NuHAG Numerical Harmonic Analysis Group

Quantum Harmonic Analysis via the Banach Gelfand Triple II

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The three main topics of this talk

The purpose of this talk is to present some details concerning the view on the subject (QHA) which have been developed during the preparation for the workshop in June 2023 in Trondheim and which have been further elaborated in the meantime (although not yet put together as a manuscript).

The key topics will be:

1. The use of the Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ instead of trace class operators, differences and advantages;
2. A time-frequency approach to the problem of (approximately) recovering a function $f \in \mathbf{S}_0(\mathbb{R}^d)$ from regular samples;
3. A detailed description of what I call the *Nto4N*-principle;
4. Signals and Operators viewed as Mild Distributions.

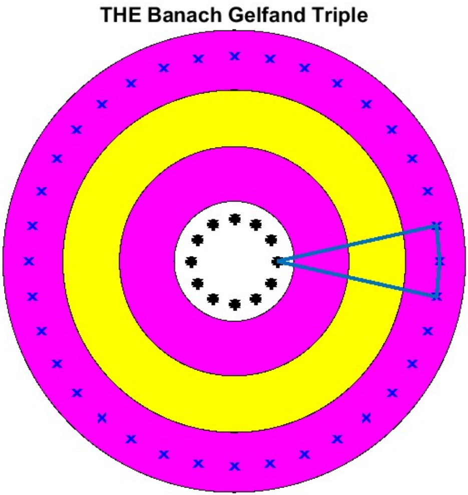
Key Objects in (Quantum) Harmonic Analysis

We have seen already in the first day that the following objects play an important role in QHA (over LCA groups)

- time-frequency shifts, STFT, convolution, Fourier transform
- Gabor expansions, Gabor multipliers,
- Wigner and cross Wigner distribution, Weyl calculus;
- Kernel Theorem, Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$;
- spreading representation and Kohn-Nirenberg calculus
- Hermite functions, Fractional FT, metaplectic group

Illustration of the BGTr I

Illustration of the BGTr II



Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

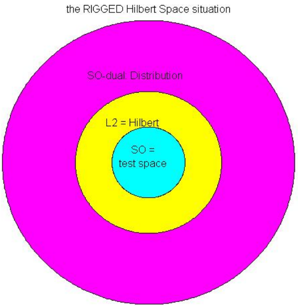


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$,

known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (**[lo83-1]**).
For $\mathbb{R}^d$ the most elegant way (which is describe in **[gr01]** or **[ja18]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}\,dy$$

and the corresponding norm

$$\|f\|_{\mathbf{S}_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|\,dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $\mathbf{S}_0(\mathbb{R}^d)$.

Theorem

1. $S_0(\mathbb{R}^d) \hookrightarrow \left(W(C_0,\ell^1)(\mathbb{R}^d), \|\cdot\|_W\right) \hookrightarrow L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$;
2. $\mathcal{F}(S_0(\mathbb{R}^d)) = S_0(\mathbb{R}^d)$ *(isometrically);*
3. *Isometrically invariant under TF-shifts*

$$\|\pi(\lambda)(f)\|_{S_0} = \|M_s T_t f\|_{S_0} = \|f\|_{S_0}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

4. $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ *is an essential double module (convolution and multiplication)*

$$L^1(\mathbb{R}^d) * S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d) \quad \mathcal{F}L^1(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d),$$

in fact a Banach ideal and hence a double Banach algebra.

5. *Tensor product property* $S_0(\mathbb{R}^d)\widehat{\otimes} S_0(\mathbb{R}^d) \approx S_0(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(S_0(G)) = S_0(H)$.

1. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ has various equivalent descriptions, e.g.
   - as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$;
   - via *atomic decompositions* of the form

$$f = \sum_{i \in I} c_i \pi(\lambda_i) g \text{ with } (c_i)_{i \in I} \in \ell^1(I).$$

2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under group automorphism;
3. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under the *metaplectic group*, and thus under the *Fractional Fourier transform* as well as the multiplication with *chirp signals*: $t \mapsto exp(-i\alpha t^2)$, for $\alpha \geq 0$.

In addition $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is quite universally useful in Classical Fourier Analysis and of course for *Time-Frequency Analysis* and *Gabor Analysis*, and as I am going to show also for QHA: Quantum Harmonic Analysis. In short, it is easier to handle than the Schwartz-Bruhat space or even the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, and since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ it is (much) bigger.

Theorem

1. $\mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$ via $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0'$.
2. *Identification of TLIS:* $\boldsymbol{H}_G(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$, $(\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \to \boldsymbol{S}_0(H) \hookrightarrow \boldsymbol{S}_0(G)$ via $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \boldsymbol{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \boldsymbol{S}_0(G) \mid \mathrm{supp}(\tau) \subset H\}$.

Theorem

1. $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}) = (\mathbf{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^\infty})$, *with* $V_g(\sigma)$ *and* $\|\sigma\|_{\mathbf{S}_0'} = \|V_g(\sigma)\|_\infty$, *hence norm convergence corresponds to uniform convergence on pahse space. Also* w^*-*convergence is uniform convergence over compact subsets of phase space.*
2. $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}) \hookrightarrow (\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$, *with density for* $1 \le p < \infty$, *and* w^*-*density in* $\mathbf{S}_0'$. *Hence, facts valid for* $\mathbf{S}_0$ *can be extended to* $\mathbf{S}_0'$ *via* w^*-*limits.*
3. *Periodic elements* $(T_h\sigma = \sigma, h \in H)$ *correspond exactly to those with* $\tau = \mathcal{F}(\sigma)$ *having* $\mathrm{supp}(\tau) \subseteq H^\perp$.
4. *The (unique) spreading representation* $T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)d\lambda$, $F \in \mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *for* $T \in \mathcal{B}$ *extends to the isomorphism* $T \leftrightarrow \eta(T)$ $\eta : \mathcal{B} \approx \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$, *uniquely determined by the correspondence with*
$$\eta(\pi(\lambda)) = \delta_\lambda, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Some conventions

Scalar product in $\mathcal{HS}$:

$$\langle T, S\rangle_{\mathcal{HS}} = \text{trace}(T * S^*)$$

In **[feko98]** the notation

$$\alpha(\lambda)(T) = [\pi \otimes \pi^*(\lambda)](T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

and the covariance of the KNS-symbol is decisive:

$$\sigma(\pi \otimes \pi^*(\lambda)(T)) = T_\lambda(\sigma(T)), \quad T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Main Applications

1. Fourier Transform as unitary BGTr-automorphism
2. Signals ARE mild distributions
3. Elements $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ represent measurements of signals (in a linear fashion);
4. Kernel Theorem: extending the Hilbert Schmidt case (linear operators are the same as "continuous matrices");
5. Convolution operators can be represented by the "impulse response" or the "transfer function";
6. Generalized Stochstic processes can be modelled as linear operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to some (stochastic) Hilbert space;
7. mild distributions can be regularized (w^*–approximation);
8. test function can be approximated from samples!

HGfei	Others
$\boldsymbol{S}_0(\mathbb{R}^d)$	$\boldsymbol{M}^1(\mathbb{R}^d)$
$\boldsymbol{S}_0'(\mathbb{R}^d)$	$\boldsymbol{M}^\infty(\mathbb{R}^d)$
$(\mathcal{N}_{w*}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$	$(\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^\infty)$

Discrete Hermite functions are treated in a large number of numerical or engineering papers. Typically one takes the *Harmonic Oscillator* and looks for a discrete approximation (by analogy) of this unbounded operator. Since one wants to create an ONB of eigenvectors of the DFT one has to make sure that the matrices built commute with the DFT matrix.

Obviously one should ask (given the multitude of approaches), which ones are the best ones? Can one prove that they converge (for increasing finite signal length and fixed number of the Hermite function) to the continuous limit (in which sense?)? What are the require properties that makes them useful, e.g. for the computation of a discrete Fractional FT (FrFT).

The approach developed by the author in the last years (which is equivalent to the method used by N. Cotfas) is based on the characterization of Hermite functions as eigenvectors of Anti-Wick operators with radial symmetric weights (following

I. Daubechies work on localization operators). For better numerical stability a multiplier of the form $(1+|t|^2+|\omega|^2)^{-s}$ for some $s>0$ is preferred. As window a discrete DFT-invariant Gaussian is used, obtained by periodization and sampling.
Many numerical experiments show that these discrete Hermite functions behave very well with respect to a number of test:

- The corresponding Hermite multiplier simulating the FrFT effectively rotates the spectrogram (with respect to a Gaussian window);
- The application to discrete chirps does not show warping effects (like others);
- The exponential representation of *coherent states* showing that they are eigenvectors to the annihilation operators works well!

The Spreading Representation I

In the finite-dimensional situation (vectors in $\mathbb{C}^n$ are viewed as functions on the cyclic group $\mathbb{Z}_n$ of unit roots of order $n \in \mathbb{N}$, it is clear that we have n cyclic shift operators $T_k, k = 0, 1, ..., n-1$ and also corresponding frequency shifts M_j, $j = 1, ..., n-1$, which are just the multiplication operators by the rows (or columns) of the DFT matrix (representing the dual group, naturally identified with $\mathbb{Z}_n$ again). Thus functions on phase space can be identified with $n \times n$-matrices! They form an n^2-dimensional vector space (equivalently described by the Hilbert-Schmidt scalar product $\langle T, S \rangle_{\mathcal{HS}} := \text{trace}(TS*)$.
Thus is it not so surprising that the n^2 TF-shifts $\pi(k,j) = M_j T_k$ form (up to the scaling factor $\sqrt{n}$) an ONB for the space of matrices. The coefficients arising in this ONB expansions describe the spreading function.

The Spreading Representation II

It is also true, that each $T \in \mathcal{M}^1$ has a (unique) representation using some $\eta(T) \in \mathbf{S}_0(\mathbb{R}^{2d})$:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(T)(\lambda)\pi(\lambda)d\lambda.$$

This relationship can be shown to be isometric with respect to the norm of the corresponding Hilbert space $\mathcal{HS}$ and $\mathbf{L}^2(\mathbb{R}^{2d})$ respectively and thus one can extend it. It is even another *unitary BGTr* isomorphism between $(\mathcal{N}_{w*}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$ and $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$.
We will use it to some extent for the big space $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$, whose elements have a spreading representation in $\mathbf{S}_0'(\mathbb{R}^{2d})$.

The Periodization and Sampling Operators I

Starting from the Fourier invariant Gauss function $g_0(t) = exp(-\pi|t|^2)$ one comes to DFT-invariant discrete Gaussians of finite length n by appling a suitable concatination of sampling and periodization. If done properly this process can be done in such a way to obtain DFT invariance of the resulting finite sequence (an appropriate restriction to the fundamental domain). This is best describe for the choise $n = k^2$ for some $k \in \mathbb{N}$. Then one can periodize with period k and sampling rate $1/k$. Since obviously $k = n \cdot 1/k$ the two operations are compatible and even commute. Moreover, making a multiplier use of Poisson's formula one can show that the resulting vector is (up the scaling) invariant under the DFT.

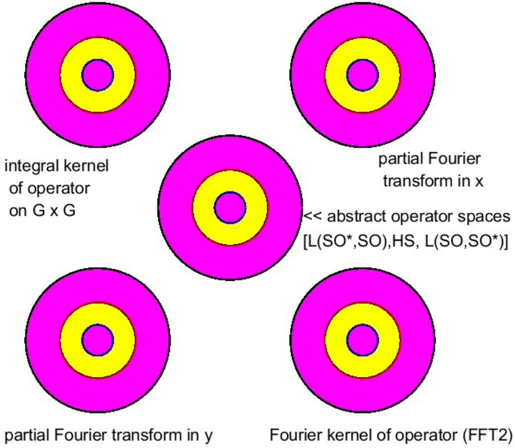


Abbildung: The spaces of operators $(\boldsymbol{B}, \mathcal{H}S, \boldsymbol{B}')$ can be identified with four different representations.

Wigner functions for even signal length I

There is an extensive literature about the Wigner distibution of a function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, which has many good properties (see the book of Maurice de Gosson), in particular due to its *covariance property* (and also perhaps the marginal properties), and certainly the fact that is real-valued, even for a general complex-valued $f \in \boldsymbol{L}^2(\mathbb{R}^d)$. It is also highly interesting that the chirp-signals (characters of second degree), namely $ch_\alpha(t) = exp(i\alpha\pi t^2)$ (on $\mathbb{R}$) have a rather simple Wigner distribution, namely a Dirac along a line in phase space. Also, the (distributional) FT of a chirp is a "complementary chirp".
There is an extensive literature on discrete version of the Wigner function and also of the group of "linear canonical transform" in the engineering literature (nothing else than the metaplectic group for a finite Abelian group, if done properly).

Wigner functions for even signal length II

BUT the usual definitions have great problems with the fact that $x \mapsto 2x := x + x$ is *not* an automorphism on many groups, such as $\mathbb{Z}_n$ with n even. In contrast, all is fine (with interpretation through finite geometry) for the case that n is a *prime number*, but even in such a case the Wigner function associated to a discrete Gaussian does not look like a discretized version of the continuous limit (because it has four peaks!).
Problems of this kind have been the reason why W. Kozek suggested to the NuHAG team to rather use the Kohn-Nirenberg Calculus for the discrete setting (which does not use dilations), also because it works well for any LCA group. In the BGTr setting one has again:

Wigner functions for even signal length III

Theorem

There are two more unitary BGTr isomorphism, namely:
1) The KNS-assignment $T \mapsto \sigma(T)$ *is an isomorphism between* $(\mathcal{N}_{w*}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$.
2) The symplectic Fourier transform established another automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$

The clasical attempt of studying the Wigner transform is to correlate it with a TF-conjugated version of the *parity operator* P, i.e. to define

$$W_\psi(\lambda) = \mathrm{trace}([\psi \otimes \psi^*] \circ [\pi(\lambda) \circ P \circ \pi(\lambda)^*]).$$

Wigner functions for even signal length IV

The realization of this program (using the so-called *displacement operators*) results in a reduction of information, since the conjugated (by displacements) parity operators only form an $n^/4$-dimensional subspace of all $n \times n$-matrices for even n. Typically people resort to some (unnatural, but working) variant, using signals with $2n$ samples, obtaining thus the correct redundancy. We take a different route, namely the one-sided combination of the parity matric (which is the square of the unitary DFT matrix!) with TF-shifts. In this way one obtains unitary matrices with the property that the product of such an $n \times n$-matrix by its conjugate is not (yet) the identity matrix, but a scalar multiple, in fact the scalar is some unit root of order n. Consequently by changing the phase by the square root of this factor (thus a unit root of order $2n$) helps to fix the problem an obtain an ONB of *symmetric matrices*.

Wigner functions for even signal length V

In this way we can establish the key properties which a (discrete) Wigner transform (in fact cross-Wigner) should have. Among others it give a real-valued symbol for hermitean matrices! However, the covariance property has to be discussed in a specific way instead.
Also there is the adjoint relationship, from the matrices to their Wigner symbols, and this is a kind of discrete Weyl-calculus.

Doubling the period and the sampling rate I

In the approximation of functions in $\mathbf{S}_0(\mathbb{R}^d)$ from (finite) samples the combination of periodization and sampling plays an important role. For us the interchange with the DFT is also important for a description.

Obviously there are many possible ways of choosing the periods and the sampling rates, but it is clear that some of them are in a nice/natural relationsship (of algebraic nature), namely if one period p_2 is the (integer) multiple of another period p_1, and similar with sampling rates s_2 could be an integer fraction of s_1.

Specifically one can ask, what happens when one starts with a given operator and then doubles the period and the sampling rate. In other words, how to the corresponding vectors of length N and $4N$ relate to each other naturally.

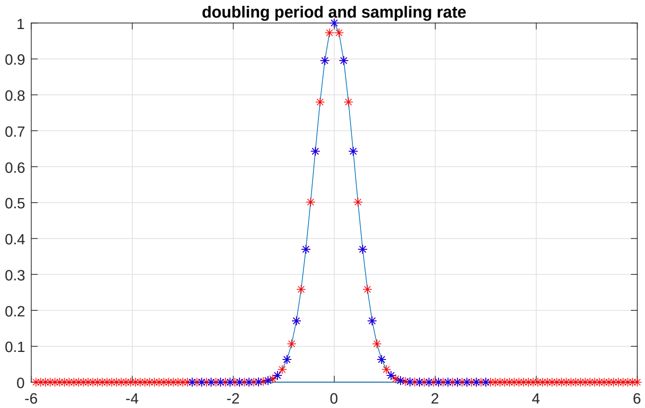


Abbildung: Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.

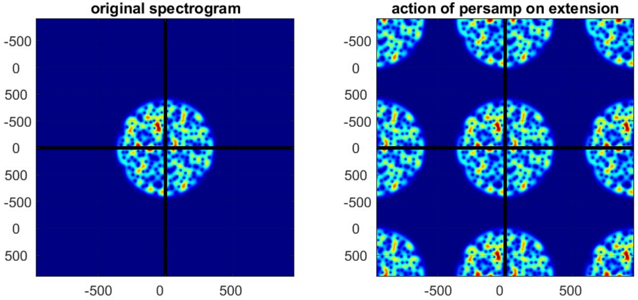


Abbildung: The picture in the STFT domain. Actually, it is the (twisted) convolution with a 2D Dirac comb with 4 points, so it is a 2D-periodization of the spectrogram.

Periodization and Sampling at the spectrogram level I

Theorem

We consider a tight Gabor family $(g_\lambda)_{\lambda\in\Lambda}$ with $\Lambda = 1/2\mathbb{Z}^d$ and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Thus any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ has a representation of the from $f = \sum_{\lambda\in\Lambda} V_g f(\lambda)\pi(\lambda)g$, and the coefficient norm of $(V_g f(\lambda))_{\lambda\in\Lambda}$ in $(\ell^1(\Lambda), \|\cdot\|_1)$ defines an equivalent norm on $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$. In addition the $\boldsymbol{S}_0'$-norm of $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ is equivalent to the $\ell^\infty(\Lambda)$-norm of its Gabor coefficients $V_g(\sigma)|_\Lambda$. In this case any periodization operator PS_p realizing a periodization with period p and sampling at the rate $1/p$, for some $p \in 2\mathbb{N}$ (any even natural number) defines a bounded operator from $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ which commutes with the Fourier transform. At the level of Gabor coefficients it corresponds to periodization with respect to the lattice $p\mathbb{Z}^{2d}$.

Periodization and Sampling at the spectrogram level II

The decisive argument can be compressed in the following chain of equations: Let us write H_p for the group $p\mathbb{Z}^d \times p\mathbb{Z}^d \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ for some even $p \in \mathbb{N}$. Consequently we have for any $\lambda \in 0.5 \cdot \mathbb{Z}^{2d}$:

$$V_g(\mathrm{PS}_p f)(\lambda) = \langle \sum h \in H_p \pi(h) f, \pi(\lambda) g \rangle =$$

$$= \langle \sum_{h \in H_p} f, \pi(\lambda - h) g \rangle = \sum_{h \in H_p} V_g(\lambda - h).$$

Theorem

Under the assumptions of Theorem 5 we have: the sequence of operators $f \mapsto (\mathrm{PS}_p f)$, with $p \in 2\mathbb{N}$ is uniformly bounded and satisfies

$$\lim_{p \to \infty} \|(\mathrm{PS}_p f) - f\|_{\boldsymbol{S}_0} = 0, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{1}$$

Discrete Hermite and Wigner: the *Nto4N* Principle

Aside from the "visually good behaviour" of the FrFT which can be based on the discrete Hermite functions computed so far the **compatibility with the *Nto4N*-principle** is one of the strongest arguments for me to go expect that the piecewise linear interpolators of the discrete Hermite functions obtained in this way (or suitable quasi-interpolation operators, using e.g. cubic B-splines) converge to the Hermite functions at least in the sense of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and probably also with respect to Sobolev or Shubin class norms (work in progress).
Concerning the (new variant) of Wigner functions associated with discrete signals of any length we have the same **compatiability with the *Nto4N*-principle**. One should not expect the marginal properties of the discrete Wigner to apply IN THE DISCRETE DOMAIN (impossibility results!) but only use dilation when one puts discrete data back to the continuous domain!

Good arguments for mild distributions I

While for many applications it is good to take $\mathcal{L}(\mathcal{H})$, the space of bounded linear operators as the *universe* to start with (comparable to the tempered distributions for function spaces in the spirit of Hans Triebel), this is sometime restrictive, as we want to explain. Recall that for $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ the space $\mathcal{L}(\mathcal{H})$, endowed with the operator norm is not only a C^*-algebra, but it is the dual space of all *trace class operators* $\mathcal{T}^1(\boldsymbol{L}^2(\mathbb{R}^d))$. Trace class operators (in fact finite rank operators) are dense in the space of all Hilbert-Schmidt operators, which is in turn a Hilbert space with the scalar product

$$\langle T, S\rangle_{\mathcal{HS}} = \operatorname{trace}(TS^*), \quad T, S \in \mathcal{HS}.$$

Altogether they also form a Banach Gelfand Triple of the form

Good arguments for mild distributions II

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Assuming our principle interest is in dealing with the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ and operators related to it, or Wiener's algeba $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (because it can be defined without recurrence to the existence of a Haar measure), there are still good reasons to introduce tools involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$!

1. If you are asking for the bounded linear operators on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ it is not clear whether they have a kernel (unless they are (compact) Hilbert-Schmidt operators $(K \in \boldsymbol{L}^2(\mathbb{R}^{2d}))$;
2. Even for pointwise our Fourier multipliers you have to use functions in $\boldsymbol{L}^\infty(\mathbb{R}^d)$ (and their Fourier transforms);

Good arguments for mild distributions III

3. Given $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and a lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ the mapping $f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$ is not anymore bounded from $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ to $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$.
4. The classical Hausdorff-Young Theorem shows strong limitations, one can only take $p \in [1,2]$. But how "nasty" are distributional Fourier transforms of functions of the form $\widehat{f}, f \in \boldsymbol{L}^p(\mathbb{R}^d)$ for $p \in (2, \infty)$.
5. How should one define spec(f), the *spectrum of a function in* $\boldsymbol{L}^\infty(\mathbb{R}^d)$? For example, for trigonometric functions (anharmonic case!) it should be the collection of non-trivial frequencies in $\widehat{\mathbb{R}^d}$ occurring in the "signal" (same for AP-functions).

Good arguments for mild distributions IV

6. Engineers are very familiar with the principle that sampling on the time side corresponds to periodization on the frequency side. For band-limited functions this allows to exactly recover the original spectrum $\widehat{f}$ by pointwise multiplication. Equivalently we can recover f from f or better $\sum_{\lambda\in\Lambda} f(\lambda)\delta_\lambda$ by convolution with a suitable "kernel" ($>$ Shannon representation).

$$f = \sum_{\lambda\in\Lambda} f(\lambda) T_\lambda g.$$

with convergence in $\boldsymbol{W}(\boldsymbol{C}_0, \ell^2)(\mathbb{R}^d)$, hence in $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{C}_0(\mathbb{R}^d)$.

Good arguments for mild distributions V

7. Even larger is the space of multipliers (operators commuting with translations) from $\left(\boldsymbol{W}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$ to its dual $\left(\boldsymbol{W}^*(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}^*(\mathbb{R}^d)}\right)$, we write $\boldsymbol{H}_G(\boldsymbol{W}, \boldsymbol{W}^*)$.
8. The space of *tempered elements* in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ which can also be considered to define bounded convolution operators on $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is in fact a Banach algebra. It can be defined as $\boldsymbol{L}^p(\mathbb{R}^d) \cap \boldsymbol{H}_G(\boldsymbol{L}^p(\mathbb{R}^d))$, or by pointwise existence of convolution products combined with bounedness of the resulting convolution operator. For $1 \leq p \leq 2$ this is the same, but for $p > 2$ we have a problem.

Good arguments for mild distributions VI

9. In the theory of *Anti-Wick operators* one typically starts with multipliers in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ or $\boldsymbol{L}^\infty(\mathbb{R}^{2d})$, giving operators of the form

$$GM_m(f) = \int_{\mathbb{R}^d} m(\lambda) V_g f(\lambda) \pi(\lambda) g \, d\lambda,$$

which are Hilbert-Schmidt resp. just bounded operators on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. But for Gabor multipliers, where one has an "upper symbol" (Berezin) of the form $\sigma = \sum_{\lambda \in \Lambda} m(\lambda)\delta_\lambda$, thus giving

$$GM_\sigma(f) = \sum_{\lambda \in \Lambda} m(\lambda) V_g f(\lambda) \pi(\lambda) g = \left(\sum_{\lambda \in \Lambda} m(\lambda) P_\lambda \right)(f).$$

Insights and Take Home Message

THE OVERALL PROGRAM:

- Signals and Operators ARE mild distributions ($\sigma \in \boldsymbol{S}_0'$);
- They can be approximated by regularization, e.g. local Gabor expansions (in time and frequency)
- These finite Gabor sums (or elements from $\boldsymbol{S}_0$) can be approximated by finite dimensional objects which are accessible by computation (MATLAB)
- structure preseving approximation is the key
- every signal equals the collection of all of its finite dimensional approximations (think of martingales)
- We are up for quantitative methods!

THANKS!

Thanks for your attention!

www.nuhag.eu/feitalks: many talks on related subjects including my Trondheim talk of June 2023.

Various Lecture Notes can be downloaded from https://www.univie.ac.at/nuhag-php/home/skripten.php and in particular the course note of the ETH Course 2020: www.nuhag.eu/ETH20

This talk describes some aspects (the algebraic relations between different finite-dimensional approximations of a continuous situation) of the overall program of "Conceptual Harmonic Analysis".