they apply to all functions in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Let us first give an explanation for the case $d = 1$, i.e. for the real line, where all the discrete (cocompact) subgroups $\Lambda \lhd \mathbb{R}$ are of the form $\Lambda = \alpha\mathbb{Z}$, for some $\alpha > 0$.

Definition

We call a sequence of pairs $(\alpha_k, \beta_k)_{k\geq 1}$ with $d_k = \beta_k/\alpha_k \in \mathbb{N}$ *exhausting* if they satisfy

$$\lim_{k\to\infty} \alpha_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \beta_k = \infty$$

1. A natural, bounded sequence of operators SP_k from $(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0})$ to $(\boldsymbol{S}_0'(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0'})$ is then given by **s**ampling combined with **p**eriodization:

$$\mathrm{SP}_k(f) := \alpha_k \sqcup\!\sqcup_{\alpha_k} \cdot (f * \sqcup\!\sqcup_{\beta_k}) = [\alpha_k \sqcup\!\sqcup_{\alpha_k} \cdot f] * \sqcup\!\sqcup_{\beta_k}$$

2. There exists a sequence of operators R_k with

$$\lim_{k\to\infty} \|\mathrm{R}_k(\mathrm{SP}_k(f)) - f\|_{\boldsymbol{S}_0} = 0.$$

Note: it is plausible that the sequence $\mathrm{SP}_k(f)$ contains all the required information about f because one has for $k \to \infty$:

$$w^*\text{-}\lim_{k\to\infty} \alpha_k ⊔⊔_{\alpha_k} = \mathbf{1} \quad \text{and} \quad w^*\text{-}\lim_{k\to\infty} f ⊔⊔_{\beta_k} = \delta_0,$$

so that one has in fact for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$:

$$f = (f \cdot \mathbf{1}) * \delta_0 = w^*\text{-}\lim_{k\to\infty} \mathrm{SP}_k(f) = (f * \delta_0) \cdot \mathbf{1}.$$

Lemma

It is clear that each of the periodic discrete signals which are in the range of SP_k can be viewed as an element of the cyclic group $G_k = \mathbb{Z}_{d_k}, k \geq 1$ of order d_k. Moreover the Fourier transform in the $\boldsymbol{S}_0'$-sense of these discrete periodic functions corresponds to the DFT/FFT on the corresponding finite group G_k.

Written in formulas (and ignoring the explicit formulation of the isomorphism, even if one has to be careful in practice) this means, let us assume for simplicity that $\alpha_k = 1/\beta_k$, with $\beta_k \in \mathbb{N}$ tending to infinity, like $\beta_k = 2^k$:

$$\mathrm{SP}_k(\widehat{f}) = \mathrm{FFT}(\mathrm{SP}_k(f)). \tag{2}$$

The approximation result of a joint paper with N. Kaiblinger (**[feka07]**, then gives:

Theorem

$$\lim_{k\to\infty} \|\widehat{f} - \mathrm{R}_k(\mathrm{FFT}(\mathrm{SP}_k f))\|_{\boldsymbol{S}_0} = 0, \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

For the computation of dual FTs and dual Gabor atoms this has been used by N. Kaiblinger in **[ka05]**.

Recipes for transfer

The question addressed here is the transfer between insight (e.g. MATLAB simulations) in the finite-discrete case and related continuous problems:

1. By taking the AHA view-point one can expect that replacing sums by integrals and the FFT by the continuous Fourier transform will give an heuristic starting point.
2. At the level of $\mathcal{B}$ (inner kernel theorem) this is actually true for all the cases discussed in this paper
3. Showing unitarity one can extend to the $\mathcal{HS}$ level
4. By duality or a sequential approach to mild distributions one can extend it to the outer layer.
5. NEW: one can expect to get for the "inner case" good numerical evidence by applying the above SP-principles.

Some not so obvious references I

[fegrla79] H. G. Feichtinger, C. C. Graham, and E. H. Lakien.
Nonfactorization in commutative, weakly self-adjoint Banach algebras.
Pacific J. Math., 80:117–125, 1979.

[feko98] H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

This paper provides the BGTr view on operators!

[fe02] H. G. Feichtinger.
Spline-type spaces in Gabor analysis.
In D. X. Zhou, editor, *Wavelet Analysis: Twenty Years Developments Proceedings of the International Conference of Computational Harmonic Analysis, Hong Kong, China, June 4–8, 2001*, volume 1 of *Ser. Anal.*, pages 100–122. World Sci.Pub., River Edge, NJ, 2002.

Some not so obvious references II

H. G. Feichtinger, W. Kozek, and F. Luef.
Gabor Analysis over finite Abelian groups.
Appl. Comput. Harmon. Anal., 26(2):230–248, 2009.

[ka05]: N. Kaiblinger.
Approximation of the Fourier transform and the dual Gabor window.
J. Fourier Anal. Appl., 11(1):25–42, 2005.

[feka07] H. G. Feichtinger and N. Kaiblinger.
Quasi-interpolation in the Fourier algebra.
J. Approx. Theory, 144(1):103–118, 2007.

[cofelu08] E. Cordero, H. G. Feichtinger, and F. Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lect. Notes Math.*, pages 1–33. Springer, Berlin, 2008.

Some not so obvious references III

[fe19-2] H. G. Feichtinger.
Gabor expansions of signals: computational aspects and open questions.
In *Landscapes of Time-Frequency Analysis*, volume ATFA17, pages 173–206. Birkhäuser/Springer, 2019.

[fe20-1] H. G. Feichtinger.
A sequential approach to mild distributions.
Axioms, 9(1):1–25, 2020.

[fe09] H. G. Feichtinger.
Banach Gelfand triples for applications in physics and engineering.
volume 1146 of *AIP Conf. Proc.*, pages 189–228. Amer. Inst. Phys., 2009.

[fe23] H. G. Feichtinger.
Sampling via the Banach Gelfand Triple.
In :, :, Stephen D. Casey, M. Maurice Dodson, Paulo J. S. G. Ferreira, Ahmed Zayed, Stephen D. Casey, M. Maurice Dodson, Paulo J. S. G. Ferreira, Ahmed Zayed, and xxx, editors, *Rowland Higgins Comemorial Volume*, pages 1–30. Birkhäuser/Springer, Cham, 2023.

A non-factorization theorem I

It is a simple consequence of the Hilbert-Schmidt kernel theorem and the characterization of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ as $\boldsymbol{L}^2(\mathbb{R}^d) * \boldsymbol{L}^2(\mathbb{R}^d)$ which gives the following factorization theorem:

Theorem

$HS \star_{OP} \mathcal{H}S = \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^{2d})$.

A non-factorization theorem II

Theorem

For any pair of operators $T1, T2$ *in* $\mathcal{L}(w^*\boldsymbol{S}_0', \boldsymbol{S}_0)$ *we have*

$$F = T_1 *_{OP} T_2 \in \boldsymbol{S}_0(\mathbb{R}^{2d}), \text{ with } \|F\|_{\boldsymbol{S}_0} \leq C\|T_1^n\|_{\boldsymbol{\mathcal{B}}} \|T_2^n\|_{\boldsymbol{\mathcal{B}}}$$

but the finite linear combinations of such functions do not exhaust all of $\boldsymbol{S}_0(\mathbb{R}^{2d})$. *On the other hand there exists a constant* $C_1 > 0$ *(depending in the choices of norms on the different spaces) such that one can find for every* $F \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ *two sequences of operators* $(T_1^n)_{n\geq 1}$ *and* $(T_2^n)_{n\geq 1}$ *such that*

$$\sum_{n=-\infty}^{\infty} \|T_1^n\|_{\boldsymbol{\mathcal{B}}} \|T_2^n\|_{\boldsymbol{\mathcal{B}}} < C_1\|F\|_{\boldsymbol{S}_0}, \quad \text{and} \quad F = \sum_{n=-\infty}^{\infty} T_1^n *_{OP} T_2^n.$$

Periodization of operators I

In the discussion of Gabor frame operators one has to consider the Gabor frame-operator

$$S_{g,\Lambda} := \sum_{\lambda\in\Lambda} P_{g_\lambda} = \sum_{\lambda\in\Lambda} \pi\otimes\pi^*(\lambda)P_g.$$

In the QHA context this can be written as convolution of the mild distribution $⊔⊔_\Lambda$ with the rank-one operator P_g: $S_{g,\Lambda} = ⊔⊔_\Lambda \star P_g$. According to the FT-rules (symplectic Fourier transform for us) this means (see **[feko98]**!) that

$$\mathcal{F}_W(S_{g,\Lambda}) = (C_\Lambda ⊔⊔_{\Lambda^\circ} \cdot \eta(P_g) = C_\Lambda \sum_{\lambda^\circ\in\Lambda^\circ} V_g(g)(\lambda^\circ)\delta_{\lambda^\circ} \ ,$$

or taking the (inverse) symplectic FT we get

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ\in\Lambda^\circ} V_g(g)(\lambda^\circ)\pi(\lambda^\circ).$$ Janssen representation.

Periodization of operators II

Since $V_g(g)(0,0) = \langle, g, \pi(0,0)g\rangle_{\boldsymbol{L}^2} = \|g\|_2{}^2 = 1$ for normalized windows this gives access to the invertibility $S_{g,\Lambda}$ (Gabor frame property) if $\sum_{\lambda^\circ \in \Lambda^\circ, \lambda^\circ \neq 0} |V_g(g)(\lambda^\circ)| < 1$.
Double preconditioning provides methods to reach this status within the algebra of Λ-invariant operators.
For the case that $\Lambda^\circ \lhd \Lambda$ this lattice is a commutative lattice and Zak transform (Gelfand transform) methods apply. For $d = 1$ this corresponds to the case of integer redundancy (see Gestur).
Since $V_g(g) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ this corresponds to the well-known principle that periodization of a function corresponds to sampling on the Fourier side. E.g. every periodic function in $h \in \left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$ is the $\mathbb{Z}$ periodization of some function in $f \in \boldsymbol{S}_0(\mathbb{R})$ and the Fourier coefficients of h are just the samples $(\widehat{f}(n))_{n\in\mathbb{Z}}$, or $h(t) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi int}$.

THANKS to the audience

THANKS you for your attention

maybe you visit www.nuhag.eu

and checkout other talks and related material. (password protected!) hgfei

`www.nuhag.eu/bibtex`: all papers

`www.nuhag.eu/talks`: many talks on related subjects

NOTE: Talks require access code:

"visitor" and "nuhagtalks"