NuHAG Numerical Harmonic Analysis Group

How to make use of Wiener Amalgam Spaces

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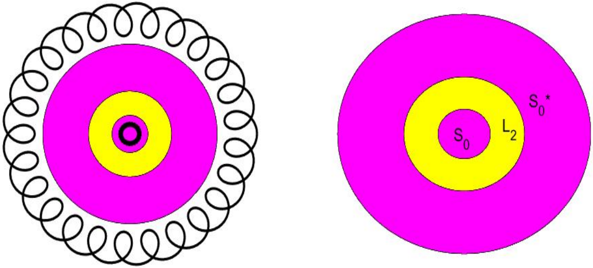
The Essence: Reducing to the Banach Gelfand Triple



R. C. Busby and H. A. Smith.
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Motivation, Non-inclusions between $L^p(\mathbb{R}^d)$-spaces I

Classical Fourier Analysis is based on the use of the spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, defined with the help of the *Lebesgue spaces* $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ over the Euclidean space $\mathbb{R}^d$. Recall that for $p < \infty$ the space $\boldsymbol{C}_c(\mathbb{R}^d)$ of compactly supported complex-valued continuous functions forms a dense subspace of $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, and that the ordinary Riemann integral (Haar measure) is OK. Recall that integrability is required for the definition of the forward/inverse Fourier transform or convolution in $\boldsymbol{L}^1(\mathbb{R}^d)$:

$$h(s) = \int_{\mathbb{R}^d} f(y) e^{\mp 2\pi i s y} dy,$$

$$f * g(z) = \int_{\mathbb{R}^d} g(x-y) f(y)\, dy.$$

The Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ allows to describe the FT (defined by approximation only) as *unitary automorphism*.

Motivation, Non-inclusions between $L^p(\mathbb{R}^d)$-spaces II

BUT aside from the level of sophistication required (recall that the Lebesgue integral is now a well established tool for 100 years) users have to get used to work with *equivalence classes of measurable functions*.

Another drawback is the fact, that the Riemann-Lebesgue Lemma implies that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) := \mathcal{F}(\boldsymbol{L}^1(\mathbb{R}^d)) \subset \boldsymbol{C}_0(\mathbb{R}^d)$, with

$$\|\widehat{f}\|_\infty \leq \|f\|_1, \quad f \in \boldsymbol{L}^1(\mathbb{R}^d),$$

is in fact a Banach algebra (with pointwise multiplication) with the norm $\|\widehat{f}\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)} := \|f\|_{\boldsymbol{L}^1(\mathbb{R}^d)}$, known as the *Fourier Algebra*.

By Plancherel's Theorem it can be characterized as $\boldsymbol{L}^2(\mathbb{R}^d) * \boldsymbol{L}^2(\mathbb{R}^d)$. Since the convolution square of a boxcar function equals the triangular function Δ it belongs to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R})$, in fact $\widehat{\Delta} = \mathrm{SINC}^2$.

SINC in the discrete/finite setting

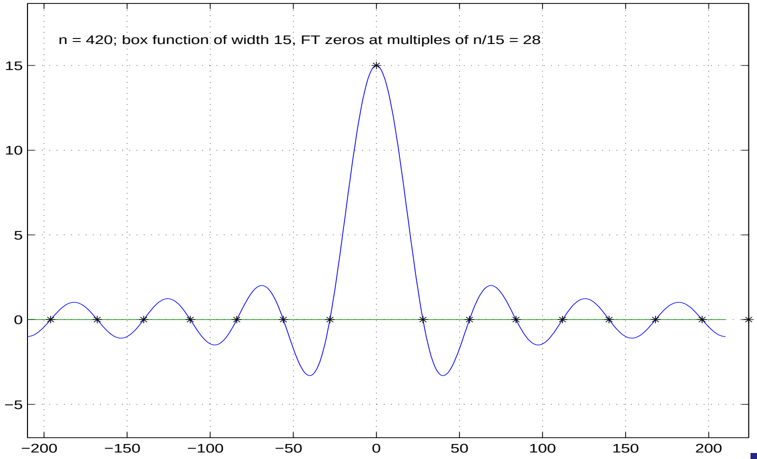
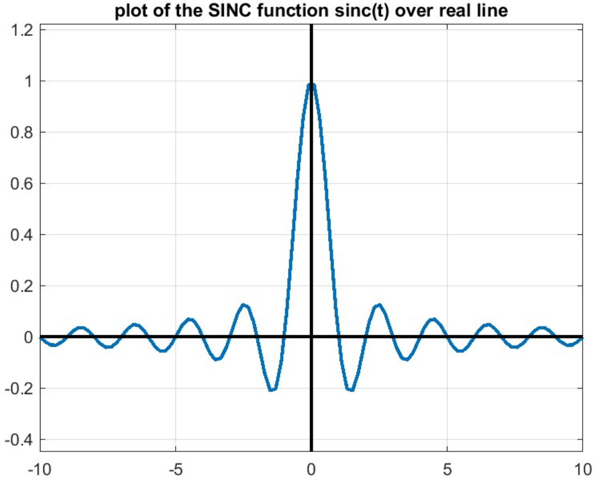
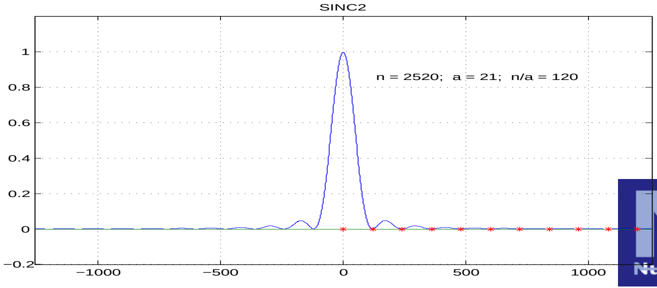


Abbildung: SINC = Fourier transform of boxcar $\mathbf{1}_{[-1/2,1/2]}$

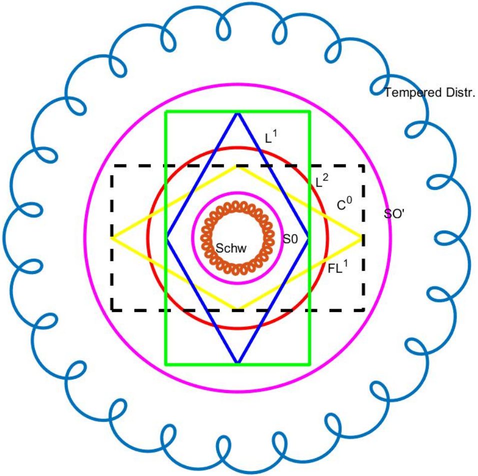
Plot over a long interval of the real line



Demonstrating better decay of SINC^2 function, in $L^1(\mathbb{R})$



A Zoo of Banach Spaces for Fourier Analysis



Non-Inclusion Results

One of the problems with the scale of Banach spaces of the from $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ (with the norm defined in the usual way) is the fact that there are no mutual inclusions.
Given any two parameters $p, q \in [1, \infty]$ one can find functions $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ which does not belong to $\boldsymbol{L}^q(\mathbb{R}^d)$.

One has to distinguish between two cases: For $p < q$ it is the lack of decay at infinity (*global behaviour*) which makes problems. E.g. SINC behaves like $1/x$ for $|x| \geq 1$, with SINC $\notin \boldsymbol{L}^1(\mathbb{R}^d)$ but it belongs to $\boldsymbol{L}^p(\mathbb{R})$ for any $p > 1$. On the other hand, for $q < p$ there are functions $x^{-\alpha}$ over $[-1, 1]$, which provide *local obstacles*.

Local and Global Features

Recall that we have an embedding of $(\ell^p, \|\cdot\|_p)$ into $(\boldsymbol{L}^p(\mathbb{R}), \|\cdot\|_p)$ via step functions taking constant values over the intervals $[n, n+1)$, while also $\boldsymbol{L}^p([-1/2, 1/2])$ or $(\boldsymbol{L}^p(\mathbb{T}), \|\cdot\|_p)$ can be viewed as subspace of $(\boldsymbol{L}^p(\mathbb{R}), \|\cdot\|_p)$, by extending such compactly supported functions to all of $\mathbb{R}$ in a trivial way (setting it to zero outside).

But it is also easy to show to check that in the second case boundedness implies integrability, hence $(\boldsymbol{L}^p(\mathbb{T}), \|\cdot\|_p) \hookrightarrow (\boldsymbol{L}^q(\boldsymbol{T}), \|\cdot\|_q)$ for the case $p \geq q$.

In contrast summable sequences are bounded, or $\ell^1 \hookrightarrow \ell^\infty$, or $\ell^p \hookrightarrow \ell^q$ which suggests (correctly) $p \leq q$.

Features of Wiener Amalgams

We treat $1 \leq p, q \leq \infty$, the Banach space case.
The guiding principles for the use of Wiener amalgams is the fact that they allow to control in some sense local and global properties of a function (? *separately*).
For step functions or smooth functions membership of $f \in \boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$ is the same as membership in $\boldsymbol{L}^q(\mathbb{R}^d)$.
For compactly supported functions $f \in \boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$ one has just another norm, equivalent to the (local) $\boldsymbol{L}^p$-norm.
A BETTER description is: The norm of a function in a Wiener amalgam space describes the global behaviour (described by the sequence space ℓ^q) OF A local quantity, or norm, expressed by the *local* (here) $\boldsymbol{L}^p$-norm.

Nice inclusion results

The classical norms for Wiener amalgams decompose $\mathbb{R}^d$ into cubes, obtained by shifting the *fundamental domain* of $\mathbb{R}^d$ with respect to the standard lattice $\mathbb{Z}^d$, and thus takes the $\boldsymbol{L}^p$-norm over all these cubes and then takes a global ℓ^q-sum.

The extreme cases are to take a local sup-norm and a global ℓ^1-norm. The space (of continuous functions inside of) $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)(\mathbb{R})$ consists of all continuous functions with finite upper Riemannian sum (we write $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R})$).

Correspondingly the largest of these classical spaces is $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)(\mathbb{R})$ which consists of all locally integrable functions of "uniform density" in the sense of boundedness of the local integrals.

This space is a closed subspace of the dual of $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$, which is the space of *translation bounded* Radon measures, we write $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R})$, e.g. $f \mapsto \int_{\mathbb{R}^d} f(x)dx$ (Riemann integral!).

Discrete versus continuous norm

On the positive side, all the spaces $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ are translation invariant, with a uniform bound (depending mostly on d) on the norm of translation operators.
However, especially the norm in $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (now for $d \geq 1$) shows a slightly annoying behaviour. Given a bump function supported by say (we look at $d = 2$) one fundamental domain may be split into 4 different pieces, so that the norm is multiplied by $4 = 2^d$ in such a case. One can thus define a new norm, by taking

$$\|f\|_{new} := \sup_{x \in \mathbb{R}^d} \|T_x f\|_{old}$$

which is not really computable and more of theoretical interest. For example, with such a norm $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ is a so-called *Segal algebra* (see H. Reiter's book of 1968).

Discrete versus continuous norms

One of the early ideas introduced into the area was based on a simple consideration: Why not "measures" a function in a continuous way, and describe the continuous behaviour of the local property. Then, instead of the indicator function $\mathbf{1}_Q$, for the cubic domain $Q = [0,1)^d$, one can take any bump function.
This we look (for $q < \infty$) at the *continuous norm*

$$\|f\|_{cont} := \left(\int_{\mathbb{R}^d} \|T_x \varphi \cdot f\|^q_{\boldsymbol{L}^p(\mathbb{R}^d)} \right)^{1/q}.$$

IT IS an EASY EXERCISE to verify that this norm is *equivalent* to the discrete norm (OLD or NEW), and strictly translation invariant:

$$\|T_x f\|_{cont} = \|f\|_{cont}, \quad \forall x \in \mathbb{R}^d.$$

The continuous variant of the norm on $W(C_0, \ell^1)(\mathbb{R})$

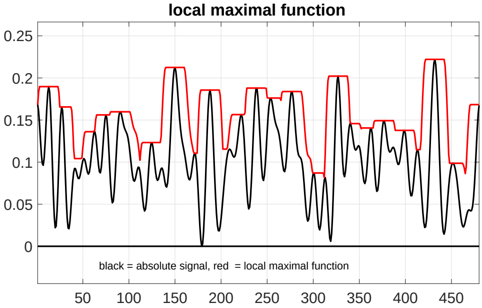


Abbildung: The red plot describes a local maximal function.

Which symbols to use?

First of all let us recall that there is a continuous and a discrete description of the classical Wiener amalgams. This suggest to use the symbols $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ (thinking of the discrete norm) and $\boldsymbol{W}(\boldsymbol{L}^p, \boldsymbol{L}^q)(\mathbb{R}^d)$, ifvf one has the continuous norm in mind.
We prefer to use the discrete version of the norm most of the time, because then it is clear that $\boldsymbol{W}(\boldsymbol{L}^{p_1}, \ell^{q_1}) \subset \boldsymbol{W}(\boldsymbol{L}^{p_2}, \ell^{q_2})$ if and only if $\boldsymbol{L}^{p_1}(Q) \subset \boldsymbol{L}^{p_2}(Q)$ for compact sets, i.e. $p_2 \leq p_1$, and $\ell^{q_1} \subset \ell^{q_2}$, or $q_1 \leq q_2$.
As a consequence spaces are equal if and only if their parameters are equal, i.e. $p_1 = p_2$ and $q_1 = q_2$.

The Magic Square for Wiener Amalgams

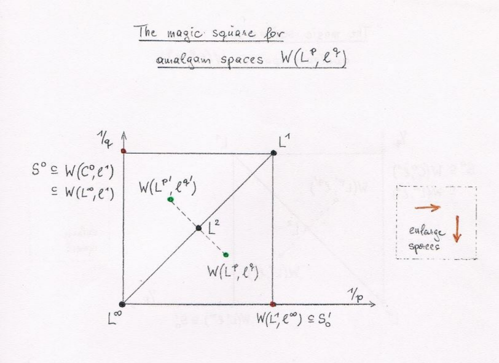


Abbildung: The inclusion relations: magic square

BUT overall classical Wiener Amalgams do not behave well under the Fourier transform!

Further references I

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In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, 1983.
(BUPU characterization, multiplication and **convolution** results)

H. G. Feichtinger and P. Gröbner.
Banach spaces of distributions defined by decomposition methods. I.
Math. Nachr., 123:97–120, 1985. (providing duality results)

H. G. Feichtinger.
Banach spaces of distributions of Wiener's type and interpolation.
In P. Butzer, S. Nagy, and E. Görlich, editors, *Proc. Conf. Functional Analysis and Approximation, Oberwolfach*, 153–165. Birkhäuser Boston, Basel, 1981.
interpolation results

T. Dobler.
Wiener Amalgam Spaces on Locally Compact Groups.
Master's thesis, University of Vienna, 1989.
completely continuous derivations

The Hausdorff-Young Theorem for Wiener Amalgams

Instead of using local $\boldsymbol{L}^p$-norm, one may take local Lorentz or Orlicz space norms, and instead of global ℓ^q-norms one may take weighted sequence space norms, this is quite easy to do.
For such simple spaces it is easy to establish the expected duality or (complex) interpolation results!
It is also true what $\boldsymbol{W}(\boldsymbol{L}^p, \ell^p) = \boldsymbol{L}^p(\mathbb{R}^d)$ with equivalence of norms, for $1 \leq p \leq \infty$, and there is a *Hausdorff-Young Theorem* (HDY)

Lemma

For $1 \leq p, q \leq 2$ *one has*

$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)) \subset \boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'}).$$

This is consistent with the general fact that *local properties* of f correspond to *global* properties of $\widehat{f}$.

The Hausdorff-Young Result for Amalgams

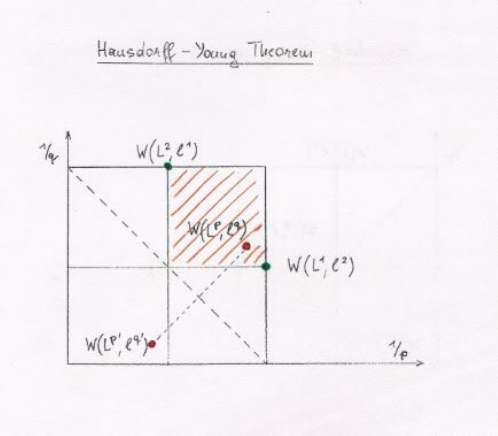


Abbildung: Hausdorff-Young theorem for Wiener amalgams

Justifying the Terminology I

The original name for Wiener Amalgam Spaces was *Wiener-type spaces* (see **fe83 :[1]**). In fact, Norbert Wiener has introduced so many important concept that there anything named after him should be related to the area in which it is applied.

The classical *Wiener algebra* $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$ of absolutely convergent Fourier series is creating one line of such a terminology. It is *Wiener's Inversion Theorem* where this Banach algebra, in fact a Segal Algebra inside of $\left(\boldsymbol{L}^1(\mathbb{T}), \|\cdot\|_1\right)$, plays a role. Given any $h \in \boldsymbol{A}(\mathbb{T})$ such that $h(t) \neq 0$ for all $t \in \mathbb{T}$ he shows that $1/h$ also belongs to $\boldsymbol{A}(\mathbb{T})$ (however without control of the constants in the general case).

In the sequel the term Wiener algebra is sometimes used for the corresponding space over LCA groups, so for the case $G = \mathbb{R}^d$ one would use it for $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ (what is better called the *Fourier algebra*), cf. P. Eymard.

Justifying the Terminology II

Correspondingly it is a good terminology to call $\mathcal{F}\boldsymbol{L}^1_w(\mathbb{R}^d)$ the *Fourier Beurling algebra*, the Fourier version of a weighted $\boldsymbol{L}^1$-space, with respect to a *submultiplicative Beurling weight*, satisfying

$$w(x+y) \leq w(x)w(y), \quad x,y \in \mathbb{R}^d.$$

More recently several authors use the name Wiener algebra for the domain of the Fourier Inversion Theorem, which we would describe as $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ (with natural norm). Again, for $G = \mathbb{T}$ it coincides with Wiener's algebra $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$.
In the Time-frequency (Gabor Analysis, Coorbit Theory) and wavelet community Reiter's terminology concerning Wiener's algebra, which we denote by $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$, is widely spread and very popular. Also for the theory of *spline-type spaces* (usually named *shift-invariant spaces*, according to Ron-Shen,DeBoor),

Justifying the Terminology III

they play an important role. It can be described in the (natural) context of LCA groups, for an alternative approach to Fourier Analysis (see [fe17]);

[fe17] H. G. Feichtinger.
A novel mathematical approach to the theory of translation invariant linear systems.
In I. Pesenson, et. al. reditors, *Recent Applications of Harmonic Analysis to Function Spaces, Differential Equations, and Data Science.*, ANHA, pages 483–516. Birkhäuser, Cham, 2017.

This case in Reiter's book was the starting point, by observing the minimality property of $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$ was established in

[fe77-3] H. G. Feichtinger.
A characterization of Wiener's algebra on locally compact groups.
Arch. Math. (Basel), 29:136–140, 1977.

It is a Segal algebra, hence $\boldsymbol{L}^1 * \boldsymbol{W} \subseteq \boldsymbol{W}$, but also a pointwise $\boldsymbol{C}_0(G)$-module: $\boldsymbol{C}_0(G) \cdot \boldsymbol{W} \subseteq \boldsymbol{W}$, and the smallest in this family.

Connection to Tauberian Theorems I

In his work on Tauberian Theorems (see his book on the Fourier Integral and Certain of it's Applications, from 1932) Norbert Wiener formulates the usual Tauberian Theorem, but also two variants. I call them the Second (using $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R})$) and the *Third Tauberian Theorem*. The least known is the third one, which also can be based on Banach algebra methods, see

[fe88] H. G. Feichtinger.
An elementary approach to Wiener's third Tauberian theorem for the Euclidean *n*-space.
In *Symposia Math.*, volume XXIX of *Analisa Armonica*, pages 267–301, Cortona, 1988.

[fegu23] H. G. Feichtinger and A. Gumber.
Approximation by linear combinations of translates in invariant Banach spaces of tempered distributions via Tauberian conditions.
J. Approximation Theory, pages 1–17, 2023.

The Wiener algebra $W(C_0, \ell^1)(\mathbb{R}^d)$ and $S_0(\mathbb{R}^d)$

The so-called *Wiener algebra*, the closure of $\mathcal{S}(\mathbb{R}^d)$ or $C_c(\mathbb{R}^d)$ in $W(L^\infty, \ell^1)(\mathbb{R}^d)$, or just $W(C_0, \ell^1)(\mathbb{R}^d)$ (having in mind that $C_0(\mathbb{R}^d)$ is the closure of $\mathcal{S}(\mathbb{R}^d)$ resp. $C_c(\mathbb{R}^d)$ in $\left(L^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$!), is the smallest among all isometrically translation invariant spaces allowing pointwise multiplication with $C_0(\mathbb{R}^d)$. Its dual is $\left(W(M, \ell^\infty)(\mathbb{R}^d), \|\cdot\|_W\right)$, the space of *translation-bounded Radon measures*.

In a similar way the Segal algebra $S_0(\mathbb{R}^d) := W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$ has been introduced (in 1979) as the smallest translation invariant Banach space of functions which allows pointwise multiplication with the Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$.

By duality the $S_0'(\mathbb{R}^d)$ which equals $W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of tempered distributions σ which are locally pseudomeasures, uniformly bounded over $\mathbb{R}^d$. We call them mild distributions, with $\|\sigma\|_{S_0'} = \sup_{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_{g_0}\sigma(\lambda)|$.

Translation bounded measures in the literature I

Translation-bounded measures, i.e. $\mu \in \boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d)$ appear frequently in the literature. Sometimes just implicitly, sometimes by this name. Let us give a few examples:
First let me mention its use in the theory of *Generalized Stochastic Processes*, where the standard approach makes use of vector-valued integration. An alternative was given in the PhD thesis of Wolfgang Hörmann (1989), see

[feho14] H. G. Feichtinger and W. Hörmann.
A distributional approach to generalized stochastic processes on locally compact abelian groups.
In G. Schmeisser and R. Stens, editors, *New Perspectives on Approximation and Sampling Theory. Festschrift in honor of Paul Butzer's 85th birthday*, pages 423–446. Cham: Birkhäuser/Springer, 2014.

Translation bounded measures in the literature II

Translation-bounded measures show up in the theory of *transformable measures* by L. Argabright and J. Gil de Lamadrid. They can be described as tempered distributions which are also Radon measures (on the time side) and translation-bounded measures on the Fourier transform side. So we note some built-in asymmetry with respect to the FT.

[argi74] L. N. Argabright and J. Gil de Lamadrid. *Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups*. Mem. Amer. Math. Soc. vol. 145. 1974.

Their theory plays also a key role in the mathematical theory of quasi-crystals. The observation that "mild distributions" (the space $\boldsymbol{S}_0'(\mathbb{R}^d)$) can be used to simplify arguments in this domain is the subject of an upcoming paper

Translation bounded measures in the literature III

[feriscst24], H. G. Feichtinger, C. Richard, C. Schumacher, and N. Strungaru.
On diffraction by translation bounded measures,
volume 3: Model Sets and Dynamical Systems, pages 1–48, 2024.

Obviously a Dirac comb over a discrete, cocompact lattice is a translation bounded measure, and by Poisson's formula (valid for $\mathbf{S}_0(\mathbb{R}^d)$, not just for $\mathcal{S}(\mathbb{R}^d)$) one can show that its Fourier transform coincides, up to normalization, with the counting (i.e. Haar measure) on the orthogonal lattice. So if $\Lambda = \mathbf{A} * \mathbb{Z}^d$ we have $\Lambda^\perp = (\mathbf{A}^t)^{-1} * \mathbb{Z}^d$. But even for more general closed subgroups $H \lhd \mathbb{R}^d$ one can make similar statements.
A summary of results concerning *regular sampling* (i.e. along lattices) is given in

Translation bounded measures in the literature IV

[fe24] H. G. Feichtinger.
Sampling via the Banach Gelfand Triple.
In Stephen D. Casey, Maurice Dodson, Paulo J.S.G. Ferreira, and Ahmed Zayed, editors, *Sampling, Approximation, and Signal Analysis Harmonic Analysis in the Spirit of J. Rowland Higgins*, Appl. Num. Harm. Anal., pages 211–242. Cham: Springer International Publishing, 2024.

But translation bounded measures also help to understand the sampling procedure in the *irregular case*. Here the notion of *relatively separated set* $X = (x_i)_{i \in I}$ is relevant.
They can be characterized as finite unions of *uniformly separated sets*, or as those families where the corresponding Dirac-comb $\sqcup\!\sqcup_X := \sum_{i \in I} \delta_{x_i} \in$. In fact, this simple geometric fact has inspired the formulation of the "Feichtinger Conjecture".
On $\boldsymbol{B}_p^{\Omega}$, the (band-limited) functions in $\boldsymbol{L}^p(\mathbb{R}^d)$ with $\mathrm{supp}(\widehat{f}) \subseteq \Omega$ we describe the sampling as the transition from f to $S(f) := \sum_{i \in I} f(x_i)\delta_{x_i} = f \cdot \sqcup\!\sqcup_X$.

Translation bounded measures in the literature V

Let us assume $\Omega \subseteq B_R(0) \subset \mathbb{R}^d$:

Lemma

Given $R > 0$ there exists $C_R > 0$ such that for $p \in [1, \infty]$

$$\|f\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^p)} \leq C_R \|f\|_{\boldsymbol{L}^p}\,, \quad \forall f \in \boldsymbol{B}_p^{\Omega}. \tag{1}$$

Given now a relatively separated set X we find another constant C_X (depending only on the maximal density of the set...) such that for some generic constant $C_0 > 0$ one has for all $f \in \boldsymbol{B}_p^{\Omega}$:

$$\|Sf\|_{\boldsymbol{W}(\boldsymbol{M},\ell^p)} \leq C_0 \|f\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^p)} \|\sqcup\!\sqcup_X\|_{\boldsymbol{W}(\boldsymbol{M},\ell^\infty)} \leq \tag{2}$$

$$\leq C_0 C_R C_X \|f\|_{\boldsymbol{L}^p}\,. \tag{3}$$

Translation bounded measures in the literature VI

The first estimate (1) comes again from simple convolution relations. For any test function $h \in \mathcal{D}(\mathbb{R}^d)$ (or even just $\mathcal{F}(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)))$ with $h(\omega) \equiv 1$ on Ω we have $\widehat{f} \cdot h = \widehat{f}$ for all $f \in \boldsymbol{B}_p^\Omega$. Let us fix such a (smooth plateau-type) function h and write $g = \mathcal{F}^{-1}(h) \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$.
Then it is clear that we have $f = g * f$, and thus

$$f \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1) * \boldsymbol{L}^p \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^1) * \boldsymbol{W}(\boldsymbol{L}^1, \ell^p) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^p),$$

with corresponding norm estimates from the convolution relations for Wiener amalgams. Obviously the constants arising depend only on $\|g\|_{\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)}$, and is independent of p!

Translation bounded measures in the literature VII

The stability of reconstruction in the regular case, using some oversampling, due to the fact that $\mathrm{supp}(h) \supset \Omega$ (by some margin, due to the continuity of h), or the control on the first step of the iterative reconstruction methods (of the first generation) arises from the following estimate of the approximation operator A, given as Shannon-like (unconditional) series:

$$Af = \sum_{i \in I} f(x_i) T_{x_i} g = (f \cdot \sqcup\!\sqcup_X) * g.$$

$$\|Af\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^p)} \leq C_0 \|f \cdot \sqcup\!\sqcup_X\|_{\boldsymbol{W}(\boldsymbol{M},\ell^p)} \|g\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^1)} \leq \qquad (4)$$

$$\|Af\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^p)} \leq C_2 C_X C_R \|f\|_{\boldsymbol{L}^p}, \quad \forall f \in \boldsymbol{B}_p^{\Omega}. \qquad (5)$$

Usually such estimates are described (equivalently) as

$$\|(f(x_i))_{i \in I}\|_{\ell^p} \leq C_3 \|f\|_{\boldsymbol{L}^p}, \quad f \in \boldsymbol{B}_p^{\Omega}.$$

Translation bounded measures in the literature VIII

Improved estimates are used to compensate variable density by using suitable (adaptive) weights for the Dirac measures δ_{x_i}.

[fegrst95] H. G. Feichtinger, K. Gröchenig, and T. Strohmer.
Efficient numerical methods in non-uniform sampling theory.
Numer. Math., 69(4):423–440, 1995.

An important observation is also the one has for each $f \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ that $\mathrm{osc}_\delta(f) \in$ and that

$$\lim_{\delta \to 0} \|\mathrm{osc}_\delta(f)\|_{\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)} = 0.$$

This accounts others for *jitter stability* of the reconstruction, since

$$\lim_{\delta \to 0} \sum_{i \in I} \mathrm{osc}_\delta(f)(x_i) = 0.$$

Translation bounded measures in the literature IX

Similar arguments/techniques apply to spline-type space (usually called shift-invariant) spaces (with one or finitely many generators), but also to the reconstruction from averages problem.

Of course one can use weighted version of such estimates, or use alternative function spaces instead of $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, and so on. There is a huge theory concerning the irregular sampling problem making use of variants of these ideas in the last 30 years, using e.g. mixed norm spaces, Orlicz spaces and so on. But almost all of them could be treated using Wiener amalgam techniques (at least the qualitative level), with constants depending on the relevant parameters only!

BUPUs for (general) Wiener Amalgams (1980) I

Next we discuss BUPUs (Bounded Uniform Partitions of Unity): This are families of the form $T_k\varphi, k \in \mathbb{Z}^d$, arising from a compactly supported function (of some *regularity*), with

$$\sum_{k\in\mathbb{Z}^d} \varphi(x-k) \equiv 1, \quad x \in \mathbb{R}^d.$$

Given a Banach space $\boldsymbol{B} \hookrightarrow \boldsymbol{L}^1_{loc}(\mathbb{R}^d)$ we assume that the action of the members of this family is uniformly bounded: $\exists C > 0$ with:

$$\sup_{k\in\mathbb{Z}^d} \|T_k\varphi \cdot f\|_{\boldsymbol{B}} \leq C\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{6}$$

In this situation we call the family $(T_k\varphi)_{k\in\mathbb{Z}^d}$ a $\boldsymbol{B}$-BUPU, or a BUPU for $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. For the continuous version we assume $\sup_{k\in\mathbb{R}^d} \|T_k\varphi \cdot f\|_{\boldsymbol{B}}$ in instead of (6).

Definition of $W(B,Y)$ I

With this terminology, and given some **solid BK-space** $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ on the lattice $\mathbb{Z}^d$ we define

Definition

$$\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y}) := \{f \in \boldsymbol{B}_{loc} \mid \|f \cdot T_k\varphi\|_{\boldsymbol{B}} \in \boldsymbol{Y}\} \tag{7}$$

endowed with the natural norm

$$\|f\|_{\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y})} := \|(\|f \cdot T_k\varphi\|_{\boldsymbol{B}})_{k\in\mathbb{Z}^d}\|_{\boldsymbol{Y}} .$$

We will write φ_k for $T_k\varphi$ in the future.

A slightly smoothed histogram

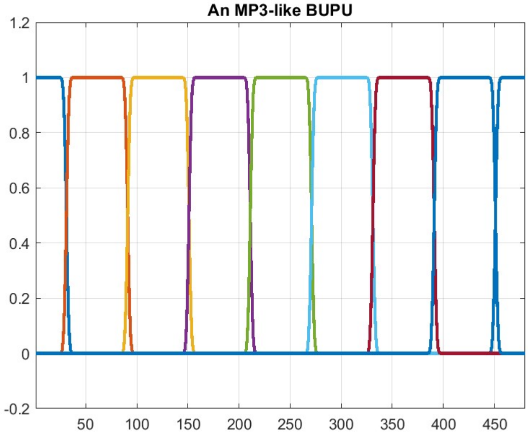


Abbildung: Indicator Functions with smooth boundaries

Linear and Cubic B-splines

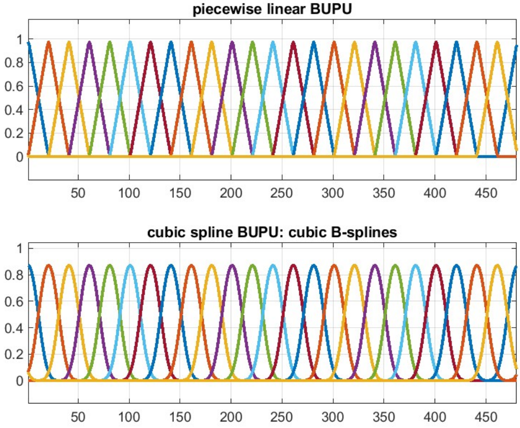


Abbildung: Two natural types of BUPUs, refinable by dilation

With this terminology, and given some **solid BK-space** $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ on the lattice $\mathbb{Z}^d$ we define

Definition

$$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{Y}) := \{f \in \boldsymbol{B}_{loc} \mid (\|f T_k \varphi\|_{\boldsymbol{B}})_{k \in \mathbb{Z}^d} \in \boldsymbol{Y}\} \tag{8}$$

which can be endowed with the natural norm

$$\|f\|_{\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y})} := \|(\|f \cdot T_k \varphi\|_{\boldsymbol{B}})_{k \in \mathbb{Z}^d}\|_{\boldsymbol{Y}} \,.$$

Of course one can show that different $\boldsymbol{B}-$BUPU define the same space and equivalent norms. It is also not important to choose $\mathbb{Z}^d$ for a BUPU, there is a more general definition, which includes not only different lattices in $\mathbb{R}^d$, but also irregular BUPUs, with $\text{supp}(\psi_i) \subseteq B_\delta(x_i)$ with controlled overlap of these balls.

$\mathcal{F}L^p(\mathbb{R}^d)$ as a local component I

The above setting is well suited in order to define (and study) Wiener amalgam spaces of the form . These spaces correspond to the Triebel-Lizorkin spaces with respect to wavelet, while modulation spaces (below) are the analogue of Besov spaces. They can be characterized using mixed norms in a Gabor setting.

[fe90], H. G. Feichtinger.
Generalized amalgams, with applications to Fourier transform.
Canad. J. Math., 42(3):395–409, 1990.

A useful (!) variant of the Hausdorff-Young Theorem reads then:

Theorem

One has for $1 \leq r \leq p \leq \infty$*:*

$$\mathcal{F}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^r)\right) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^r, \ell^p).$$

The unweighted modulation spaces $M^{p,q}(\mathbb{R}^d)$ I

The unweighted modulation spaces are defined via inverse Fourier transform of spaces of the form $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$. We set $p, q \in [1, \infty]$:

Definition

$\boldsymbol{M}^{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d))$, with transport of norms.

It is natural to write $\left(\boldsymbol{M}^{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^{p,q}}\right)$ for the normed space, in fact Banach space of mild distributions (or otherwise, with polynomial weights, of tempered distributions).
For $1 < p, q < \infty$ these space are reflexive, and the dual space is obtained by taking the dual parameters p' and q' respectively:

$$(\boldsymbol{M}^{p,q}(\mathbb{R}^d))^* = \boldsymbol{M}^{p',q'}(\mathbb{R}^d).$$

Note that $\boldsymbol{M}^{\infty,1}(\mathbb{R}^d)$ coincides with *Sjoestrand's algebra*.

The unweighted modulation spaces $M^{p,q}(\mathbb{R}^d)$ II

Note that convolution results as well as pointwise multiplications for modulation spaces can thus be inferred from the corresponding pointwise respectively convolution results of Wiener Amalgam spaces (Sjoestrand's algebra is a special case).

Theorem

For each $p \geq 1$ we have: $\boldsymbol{M}^{p,1}$ is a Banach algebra with respect to pointwise multiplication. Locally each of these algebras coincides with the usual Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. Nevertheless these algebras are different for different values of p.

The investigation of spectral problems in these Banach algebras is the subject a recent paper with M. Kobayashi (2024).

The unweighted modulation spaces $M^{p,q}(\mathbb{R}^d)$ III

Proof.

Instead of the pointwise relation we consider the corresponding convolution results on the Fourier transform side.

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^1) \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1),$$

together with the corresponding norm inequality, simply because we have $\mathcal{F}\boldsymbol{L}^p \hookrightarrow \mathcal{F}\boldsymbol{L}^\infty$ *locally*! for every $p \leq \infty$, see below

$$\mathcal{F}\boldsymbol{L}^p * \mathcal{F}\boldsymbol{L}^\infty = \mathcal{F}(\boldsymbol{L}^p \cdot \boldsymbol{L}^\infty) \subseteq \mathcal{F}(\boldsymbol{L}^p),$$

due to the solidity of $\boldsymbol{L}^p(\mathbb{R}^d)$. In fact, HDY gives $\mathcal{F}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1)) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^p) \subset \boldsymbol{C}_0(\mathbb{R}^d) \subset \boldsymbol{L}^\infty$, thus $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1) \hookrightarrow \mathcal{F}\boldsymbol{L}^\infty$, hence $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^1) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^1)$. □

Pointwise multiplication and convolution

We do not formally describe the abstract relations, which hold in the most general form, but instead illustrate them by examples.

Lemma

The spaces $\left(\boldsymbol{M}^p(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^p}\right) := \left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p), \|\cdot\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)}\right)$ *are Banach spaces of distributions which are invariant under the Fourier transform, with*

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^{p_1}(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^{p_2}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$$

if and only if $1 \leq p_1 \leq p_2 \leq \infty$.

Proof: $p = 1$ direct, $p = \infty$ by duality, then interpolation.

Basic Facts concerning Wiener Amalgams

We have established the following facts:

- One replace the discrete norm by a continuous one;
- Natural duality results, or complex interpolation;
- Pointwise multiplication operators;
- Convolution operator, also coordinatewise

More or less all these questions have been answered in the early papers on the subject, between 1980 (time of writing) and 1985 (publication). At the same time the theory of *modulation spaces* $\boldsymbol{M}^{s}_{p,q}(\mathbb{R}^d)$ has been developed, with the idea that they should be viewed as $\mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q{}_{v_s})) >>$ PART II.
Nowadays the more elegant approach using the STFT is prevalent in the literature and often the only known variant.

More general Local Components I

The idea behind the spaces now called *Wiener amalgam spaces* with *more general local components* (like Lipschitz or Besov norms) can be easier expressed by the continuous norm: We just replace the local $\boldsymbol{L}^p$-norm by such a more general norm. However, then the question of the independence of the norm (up to equivalence) on the bump-function arises. It turns out that there is a rather simple condition for a local component $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$:

$$\|T_x\varphi \cdot f\|_{\boldsymbol{B}} \leq C\|f\|_{\boldsymbol{B}}, \quad \forall f \in \boldsymbol{B}. \tag{9}$$

If the norm on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is translation invariant this simply means that φ is a pointwise multiplier for $\boldsymbol{B}$, or that φ is sufficiently smooth (and compactly supported).

Hausdorff-Young Theorem and Sobolev algebras

The proof of Sobolev Embedding's Theorem can be realized as follows using the Cauchy inequality, whenever $s > d/2$. We write $v_s(t) = \langle t \rangle = (1 + |t|^2)^{1/2}$. It satisfies $v_s(t+s) \leq C v_s(t) v_s(s)$.

$$\mathcal{H}_s(\mathbb{R}^d) = \mathcal{F}^{-1}(\boldsymbol{L}^2_{v_s}(\mathbb{R}^d)) = \mathcal{F}^{-1}(\boldsymbol{W}(\boldsymbol{L}^2, \ell^2_{v_s})) \hookrightarrow \mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^1))$$

But since obviously $1 \leq 2$ we can apply the Hausdorff- Young Theorem 5 and obtain

$$\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^2) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \ell^2) \hookrightarrow \boldsymbol{L}^2 \cap \boldsymbol{C}_0.$$

In a similar way one find $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{H}_s, \ell^2)$, or weighted version of this claim, using weighted $\boldsymbol{L}^p$-spaces.

Pointwise multipliers of Sobolev spaces

In order to verify that not only $(\mathcal{H}_s(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_s}) \hookrightarrow (\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ for $s > d/2$, but that it is in fact closed under pointwise multiplication we just have to find out that $\boldsymbol{L}^2 * \boldsymbol{L}^2 \subset \boldsymbol{C}_0$ and that $\ell^2_{v_s} = \ell^1 \cap \ell^2_{v_s}$ is closed under convolution, which can be easily derived using the WSA (*weakly sub-additive*) property of $v_s(x) = (1 + |x|)^s$:

$$v_s(x+y) \leq C_s(v_s(x) + v_s(y)), \quad x, y \in \mathbb{R}^d.$$

Since $\mathcal{F}\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^2_{v_s})$ we also see that $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2_{v_s}, \ell^2) = \boldsymbol{W}(\boldsymbol{H}, \ell^2)$. As a consequence it is not surprising that one has

$$\mathcal{M}(\mathcal{H}_s(\mathbb{R}^d)) = \mathcal{M}(\boldsymbol{W}(\mathcal{H}_s, \ell^2)) = \boldsymbol{W}(\mathcal{H}_s, \ell^\infty).$$

Product-Convolution Operators

Wiener amalgam for the study of product-convolution and convolution-product operators, i.e. operators of the form

$$f \mapsto g * (h \cdot f) \quad \text{or} \quad f \mapsto h \cdot (g * f).$$

Such operators are often as regularization operators, because convolution produces smoothness and pointwise multiplication produces decay. It is well known that

$$\mathcal{S}(\mathbb{R}^d) * (\mathcal{S}(\mathbb{R}^d) \cdot \mathcal{S}'(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$$

$$\mathcal{S}(\mathbb{R}^d) \cdot (\mathcal{S}(\mathbb{R}^d) * \mathcal{S}'(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d).$$

Corresponding rules apply for $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ via amalgam arguments on convolution and multiplication.

Connections to the Riesz Compactness Criterion

For most of the function spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ described so far, e.g. $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, $(\boldsymbol{M}^{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^{p,q}})$, and so on, which are both pointwise modules over $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ or at least $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ and Banach convolution modules over $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ we can describe the compact sets among all closed and bounded subsets M which are *uniformly tight* and *uniformly equicontinuous* in their respective norms.

[fe84] H. G. Feichtinger.
Compactness in translation invariant Banach spaces of distributions and compact multipliers.
J. Math. Anal. Appl., 102:289–327, 1984.

It is an important feature of PC/CP operators, applied to come bounded set, that the convolution part creates uniform continuity (while preserving tightness) and the pointwise multiplier (by a summability kernel) creates decay (preserving equicontinuity). Thus such operators are typically compact operators.

Quick summary of Gabor Analysis I

The basic idea of **Gabor Analysis** is the "representation of functions or distributions" (signals) using time-frequency shifted copies of a template, also called *Gabor atom*, typically along some lattice in the Time-Frequency plane (e.g. $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, also known as *phase space* or TF-plane). This is the *atomic viewpoint*: Given the building blocks or atoms, the question is how to find coefficients and establish convergence.

Equivalently one can look into this problem from a frame theoretic viewpoint. Given as set of frame-coefficients, i.e. a collection of scalar products of a given distribution with all the elements from a (to-be) frame, can we actually (re)construct this distribution from the coefficients (in a linear and stable way). All this is captured in the setting of Gabor frames respectively the reconstruction of f from samples of the STFT (Short Time Fourier Transform).

The KEY PLAYERS

1. Time-Frequency Shifts
2. STFT (the Short-Time Fourier Transform)
3. Gabor families (regular and irregular)
4. Gabor frames, dual Gabor atom
5. Spreading and Kohn-Nirenberg description
6. Gabor multipliers, underspread operators
7. Anti-Wick Operators
8. Gaborian Riesz sequences
9. Finite variants over $\mathbb{C}^n = \ell^2(\mathbb{Z}_n)$.

BASIC FACTS concerning STFT

The STFT $(f, g) \mapsto V_g f$ is a continuous, sesquilinear mapping from $\boldsymbol{L}^2(\mathbb{R}^d) \times \boldsymbol{L}^2(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, satisfying

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

We also have Moyal's Equality (energy preservation), but not automatically boundedness into $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$ whenever we restrict the STFT to some lattice Λ, typically to $a\mathbb{Z}^d \times b\mathbb{Z}^d$.
This means on the one hand for some g one may obtain coefficients from certain functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ which are *not* which are just bounded by NOT in $\ell^2(\Lambda)$, while on the other hand for such "bad" functions $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ also the synthesis mapping will fail to be bounded (they violate the Bessel condition).

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

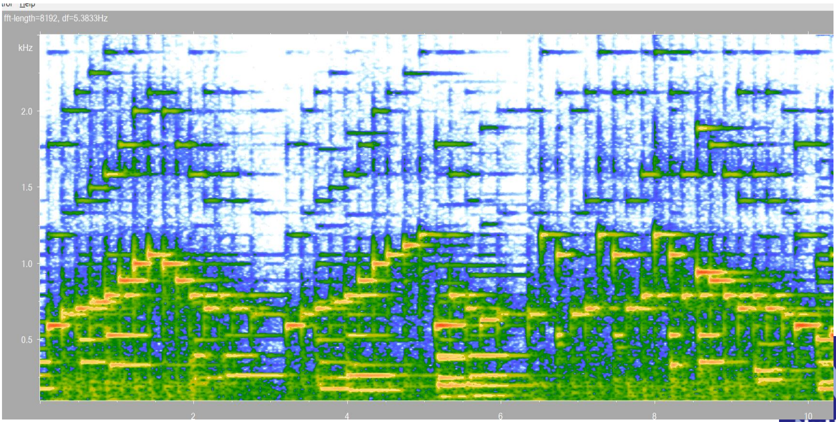
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t,\omega);$$

A Typical Musical STFT

A typical piano spectrogram (Mozart), from recording



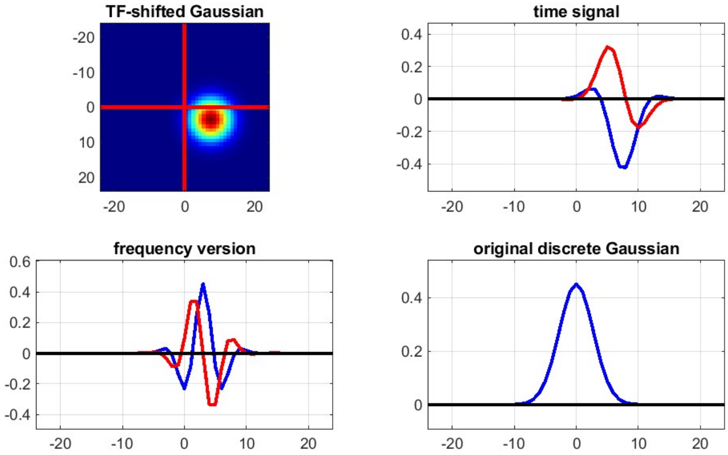
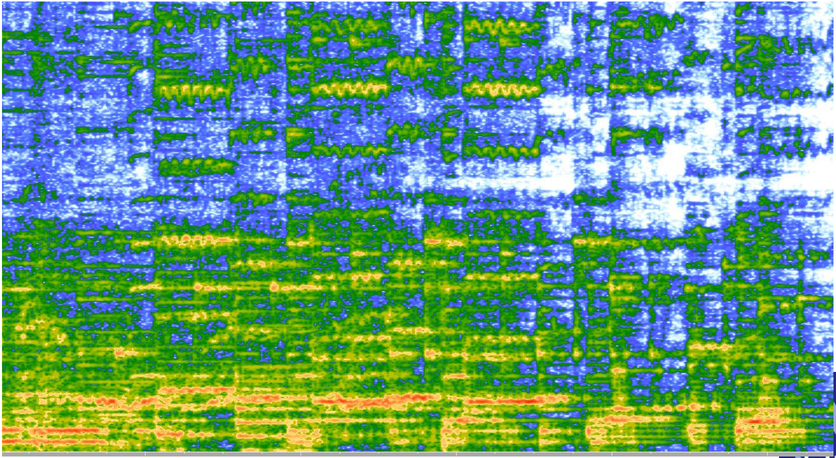
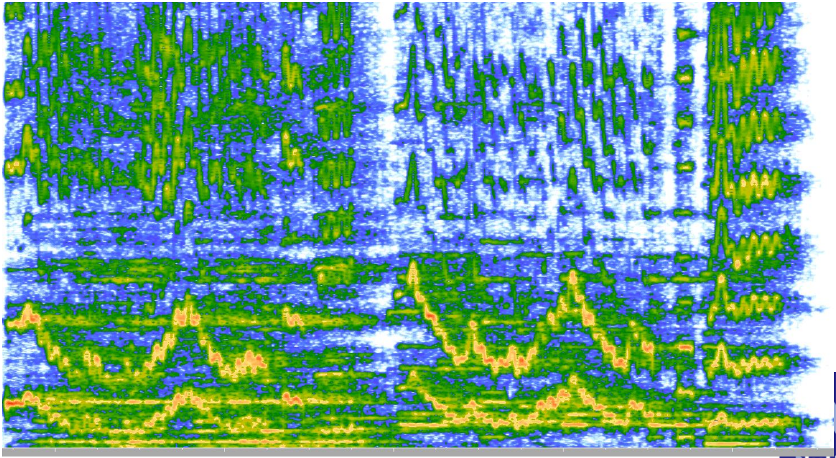


Abbildung: g48TFshifts.jpg

A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \le \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of V_g is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$. Assuming $\|g\|_2 = 1$ we have the *reconstruction formula*:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda)g,$$

which can be approximated in $\boldsymbol{L}^2$ by Riemannian sums.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ *(dense embedding);*
2. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

Standard Gabor Frame Terminology

A *Gabor family* $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is obtained from an "atom" $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ by applying a (discrete, countable) set Λ of TF-shifts $\pi(\lambda) = M_s T_t$, with $\lambda = (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Such a family is called a *regular Gabor family* if $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a discrete (compact) lattice in $\mathbb{R}^{2d}$, i.e. if it is of the form $\boldsymbol{B} * \mathbb{Z}^{2d}$, where $\boldsymbol{B}$ is some non-singular $2dd$-matrix.

It is called a *Gabor frame* respectively *Gabor Riesz sequence* if it is a frame for the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (resp. a Riesz basic sequence, meaning a Riesz basis for its closed linear span).

Both the *frame* and the *Riesz* property can be described by (meanwhile well-known) pairs of inequalities, but it is more instructive (and strengthens the analogy to linear algebra, see the FOUR SPACES principle by Gilbert Strang!) to express these properties via commutative diagrams.

Gabor Frame operators I

For fixed (g, Λ) the composition of the coefficient mapping, assumed to be bounded, from $\boldsymbol{L}^2(\mathbb{R}^d)$ to $\ell^2(\Lambda)$ (or meeting the Bessel condition) $f \mapsto V_g f|_\Lambda$ with the synthesis mapping (its adjoint) $(c_\lambda)_{\lambda\in\Lambda} \to \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$ is a bounded operator denoted by $S_{g,\Lambda}$ on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.
The Gabor family g_λ (short-hand notation!) defines a *Gabor frame* is $S_{g,\Lambda}$ is an invertible(!) operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.
We concentrate here on the regular case, because in this case one can show that the frame-like operator $S_{g,\Lambda}$ commutes with the TF-shifts defining it:

$$\pi(\lambda) \circ S_{g,\Lambda} = S_{g,\Lambda} \circ \pi(\lambda), \quad \lambda\in\Lambda.$$

Obviously this implies that also the inverse commutes:

$$\pi(\lambda) \circ S_{g,\Lambda}^{-1} = S_{g,\Lambda}^{-1} \circ \pi(\lambda), \quad \lambda\in\Lambda.$$

Gabor Frame operators II

But this in turn implies that there is some (canonical!) dual Gabor atom $\widetilde{g} = S_{g,\Lambda}^{-1}(g)$ (depending on (g, Λ)) such that

$$S_{g,\Lambda}^{-1} = S_{\widetilde{g},\Lambda}.$$

In practice on seeks to find the solution of the linear equation

$$S_{g,\Lambda}(\widetilde{g}) = g,$$

observing that $S_{g,\Lambda}$ is a positive definite operator (matrix in the finite dimensional case).
The samples of the STFT $V_{\widetilde{g}}(f)$ over the lattice provides in a linear way suitable coefficients, in fact, it provides
the minimal norm solution in $\ell^2(\Lambda)$.

Basic concepts on Segal Algebras I

A Segal algebra $S(G)$ on a locally compact abelian group G is a subspace of $L^1(G)$ that satisfies the following conditions:

1. **Banach Space Continuously Embedded in $L^1(G)$**: $S(G)$ is a Banach space with its own norm, denoted by $\|\cdot\|_S$, and it is continuously embedded in $L^1(G)$ as a dense subspace. This means there exists a constant C such that for all $f \in S(G)$,

$$\|f\|_{L^1} \leq C\|f\|_S.$$

2. **Isometric Translation Invariance**: $S(G)$ is invariant under translations, and the translation operator is an isometry. For any $f \in S(G)$ and any $x \in G$, the translated function $T_x f$ defined by $(T_x f)(y) = f(y - x)$ is also in $S(G)$ and $\|T_x f\|_S = \|f\|_S$.

Basic concepts on Segal Algebras II

3. **Continuous Dependence on Translations**: The norm $\|T_x f - f\|_S \to 0$ as $x \to 0$.

As a consequence of these conditions, $S(G)$ has the following properties:

1. **Ideal Property**: $S(G)$ is an ideal in $L^1(G)$. For all $f \in S(G)$ and $g \in L^1(G)$, there exists a constant C' such that

$$\|f * g\|_S \le C' \|f\|_S \|g\|_{L^1}.$$

2. **Banach Algebra**: $S(G)$ is a Banach algebra with respect to convolution.

Basic concepts on Segal Algebras III

3. **Submultiplicativity**: The norm is submultiplicative with respect to convolution, meaning there exists a constant C'' such that for all $f, g \in S(G)$,

$$\|f * g\|_S \leq C''\|f\|_S\|g\|_S.$$

4. The density combined with the continuous translation also implies that any Segal algebra contains (unbounded) approximate units.
5. In fact, any band-limited function belongs to any Segal algebra and consequently band-limited functions are dense in any Segal algebra with respect to its norm (not only in the $\boldsymbol{L}^1$-sense.

Basic concepts on Segal Algebras IV

6. More generally *homogeneous Banach spaces* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in the sense of Y. Katznelson ([ka76]) are more general than Segal algebras. They do not have to be dense (any closed ideal of a Segal algebra qualifies), nor do they have to be included in $\boldsymbol{L}^1(\mathbb{R}^d)$. Here again $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is a good example.

Strong Character Invariance: $S(G)$ is strongly character invariant if pointwise multiplication by a character is isometric on $S(G)$. This means that for any $f \in S(G)$ and any character χ of G,

$$\|f \cdot \chi\|_S = \|f\|_S.$$

This property holds if $S(G)$ is also a Banach function space in the sense of Luxemburg/Zaanen, such as $\boldsymbol{L}^1 \cap \boldsymbol{L}^p(\mathbb{R}^d)$ or Wiener's algebra $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$.

Basic concepts on Segal Algebras V

It has been shown in [?] that one has for any strongly character-invariant Segal algebra

$$\lim_{\chi \to 0} \|\chi \cdot f - f\|_S = 0, \quad \forall f \in S.$$

Thus, as a consequence (in the terminology of time-frequency analysis) the mapping from $(x, \chi) \in G \times \widehat{G}$ to $M_\chi T_x$ is a *strongly continuous projective representation of phase space*. By adding the torus group one obtains the *reduced Heisenberg group* and it has now an irreducible, unitary representation on $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$.
It is described in [fe81-2] (a survey is given in [ja18]) that there is a smallest member in this family (of strongly character invariant Segal algebras), which is $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Being a member of the family this representation leaves $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ invariant (note that it is a dense, hence not a closed subspace of $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$).

Basic concepts on Segal Algebras VI

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Usefulness and applications of Gabor frames:

The question of Gabor frames is of interest, when a signal (say some audio signal, or some image, cf. introduction) is to be *decomposed* into meaningful elementary building blocks, somehow like *transcription*. Ideally the distribution of *energy* in the signal goes over into an equivalent energy distribution. AND WHAT can we do with this:

a) contributions may be irrelevant (or disturbing) and can be eliminated (the bird contributing to the open air classical concert): denoising of signals

b) signals can be separated in a TF-situation

c) unimportant, small contributions can be omitted (+ masking effect): allows for efficient lossy compression schemes >> MP3.

The usefulness of $W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d) = S_0(\mathbb{R}^d)$ I

It is meanwhile established theory that $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ is helping to provide an answer to such questions (arguments will follow later). We have to following results from the last 30 years of Gabor analysis. Recall the chain of inclusions

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right).$$

Consequently all the statements to follow are valid for Schwartz functions or functions having sufficient smoothness and decay, but $\boldsymbol{S}_0(\mathbb{R}^d)$ is a much larger (Banach) space.

Note that for function $f \in \boldsymbol{L}^1(\mathbb{R})$ it is enough that a function satisfies $f', f" \in \boldsymbol{L}^1(\mathbb{R})$, or that it is piecewise linear, with a set of nodes of minimal distance (result of 2023). Any classical summability kernel belongs to $\boldsymbol{S}_0(\mathbb{R})$ (see F. Weisz).

Note also that these results are not restricted to the case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$.

The usefulness of $W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d) = S_0(\mathbb{R}^d)$ II

1. For $g \in S_0(\mathbb{R}^d)$ the family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Bessel family, hence both analysis and synthesis operators and thus the frame operator are bounded for each lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (with uniform estimates); [fezi98]
2. The dual window $\widetilde{g}$ belongs to $S_0(\mathbb{R}^d)$ by [grle04], and depends continuously on $g \in S_0(\mathbb{R}^d)$ and $\Lambda = A * \mathbb{Z}^d$. [feka04]
3. The Riemannian sums for the inversion formula converge in the sense of $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$ whenever $g \in S_0(\mathbb{R}^d)$. (F. Weisz, various papers).
4. Given $g \in S_0(\mathbb{R}^d)$ one can show that any sufficiently dense lattice Λ generates a Gabor frame. In fact [fegr89],[fezi98]

$$\lim_{\Lambda\to(0,0)} C_\Lambda S_{g,\Lambda} = \mathrm{Id}\,.$$

The Segal algebra $S_0(\mathbb{R}^d)$ as a prototype I

The definition (see [fe81-2]) of the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ (also well defined for LCA groups) as Wiener Amalgam spaces as $\left(W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d), \|\cdot\|_{W(\mathcal{F}L^1,\ell^1)}\right)$ is a good example of the usefulness of Wiener amalgams. We have a chain of continuous embeddings:

$$W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^1, \ell^\infty)(\mathbb{R}^d).$$

The largest space in this chain is in fact the space of pointwise multipliers of the algebra $S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$.
One inclusion follows from the pointwise relation

$$W(\mathcal{F}L^1, \ell^\infty) \cdot W(\mathcal{F}L^1, \ell^1) \subset W(\mathcal{F}L^1, \ell^1),$$

taking local and global components separately.

Wiener Amalgam in Action: Regularization

Convolution and pointwise multiplier results imply that

$$\boldsymbol{S}_0(\mathbb{R}^d) \cdot (\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \tag{10}$$

$$\boldsymbol{S}_0(\mathbb{R}^d) * (\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \tag{11}$$

Proof.

The key arguments for both of these regularization procedures, be it convolution followed by pointwise multiplication (a CP or product-convolution operator), or correspondingly PC operators, are based on the pointwise and convolutive behavior of generalized Wiener amalgam spaces, such as the relation
$\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)$.
Combined with the multiplier result of the last slide we are done. The second one is the Fourier version of the same claim. □

Fourier Invariance

We know that $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right) = \left(W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d), \|\cdot\|_{W(\mathcal{F}L^1,\ell^1)}\right)$ is Fourier invariant (using e.g. the characterization via STFT with Gaussian window), FT corresponds to rotation.
Obviously $W(\mathcal{F}L^2, \ell^2) = L^2(\mathbb{R}^d)$ (with norm equivalence), and thus also is Fourier invariant by Plancherel's theorem. This implies that also is Fourier invariant by complex interpolation for $1 \leq p \leq 2$ and subsequently by duality of $p \in [2, \infty]$.
Since $\left(M^{p,p}(\mathbb{R}^d), \|\cdot\|_{M^{p,p}}\right)$ is (by definition) the inverse FT of , hence equal, this implies that the spaces $\left(M^p(\mathbb{R}^d), \|\cdot\|_{M^p}\right)$ form a scale of Fourier invariant Banach spaces. For $1 \leq p \leq \infty$ we have:

$$M^1(\mathbb{R}^d) \hookrightarrow M^p(\mathbb{R}^d) \hookrightarrow M^\infty(\mathbb{R}^d).$$

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Thanks!

THANKS you for your attention

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