NuHAG Numerical Harmonic Analysis Group

Group Theoretical Principles and the use of THE Banach Gelfand Triple

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Mathematical Analysis vs. Engineering Reality I

Fourier Analysis is an important branch of **Mathematical Analysis**. Once Fourier Analysis had been developed far enough with the help of L. Schwartz's theory of *tempered distributions* PDE became accessible to a Fourier-analytic treatment. The development which has been initiated by work of L. Hörmander has a long-lasting impact in the domain of PDEs, and the study of PDEs with variable coefficients has led to the development of the theory of *pseudo-differential operators*, typically described via the Kohn-Nirenberg calculus, i.e. via a *time-variant* transfer function containing all the relevant information about the operator.
Thus in the mathematical description we have functions or distributions describing domain and range of various operators (e.g. the Fourier transform), as well as different ways to describe operators (via their distributional kernel, a kind of "continuous matrix representation") or the KNS-symbol, and so on.

Mathematical Analysis vs. Engineering Reality II

Clearly *convolution* (between functions, or functions and distributions, or functions and operators in Quantum Harmonic Analysis) are important operations, and the (appropriate) Fourier transform transfers these (usually considered *cumbersome*) operations to simple pointwise multiplication.

In physics and in particular in engineering, especially in the *digital age* Fourier Analysis is at the basis of data transmissions and **signal processing** (think if data formats used from storage or streaming of audio or image content), and therefore also highly important.

Unfortunately the connections between the mathematical developments and the real-world applications are getting more and more vague and the members of one community have a hard time to even understand what the other side is doing, or needs (in terms of tools, etc.). It is true that first year students (typically in communication theory) learn about time-invariant systems.

Mathematical Analysis vs. Engineering Reality III

They hear about different types of signals, discrete and continuous, periodic and non-periodic, and corresponding different so-called *Fourier transformations* (one for each setting!), with the FFT being the work-horse of many digital signal processing algorithms. Invariant systems are then easily understood as convolution operators by the *impulse response*, but for the continuous setting one has to invoke the mysterious Delta-function, which verifies the so-called *sifting property*

$$f(t) = \int_{-\infty}^{\infty} f(y)\delta(x-y)dy \quad (= f * \delta_0),$$

or the claim that

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} ds \quad (= \mathcal{F}^{-1}(\mathbf{1})).$$

Mathematical Analysis vs. Engineering Reality IV

Since signal processing obviously means
handle signals: move them from input side to output
one should have a clear definition of **what a signal is**.

ChatGPT provides the following answer:

`An ontological description of signals involves defining what signals are in the most fundamental terms, considering their nature, existence, and the relationships they have within various contexts. Here is a structured ontological description. Signals are entities that carry information from one point to another. They are manifestations of some underlying phenomena that convey data, commands, or other forms of communication.`

Fourier Analysis and Group Theory I

Group theory plays an important role in the context of Fourier Analysis. Historically (see the classical book of A. Zygmund) Fourier series made use of the trigonometric functions $\cos(nt)$ and $\sin(nt)$ (or often $\cos(2\pi nt)$ and $\sin(2\pi nt)$, for $n \geq 0$. The advantage of having real Fourier coefficients for real-valued functions is minor compared to the use of the complex exponential functions (say) $t \mapsto \exp(2\pi kt)$, with $k \in \mathbb{Z}$, because in this case it becomes clear that these "pure frequencies" form themselves a group (under pointwise multiplication), isomorphic to $(\mathbb{Z}, +)$ (due to the exponential law). Hilbert space techniques show that these pure frequencies form a complete ONB for $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ (the space of square integrable functions on the torus).

Fourier Analysis and Group Theory II

Moving on to the non-periodic case we find that the "pure frequencies" on $(\mathbb{R},+)$ are exactly of the form $\chi_s(t) = \exp(2\pi i s t)$. Hence the Fourier transform is of the form

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t)\overline{\chi_s(t)}dt = \int_{\mathbb{R}} f(t)\exp(-2\pi i s t)dt, \quad f \in \boldsymbol{L}^1(\mathbb{R}).$$

The group theoretical interpretation is that the *dual group* of $\mathbb{R}$ (time parameter t) is isomorphic to $\mathbb{R}$ (with frequency variable $s \in \mathbb{R}$). Similar formulas are valid for $\mathbb{R}^d$.
Obviously one has to be careful with the functions which allow such an integral (also for the inverse Fourier transform, which is valid only if $\widehat{f}$ is integrable), by assuming f (resp. $\widehat{f}$) to belong to the *Lebesgue space* $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$.

How to introduce THE Banach Gelfand Triple I

I would like to start with a bold statement:
In the same way as it is important to use appropriate "number systems" (unlike Roman numbers) for computation, it is important to use "suitable function spaces" in order to model signals, measurements and prepare for the development of [digital] signal processing tools (e.g. for mobile communication, or quantum optics).
We are all familiar with the chain of fields: $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. It is the field of rational numbers $\mathbb{Q}$ where we do actual computations (or at least approximately), we use the completeness of $\mathbb{R}$ in most of our arguments concerning analysis (existence of π, $\sqrt{2}$, etc.) and we find complex numbers and complex exponential functions a good way to describe the foundations of Fourier Analysis (pure frequences $\exp(2\pi ikx)$ instead of $\cos(2\pi kx)$ and $\sin(2\pi kx)$!

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

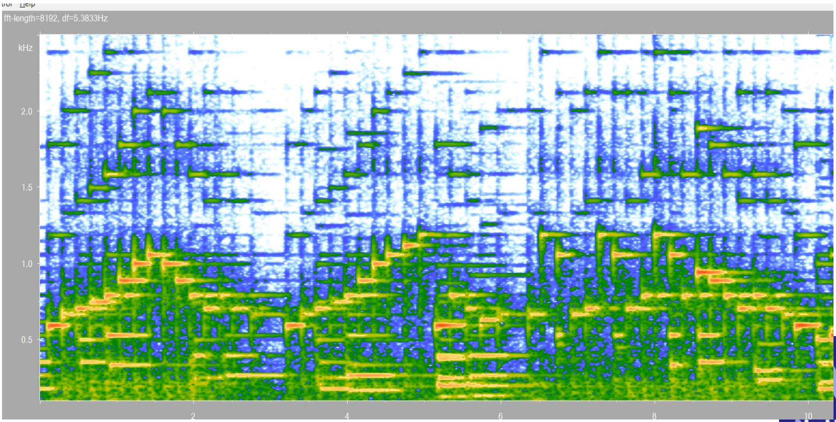
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

A typical piano spectrogram (Mozart), from recording



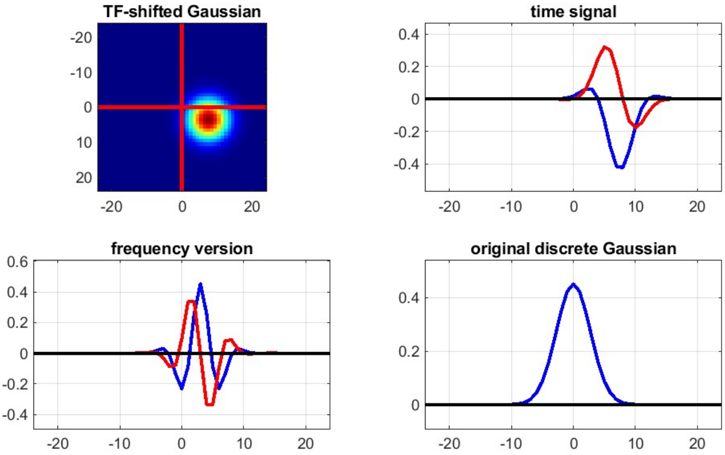
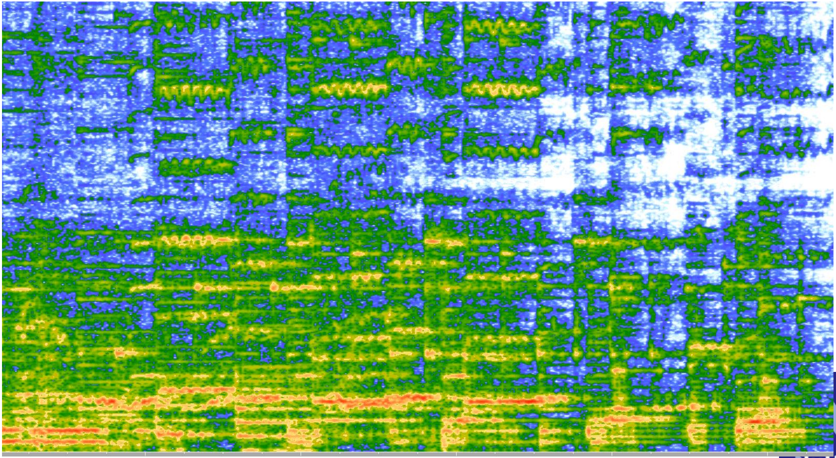
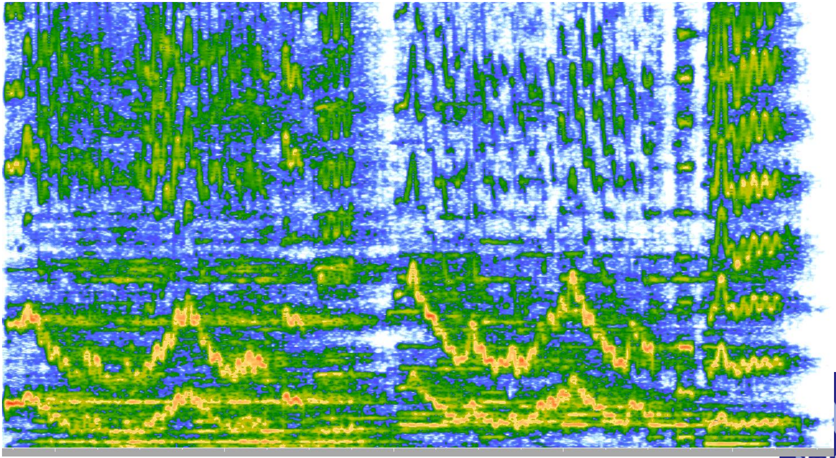


Figure: g48TFshifts.jpg

A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of V_g is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$. Assuming $\|g\|_2 = 1$ we have the *reconstruction formula*:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda)g,$$

which can be approximated in $\boldsymbol{L}^2$ by Riemannian sums.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

The Segal Algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ I

This makes it plausible to build the smallest Banach space containing all the TF-shifted functions in the following way:

Definition

$$\mathcal{S}_{atom} := \{f \in \mathbf{L}^2(\mathbb{R}^d) \,|\, f = \sum_{k=1}^{\infty} c_k \pi(\lambda_k) g_0 \, , \sum_{k=1}^{\infty} |c_k| < \infty\}$$

It is easy to verify that this space is a Banach space with the natural quotient norm

$$\|f\|_{atom} := \inf\{\sum_k |c_k|, \text{over all admiss. representations of } f\}$$

THE Banach Gelfand Triple I

Observing the fact that we have

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^p(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$$

we can call the triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ THE Banach Gelfand Triple (relevant for our discussion, and for Fourier Analysis in general) on $\mathbb{R}^d$. In fact, since $\boldsymbol{S}_0(\mathbb{R}^d)$ and all the other ingredients are well defined over general LCA groups we could use the BGTr in this more general framework, and also express the functional properties better at this level of generality.

Recall that $\boldsymbol{S}_0(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})$, Wiener's algebra of absolutely convergent Fourier series, which via the (classical Fourier transform) coincides with $\ell^1(\mathbb{Z})$. Its dual, the space of *pseudo-measures* thus corresponds to $\ell^\infty(\mathbb{Z})$.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

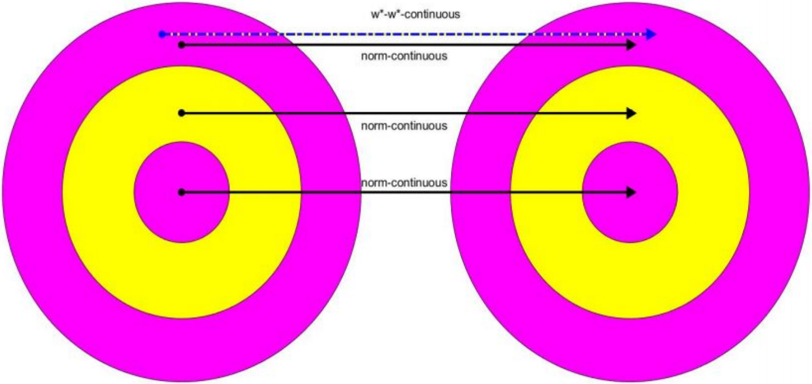
If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

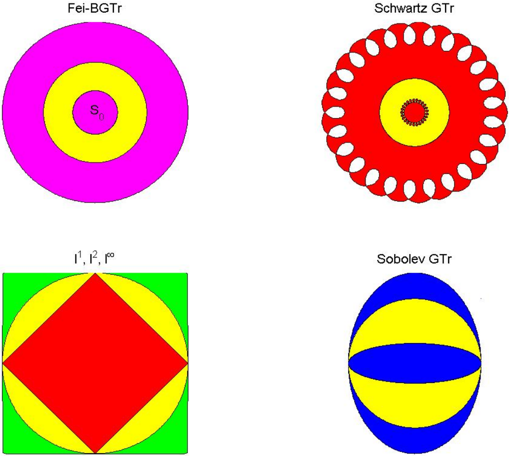
The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

Banach Gelfand Triple Morphism



Other Banach Gelfand Triples



The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \qquad (1)$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Poisson Formula

Among others the so-called Dirac comb $⊔⊔_\Lambda$ defines a bounded linear functional on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, for any lattice $\Lambda = \mathbb{A} * \mathbb{Z}^d \lhd \mathbb{R}^d$ defines an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$ via

$$⊔⊔_\Lambda(f) := [\sum_{\lambda\in\Lambda} \delta_\lambda](f) = \sum_{\lambda\in\Lambda} f(\lambda).$$

The validity of Poisson's formula for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be formulated like this: For any $\Lambda \lhd \mathbb{R}^d$ there exists $C_\Lambda > 0$ such that

$$\sum_{\lambda\in\Lambda} f(\lambda) = C_\Lambda \sum_{\lambda^\perp\in\Lambda^\perp} \widehat{f}(\lambda^\perp) \qquad (2)$$

for $\Lambda^\perp = (\boldsymbol{A}^t)^{-1} * \mathbb{Z}^d$, or equivalently, the validity (in the sense of $\boldsymbol{S}_0'(\mathbb{R}^d)$):

$$\mathcal{F}(⊔⊔_\Lambda) = C_\Lambda ⊔⊔_{\Lambda^\perp}. \qquad (3)$$

The KERNEL THEOREM for the Schwartz-space

The *kernel theorem* for the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ reads as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{4}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (4) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)

One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (also leading to a characterization via functorial properties, given by V. Losert, [lo80]) is the tensor-product factorization:

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \tag{5}$$

with equivalence of the corresponding norms.

The KERNEL THEOREM for S_0 II

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-kernels.

In fact, the operators in the inner shall (see the paper [feja22] discussing it in great detail) are trace class operators with the expected formula

$$\operatorname{trace}(T_K) = \int_G K(x,x)dx.$$

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$ and nuclear on $(\boldsymbol{S}_0'(\mathbb{R}^d), w^*)$.

In analogy to the matrix case, where the entries of the matrix

$$a_{k,,j} = T(\mathbf{e}_j)_k = \langle T(\mathbf{e}_j), \mathbf{e}_k\rangle$$

we have for $K \in \boldsymbol{S}_0$ the continuous version of this principle:

$$K(x,y) = \delta_x(T(\delta_y)), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $(\mathcal{N}_{w*}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ *(different from the trace class, HS-operator triple, with dual space* $\mathcal{L}(\mathcal{H})$*) where the first space is understood as the nuclear* w^* *to norm continuous operators from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\mathbb{R}^d)$*, the so-called regularizing operators.*

Spreading function and Kohn-Nirenberg symbol

1. For $\sigma \in S_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$$

The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}d\omega\Big)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta,u)M_\eta T_{-u}f(x)dud\eta.$$

$\widehat{\sigma}$ is called the spreading function of T_σ.

Further details concerning Kohn-Nirenberg symbol

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x + \frac{y}{2}, x - \frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y - x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

The kernel $K(x,y)$ can be described as follows:

$$K(x,y) = \mathcal{F}_2\sigma(\eta, y - x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y - x)$$
$$= \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y - x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

Kohn-Nirenberg symbol and spreading function II

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x),dt\,d\nu\,,$

Spreading representation and commutation relations

The description of operators through the spreading function and allows to understand a number of commutation relations.
If an operator is a limit (in the strong operator topology) of translation operators it is just a convolution operator with some $\tau \in \boldsymbol{S}_0'(\mathbb{R}^d)$, resp. its spreading representation is just an element concentrated on the *time axis* (more or less representing $\hat{\tau}$, the "individual frequency contributions".
Similarly, multiplication operators require just the use of modulation operators, so their spreading function is concentrated in the frequency axis of the TF-plane.
Finally typical *Gabor frame operators* arising from a family of Gabor atoms (g_λ), where $\lambda \in \Lambda$, some lattice within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ typically commute with TF-shift operators, one can say that they are obtained by periodizing the projection operator $f \mapsto \langle f, g \rangle g$ along the lattice.

The symplectic Fourier transform

The *symplectic Fourier transform* connects the Kohn-Nirenberg symbol with the spreading function, i.e.

$$\mathcal{F}_s(\sigma(T)) = \eta(T) \quad \text{resp.} \quad \mathcal{F}_s(\eta(T)) = \sigma(T). \tag{6}$$

$$(\mathcal{F}_{symp} f)(k,l) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y) e^{-2\pi i(k\cdot y - l\cdot x)}; \quad f \in \mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d). \tag{7}$$

It is completely characterized by its action on elementary tensors:

$$\mathcal{F}_{symp}(f \otimes \hat{g}) = g \otimes \hat{f}, \quad f, g \in \mathbf{S}_0(\mathbb{R}^d), \tag{8}$$

and extends from there in a unique way to a $w^* - w^*$ continuous mapping from $\mathbf{S}_0'(\mathbb{R}^{2d})$ to $\mathbf{S}_0'(\mathbb{R}^{2d})$, also $\mathcal{F}_{symp}{}^2 = Id$.

Understanding the Janssen representation

The spreading representation of operators has properties very similar to the ordinary Fourier expansion for functions! Periodization at one side corresponds to sampling on the transform side, if we understand "translation" either at the level of ordinary translation of the Kohn-Nirenberg symbol (which is the *symplectic Fourier transform* of the spreading function), OR by conjugation of an operator by the corresponding TF-shifts.
In other words: for any given operator T and $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we can **define** [recall $\pi(x,\omega) = M_\omega T_x$ for $\lambda = (x,\omega)$]

$$\pi \otimes \pi^*(T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \tag{9}$$

providing the important *covariance property* for KNS:

$$\sigma[\pi \otimes \pi^*(\lambda)(T)] = T_\lambda[\sigma(T)], \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{10}$$

Periodization goes over to sampling

If we have a "nice operator" T_0 we can form its periodic version $\sum_{\lambda\in\Lambda} \pi \otimes \pi^*(\lambda)(T_0)$ and it is still a well defined operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$. Its KNS is just the Λ-periodization of T_0. Consequently its spreading function is obtained by sampling of $\eta(T) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, over the *adjoint lattice* Λ° and obtain in this case an ℓ^1-sequence.
The adjoint lattice Λ° can be characterized by the fact that

$$\mathcal{F}_s(\sqcup\!\sqcup_\Lambda) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\circ}. \qquad (11)$$

For the projection on the Gabor atom $P_g : f \mapsto \langle f, g\rangle g$ the spreading functions is essentially

$$[\eta(P_g)](\lambda) = Vg(g)(\lambda) = \langle g, \pi(\lambda) g\rangle, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Janssen representation II

An important insight concerning the connection between the Gabor atom g, the TF-lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and the quality of the resulting Gabor frame resp. Gabor Riesz basis (e.g. condition number) clearly comes from the *Janssen representation* of the *Gabor frame operator* for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$S_{g,\Lambda}(f) = \sum_{\lambda \in \Lambda} P_{g_\lambda}(f) = \sum_{\lambda \in \Lambda} \pi \otimes \pi^*(\lambda)[P_g]. \qquad (12)$$

The periodization principle gives the Janssen representation

$$S_{g,\Lambda} = \eta^{-1}[\eta(S_{g,\Lambda})] = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g(g))(\lambda^\circ)\pi(\lambda^\circ), \qquad (13)$$

as an absolutely convergent sum of TF-shifts from Λ°.

Mild convergence in $S_0'(\mathbb{R}^d)$

As it has been mentioned w^*-convergence in the dual space of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is one of the key features which makes the "Rigged Hilbert Space" $(S_0, L^2, S_0')(\mathbb{R}^d)$ work.
Let us recall an equivalent description:
A sequence (or bounded net) of mild distributions $(\sigma_n)_{n\geq 1}$ is "mildly convergent" whenever for some (and then in fact for any) non-zero window $g \in S_0(\mathbb{R}^d)$ one has convergence

$$V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda), n \to \infty,$$

uniformly over compact subsets of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
Standard JOKE: Isn't it enough to know the audio signal for the duration of a song and up to 20 kHZ in order to sell a CD?

While for many applications it is good to take $\mathcal{L}(\mathcal{H})$, the space of bounded linear operators as the *universe* to start with (comparable to the tempered distributions for function spaces in the spirit of Hans Triebel), this is sometime restrictive, as we want to explain. Recall that for $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ the space $\mathcal{L}(\mathcal{H})$, endowed with the operator norm is not only a C^*-algebra, but it is the dual space of all *trace class operators* $\mathcal{T}^1(\boldsymbol{L}^2(\mathbb{R}^d))$. Trace class operators (in fact finite rank operators) are dense in the space of all Hilbert-Schmidt operators, which is in turn a Hilbert space with the scalar product

$$\langle T, S\rangle_{\mathcal{HS}} = \mathrm{trace}(TS^*), \quad T, S \in \mathcal{HS}.$$

Altogether they also form a Banach Gelfand Triple of the form

$$(\mathcal{T}^1(\boldsymbol{L}^2), \mathcal{HS}, \mathcal{L}(\boldsymbol{L}^2))$$

.

Assuming our principle interest is in dealing with the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ and operators related to it, or Wiener's algeba $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (because it can be defined without recurrence to the existence of a Haar measure), there are still good reasons to introduce tools involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$!

1. If you are asking for the bounded linear operators on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ it is not clear whether they have a kernel (unless they are (compact) Hilbert-Schmidt operators $(K \in \boldsymbol{L}^2(\mathbb{R}^{2d}))$;
2. Even for pointwise our Fourier multipliers you have to use functions in $\boldsymbol{L}^\infty(\mathbb{R}^d)$ (and their Fourier transforms);
3. Given $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and a lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ the mapping $f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$ is not anymore bounded from $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ to $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$.

4. The classical Hausdorff-Young Theorem shows strong limitations, one can only take $p \in [1,2]$. But how "nasty" are distributional Fourier transforms of functions of the form $\widehat{f}, f \in \boldsymbol{L}^p(\mathbb{R}^d)$ for $p \in (2, \infty)$.
5. How should one define spec(f), the *spectrum of a function in* $\boldsymbol{L}^\infty(\mathbb{R}^d)$? For example, for trigonometric functions (anharmonic case!) it should be the collection of non-trivial frequencies in $\widehat{\mathbb{R}^d}$ occurring in the "signal" (same for AP-functions).
6. Engineers are very familiar with the principle that sampling on the time side corresponds to periodization on the frequency side. For band-limited functions this allows to exactly recover the original spectrum $\widehat{f}$ by pointwise multiplication. Equivalently we can recover f from f or better $\sum_{\lambda \in \Lambda} f(\lambda)\delta_\lambda$

by convolution with a suitable "kernel" (> Shannon representation).

$$f = \sum_{\lambda \in \Lambda} f(\lambda) T_\lambda g.$$

with convergence in $\boldsymbol{W}(\boldsymbol{C}_0, \ell^2)(\mathbb{R}^d)$, hence in $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{C}_0(\mathbb{R}^d)$.

7. Even larger is the space of multipliers (operators commuting with translations) from $\left(\boldsymbol{W}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$ to its dual $(\boldsymbol{W}^*(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}^*(\mathbb{R}^d)})$, we write $\boldsymbol{H}_{\mathcal{G}}(\boldsymbol{W}, \boldsymbol{W}^*)$.

8. The space of *tempered elements* in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ which can also be considered to define bounded convolution operators on $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is in fact a Banach algebra. It can be defined as $\boldsymbol{L}^p(\mathbb{R}^d) \cap \boldsymbol{H}_{\mathcal{G}}(\boldsymbol{L}^p(\mathbb{R}^d))$, or by pointwise existence of convolution products combined with bounedness of the resulting convolution operator. For $1 \leq p \leq 2$ this is the same, but for $p > 2$ we have a problem.

9. In the theory of *Anti-Wick operators* one typically starts with multipliers in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ or $\boldsymbol{L}^\infty(\mathbb{R}^{2d})$, giving operators of the form

$$GM_m(f) = \int_{\mathbb{R}^d} m(\lambda) V_g f(\lambda) \pi(\lambda) g \, d\lambda,$$

which are Hilbert-Schmidt resp. just bounded operators on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. But for Gabor multipliers, where one has an "upper symbol" (Berezin) of the form $\sigma = \sum_{\lambda \in \Lambda} m(\lambda)\delta_\lambda$, thus giving with $P_\lambda(f) = \langle f, g_\lambda \rangle g_\lambda$:

$$GM_\sigma(f) = \sum_{\lambda \in \Lambda} m(\lambda) V_g f(\lambda) \pi(\lambda) g = \left(\sum_{\lambda \in \Lambda} m(\lambda) P_\lambda \right)(f).$$

A motivating example: convolution matrices

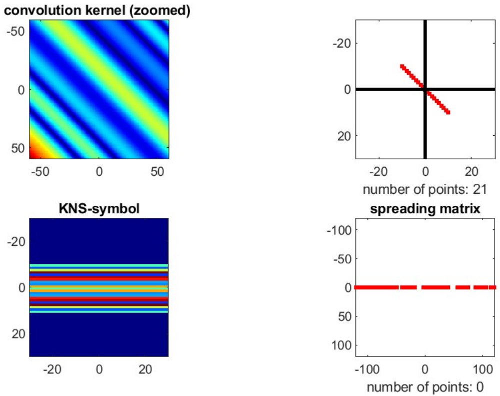
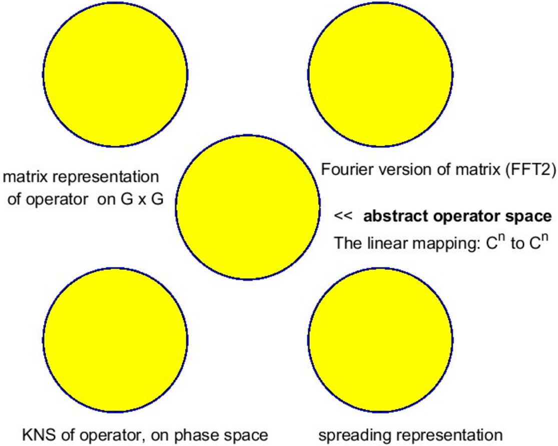
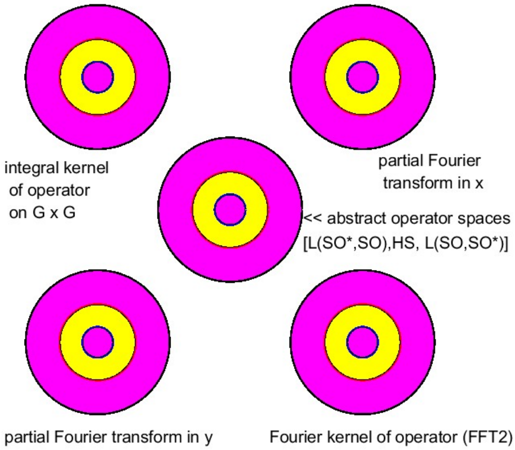


Figure: The four different representations of a convolution matrix in the form of an $n \times n$-matrix.

Matrix representations: Kohn-Nirenberg, Spreading



Different representations of operators



A more realistic scenario

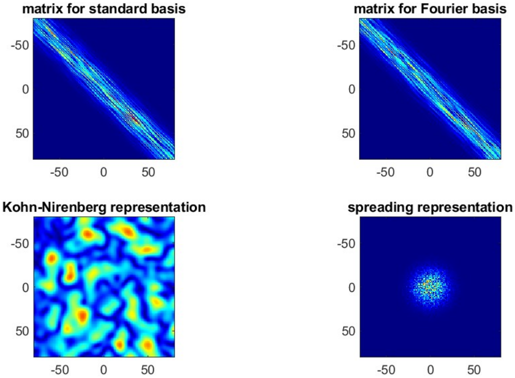
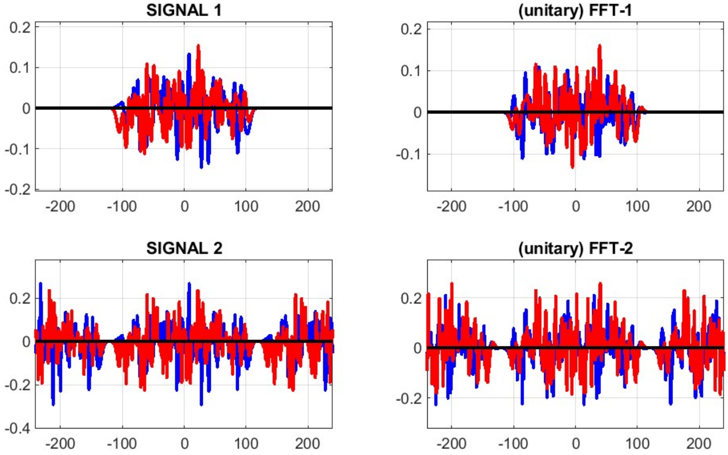


Figure: 4 version of the description of a Gabor multiplier, or Anti-Wick operator, with Gaussian window, zoomed version.

Periodization and Sampling



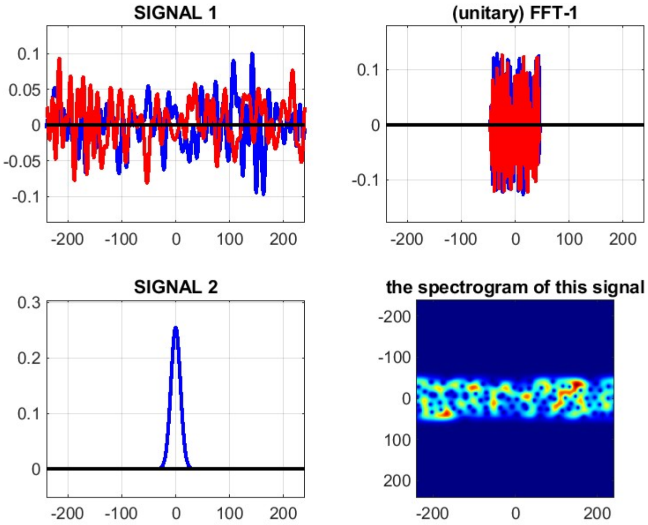


Figure: lowsigspec1.jpg

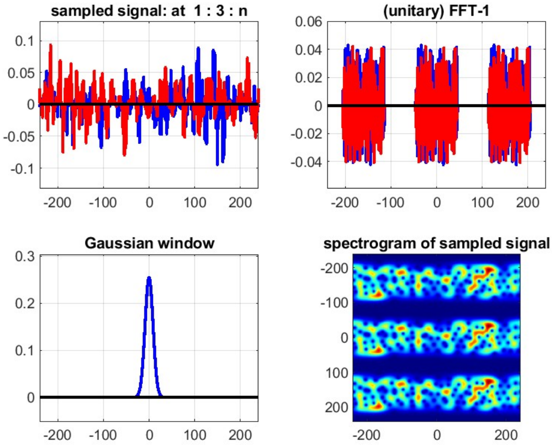


Figure: lowsigspec2.jpg

Periodization of operators I

The transition from the KNS-level to the spreading symbol has can be considered a kind of Fourier transform for operators. The important covariance property of KNS makes it interesting for Gabor Analysis. In fact, with $P_\lambda(f) = \langle f, g_\lambda \rangle g_\lambda$ we have

$$S_{g,\Lambda} = \sum_{\lambda \in \Lambda} P_\lambda = \sum_{\lambda \in \Lambda} \pi(\lambda) \circ P_g \circ \pi(\lambda)^*$$

can be rewritten (using the intertwining property) in the KNS-domain as

$$\kappa(S_{g,\Lambda}) = \sum_{\lambda \in \Lambda} T_\lambda \kappa(P_g) = \sqcup\!\sqcup_\Lambda * \kappa(P_g),$$

Periodization of operators II

and via the (symplectic) Fourier transform one gets Janssen's representation

$$\eta(S_{g,\Lambda}) = \mathcal{F}_s(\kappa(S_{g,\Lambda})) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\circ} \cdot V_g g,$$

or explicitly

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, g_{\lambda^\circ} \rangle \pi(\lambda^\circ).$$

One has $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $V_g g \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, so the good concentration of $V_g g$ over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ gives good chances for the operator to be invertible. Note that $V_g g(0) = \|g\|_2{}^2 = 1$ and $\sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\Lambda^\circ)| < \infty$.

General operators

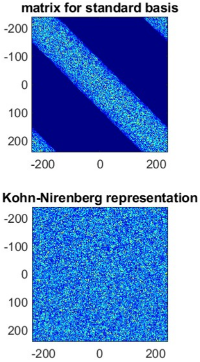


Figure: The description of an underspread operator, who is well concentrated in the spreading domain (right lower corner).

Embedding of finite cyclic groups

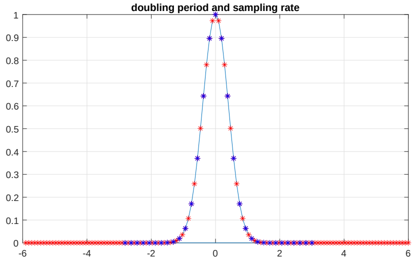


Figure: Adding the red values: period is twice as long and the sampling rate is twice as big, from blue to red.

Embeddings at the matrix level

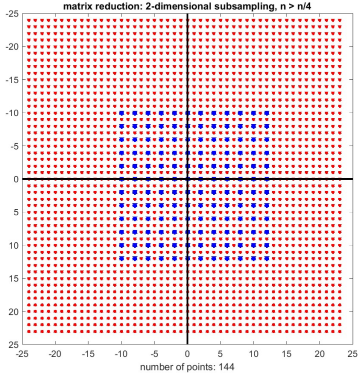


Figure: Subsampling from a matrix, using a QUARTER of each row and column, reducing the number of entries by the factor 16.

Summary of observations

The use of the BGTr $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ gives a lot of freedom, because it allows to push the analogy to linear algebra quite far, with a lot of good properties with respect to group theory (subgroups, invariance properties, etc.).
One can view mild distributions over $\mathbb{R}^d$ as "SIGNALS" and over $\mathbb{R}^{2d}$ as "OPERATORS" in whatever context (kernel, KNS, spreading, Weyl, Wigner,...).
Usual signal classes ($\mathbf{L}^p(\mathbb{R}^d)$, periodic, discrete) signals are all contained in $\mathbf{S}_0'(\mathbb{R}^d)$. Sampling in the time-domain corresponds to periodization in the frequency domain. Sampling combined with periodization allows the "mild approximation" of objects at the $\mathbf{S}_0$-level by finite discrete objects, regularization for $\mathbf{S}_0'(\mathbb{R}^d)$.
All this includes operations which are not realizable as bounded operators on the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ or using HS-operators.

Combined periodization and sampling

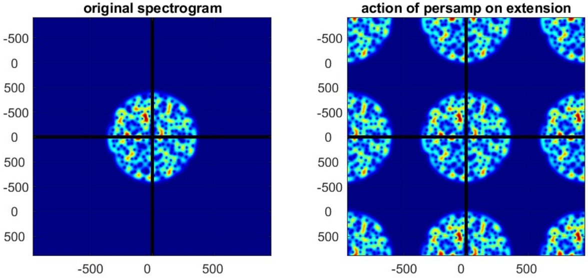


Figure: The picture in the STFT domain. Actually, it is the (twisted) convolution with a 2D Dirac comb with 4 points, so it is a 2D-periodization of the spectrogram.

Thanks!

THANK you for your attention

Best WISHES to Martin Grigoryan

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