NuHAG **Numerical Harmonic Analysis Group**

The Evolution of Time-Frequency Analysis from Fourier Analysis: The Role of Function Spaces in Gabor Analysis

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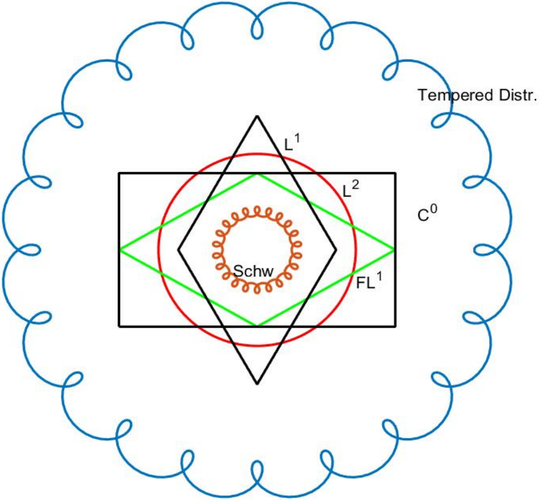
The Classical Situation



Official Abstract I

Classical Fourier Analysis, a foundational aspect of Functional Analysis, heavily relies on the properties of function spaces defined by integrability conditions. These function spaces are instrumental in the progression of classical Fourier methods. Abstract Harmonic Analysis unifies results first obtained for periodic and then for square integrable functions on Euclidean spaces. By starting with a locally compact Abelian group, endowed with a Haar measure, one can introduce the corresponding notion of Fourier transforms. In the case of finite Abelian groups, this yields the classical Discrete Fourier Transform (DFT) and its efficient computation through the Fast Fourier Transform (FFT).
From a practical standpoint, such theoretical considerations have had limited direct impact on the development of modern signal processing algorithms, such as the $MP3$ compression scheme widely used for audio signals.
These signals are not truly L^2-functions but instead have a (discrete-time) short-time Fourier transform (STFT), based on the digital version of the signal, sampled at 44,100 samples per second.

Official Abstract II

This scenario necessitates different function spaces.
Collaboration with engineers (e.g. in the area of mobile communication) has reconfirmed a gap in the understanding of the necessary mathematical framework. Bridging this gap has led to the development of new mathematical methods over the past 40 years, creating a dynamic field of mathematical analysis, known as Time-Frequency Analysis, with Gabor Analysis as a central component.
This field utilizes series expansions of non-periodic signals (such as music) into localized Fourier series expansions. The relevant function spaces in this context are modulation spaces, with "Feichtinger's algebraänd its dual, known as the space of "mild distributions being the most pertinent. Among others one can discuss new and interesting approximation theoretic questions relevant for applications.

In fact, signals ARE mild distributions!!

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The need for new function spaces

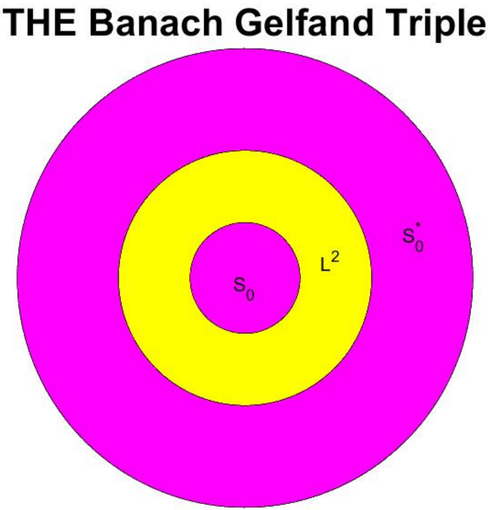
The key-point of this talk will be some exploration of function spaces concepts arising from time-frequency analysis respectively Gabor Analysis. Modulation spaces and Wiener amalgams have proved to be indispensable tools in time-frequency analysis, but also for the treatment of pseudo-differential operators or Fourier integral operators.

Given the limited time we will concentrate on the spaces $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (the space of *mild distributions*), also known as modulation spaces $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, $(\boldsymbol{M}^2(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^2(\mathbb{R}^d)})$ and $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$ respectively.

This will be a bottom up talk, with applications coming first.

In this sense the talk is an experimental one, trying to test modern tools, both mathematical tools and presentation tools.

THE Banach Gelfand Triple



More precisely, we will recall a short summary of the concepts of Wiener amalgam spaces and modulation spaces, as well as the concept of Banach Gelfand Triples, with the associated kernel theorem (in the spirit of the L.Schwartz kernel theorem). We will indicate in which sense these spaces allow to capture more precisely the mapping properties of operators which may be unbounded in the Hilbert space setting. The subfamily of translation and modulation invariant spaces plays a specific role, with naturally associated regularization operators involving smoothing by convolution and localization by pointwise multiplication.

The presentation will be in the spirit of *Conceptual Harmonic Analysis*, which is more than just the combination of *Abstract Harmonic Analysis* and *Numerical* or *Computational Harmonic Analysis*.

The role of function spaces

So what is the motivation to use function spaces (and what are function spaces). As a matter of convenience (and *personal conviction!*) I am restricting my attention to (families of) Banach spaces of (generalized) functions, because many of the interesting topological vector spaces (e.g. those used for the definition of ultra-distributions) can be based on the intersection of families of Banach spaces and their topology can be obtained by the family of (semi)norms which arise from these individual Banach spaces. You may take the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions (with the usual family of seminorms) and you will find that this space can also be viewed as the intersection of (Fourier invariant) modulation spaces, the (Fourier invariant) so-called Shubin classes $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$. The Fourier invariance of $\mathcal{S}(\mathbb{R}^d)$ thus follows easily, but on the other hand the invariance with respect to differentiation is less obvious.

The use of function spaces

Many of the “applications” of function spaces are in the description of operators and their mapping properties. Typically the scale of *Sobolev spaces* is suitable for the description of the mapping properties of the Laplace operator $f \mapsto f''$ (first) and extending it to $\mathbb{R}^d$.

As Yves Meyer once put it (in a conversation with the author): Function spaces are only good for the description of operators, not in order to study them by themselves!

At that time he had shown how wavelet expansions are a good way to understand the mapping properties of Calderon-Zygmund operators on the classical function spaces, namely the Besov-Triebel-Lizorkin spaces, which include the Sobolev spaces.

I just had *modulation spaces*, and still very few results showing that they are useful. This has changed meanwhile.

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Global Orientation I

This talk is (another) PERSPECTIVE talk of mine, trying to contribute to a timely interpretation of mathematical tasks related to Fourier Analysis in the modern world.

Classical (or later Abstract) Harmonic Analysis have been dealing with purely mathematical questions, such as Fourier Analysis over LCA (locally compact Abelian) groups which provides an good, qualitative framework.
One of the key-person (with his book L'integration dans les Groupes Topologiques et ses Applications. Hermann and Cie, (1940), Paris) was **Andre Weil**.
Unfortunately Lebesgue integration, or almost everywhere convergence of Fourier series (Carleson, 1972) so not play a role for engineering applications.

AWEIL31

Abbildung: Andre Weil, Aligarh Muslim University, 1931-1932

Lack of Connection

Very unfortunately the classical tools as such are insufficient in order to deal with the problems that have to be addressed in the world of applications. Of course they form crucial building blocks for an introduction to a modern view on harmonic analysis, e.g. for *wavelet theory* of *time frequency analysis* and *Gabor Analysis*, for non-periodic and not decaying signals, like a piece of music.

I think it is in the very spirit of this conference to indicate that modern mathematical concepts are needed and provide important opportunities for relevant research work.

In my paper `Ingredients for Applied Fourier Analysis` published in the Proceedings of the Sharda Conference of Feb. 2018 , published with Taylor and Francis in 2020 p.1-22, I outline an alternative approach to Fourier Analysis, which does NOT require to first learn about Lebesgue integration or topological vector spaces leading to *tempered distributions*.

Modern Applications

There is a large variety of real-world applications of Fourier Analysis, and in fact the FFT (the Fast Fourier transform, implementing the DFT in an efficient way) is one of the backbones of the modern digital world.

In everyday life we make (mostly unconsciously) use of the FFT:

- making phone calls, exchanging messages;
- streaming music (MP3) or movies;
- taking pictures, face recognition;
- editing and filtering images;
- Scanners and MR-imaging in medicine;
- bar-codes and QR-codes, communication,...
- online conferences such as this one!

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

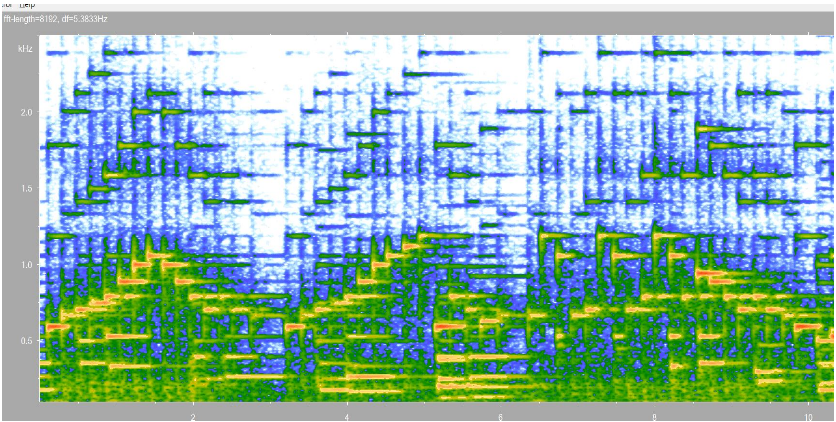
$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

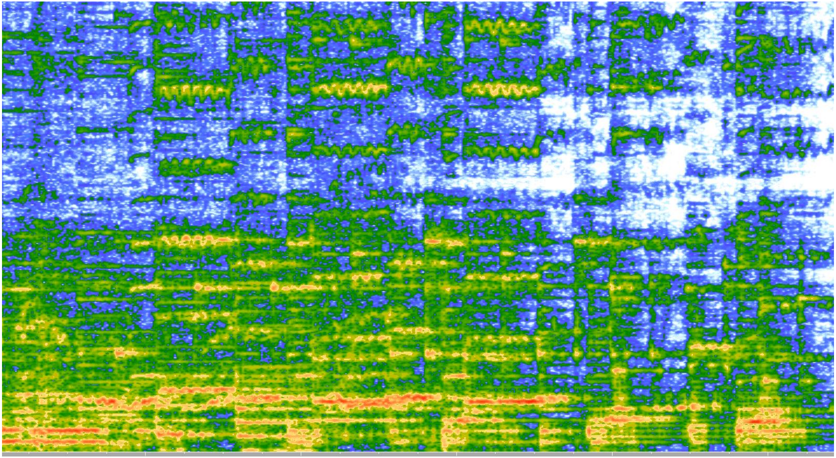
$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \; \lambda = (t, \omega);$$

A Typical Musical STFT

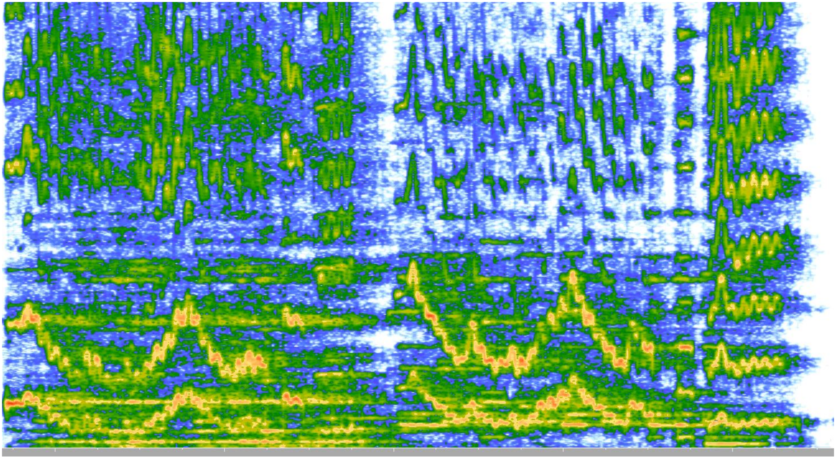
A typical piano spectrogram (Mozart), from digital recording (realized with the help of the STX software, free download)



A Musical STFT: Brahms, Cello

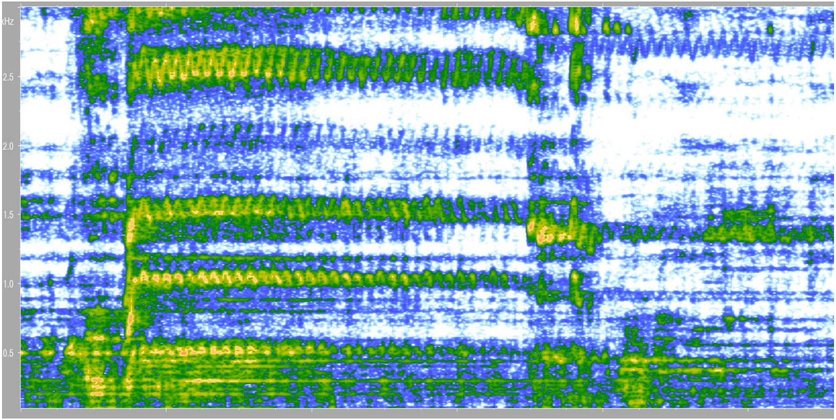


A Musical STFT: Maria Callas



A Musical STFT: Tenor: VINCERO!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{\mathbf{S}_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

cSo

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}, \quad \forall f \in \boldsymbol{S}_0$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Various Function Spaces

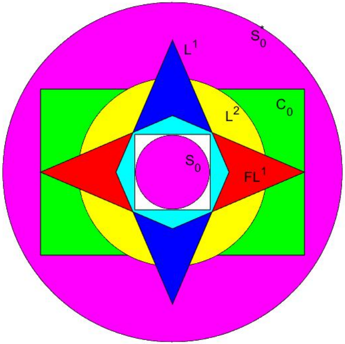


Abbildung: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

Consequences and benefits

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Varying the time-frequency lattice of Gabor frames.
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Comparison with wavelets

If we quickly compare the situation that we have for wavelets and for time-frequency or Gabor analysis, we can say:

1. For wavelets the CWT (continuous wavelet transform) is defined on the affine group ("$ax + b$"-group), which is non-unimodular;
2. For the STFT one should consider the extension of the *projective representation* $\pi(z) = M_s T_t$ (TF-shift) to the reduced Heisenberg group $\pi(t, s, \tau) = \tau M_s T_t$, $t, s \in \mathbb{R}^d$, which is unimodular and nilpotent. Usually this extended representation is called the *Schrödinger representation*;
3. For the "ax+b" case one can discretize and can find ONBs for $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ with compact support prescribed smoothness;
4. For the STFT case (in contradiction to the expectation of D. Gabor from 1946) on *cannot have* an ONB, even with a modest amount of smoothness (say in $\boldsymbol{S}_0(\mathbb{R}^d)$).

What are the key questions

Let us just summarize the key questions of Gabor analysis:

1. Under which conditions can one expand "every function" (or signal, or distribution) as an (infinite double) sum of time-frequency shifted Gabor atoms (despite the fact that such families cannot be an orthonormal basis for $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$): Gabor expansion of functions
2. Under which conditions (on the signal or on the Gabor window) can one have a *stable reconstruction* of the signal from the sampled STFT (the *regular* case covers sampling along a *lattice* $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ in *phase space*): Signal reconstructions from sampled STFT.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

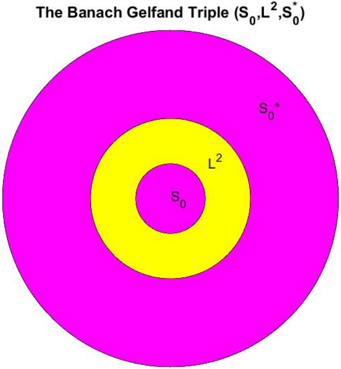
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



Banach Gelfand Triple Morphism

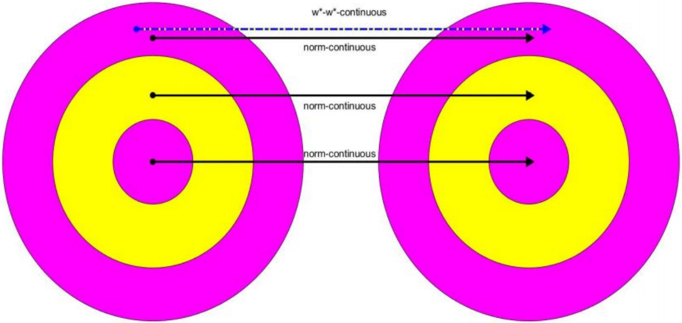


Abbildung: Operators preserving three norms as well as w^*-w^*-convergence.

Regularizing Operators

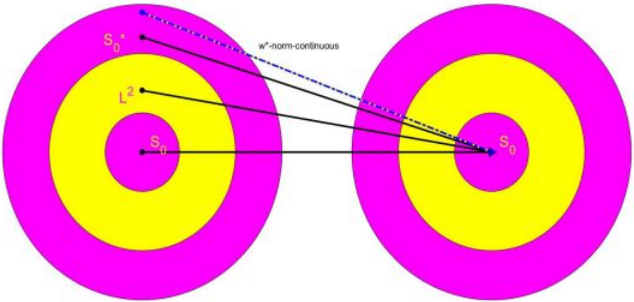


Abbildung: Regularizing operators map $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, in a w^*-to-norm continuous fashion. They are exactly operators with operator kernel (or Kohn-Nirenberg symbol) in $\boldsymbol{S}_0(\mathbb{R}^{2d})$.

The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \qquad (1)$$

par

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

What are the positive facts I

Compared to *wavelet theory* where the "mother wavelet" has to satisfy a certain *admissibility* condition, it looks as if the situation was much better for STFT: In fact, one has for any *Gabor atom* $g \in \mathbf{L}^2(\mathbb{R}^d)$ with $\|g\|_{\mathbf{L}^2} = 1$ the isometric property for V_g, i.e.

$$\|V_g f\|_{\mathbf{L}^2(\mathbb{R}^{2d})} = \|f\|_{\mathbf{L}^2(\mathbb{R}^d)}, \quad f \in \mathbf{S}_0. \tag{2}$$

Vgs

Consequently we have $V_g^* \circ V_g = \mathrm{Id}_{\mathbf{L}^2(\mathbb{R}^d)}$, i.e. the adjoint operator is the inverse to $V_g : f \mapsto V_g f$ on the range of V_g.
But one can even show that V_g maps $\mathbf{S}_0(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^{2d})$ and therefore V_g^* extends to a bounded linear mapping from $\mathbf{S}_0'(\mathbb{R}^d)$ into $\mathbf{S}_0'(\mathbb{R}^d)$, which is w^*-w^*-continuous. Since $V_g^*(\delta_z) = \pi(z)g$, one can even derive $V_g^*(\mathbf{M}_b(\mathbb{R}^{2d}))$ *subset* $\mathbf{S}_0(\mathbb{R}^d)$.

What are the positive facts II

Theorem

For normalized Gabor atoms the STFT $f \mapsto V_g f$ is an isometric embedding of $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $\left(\mathbf{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$. This implies the continuous reconstruction formula (using the adjoint operator):

$$f = V_g^*(V_g f) = \int_{\mathbb{R}^{2d}} V_g f(\lambda)\pi(\lambda)g,$$

to be understood in the weak sense!
But it is also a non-expanding mapping from $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $\left(\mathbf{C}_0(\mathbb{R}^{2d}), \|\cdot\|_\infty\right)$. Hence the range space $V_g(\mathbf{L}^2(\mathbb{R}^d))$ is a reproducing kernel Hilbert space.

Mild distributions I

Since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ as a *dense subspace* the dual space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ can be viewed as a subspace of tempered distributions. We call the elements of $\mathbf{S}_0'(\mathbb{R}^d)$ (the same as the modulation space $(\mathbf{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^\infty})$) the space of *mild distributions*. The STFT with respect to a Gaussian window is still defined via $V_g(\sigma)(\lambda) = \sigma(\pi(z)g)$, and we have:

Mild distributions II

PRd

Lemma

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ *belongs to* $\mathbf{S}_0'(\mathbb{R}^d)$ *if and only if it has a spectrogram* $V_g(\sigma)$ *in* $(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$ *(resp.* $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$*).*
Norm convergence in $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ *corresponds to uniform convergence, and a sequence* (σ_n) *is* w^**convergent to* σ_0 *if and only if*

$$V_g(\sigma_n)(z) \to V_g(\sigma_0)(z), \quad \text{uniformly over compact subsets of } \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Obviously $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ *is invariant under the extended Fourier transform, defined via* $\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in \mathbf{S}_0$*.*

Question: But if we restrict the STFT to some lattice, say $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \lhd \mathbb{R} \times \widehat{\mathbb{R}}$, can we assume that the (analysis or sampling) mapping

$$f \mapsto V_g f|_\Lambda = (V_g f(\lambda))_{\lambda \in \Lambda} \tag{3}$$

Vgf

which obviously maps $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $(c_o(\Lambda), \|\cdot\|_\infty)$, is also a bounded mapping into $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$? The answer is simply negative.

NOT every $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ generates a Bessel family of the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$.

Equivalently, due to an adjointness relation, the *synthesis mapping*

$$(c_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g \tag{4}$$

gab

is *not bounded* for an arbitrary $g \in \boldsymbol{L}^2(\mathbb{R}^d)$.

There are two ways out:

- Either one restricts that Gabor atom, i.e. one assumes that both the analysis and synthesis window belong to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Then both the analysis mapping $f \mapsto (V_g f(\lambda))_{\lambda\in\Lambda}$ and the synthesis mapping (4) are bounded mappings between the Hilbert spaces $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$. In other words, the family $(g_\lambda)_{\lambda\in\Lambda}$ is Bessel family (for any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$!);
  Furthermore, the combined mapping:

$$f \mapsto S_{g,\Lambda} := \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda \qquad (5)$$

gab

is a bounded operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$.

- Alternatively, one can restrict the domain of the analysis mapping to $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ and thus observe that the analysis mapping is bounded from $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ to $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$. On the other hand (by another adjointness argument) the synthesis mapping maps $\ell^2(\Lambda)$ back into $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ (in a bounded way). Thus one can still describe the Gabor (pre)frame operator $S_{g,\Lambda}$ as a bounded linear operator from $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ to $\left(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}\right)$.

Either way it is not difficult to verify that $S_{g,\Lambda}$ commutes with any $\pi(\lambda), \lambda \in \Lambda$.

Such considerations imply also that the mapping from $g \to S_{g,\Lambda} \in \mathcal{L}(\mathbf{L}^2(\mathbb{R}^d), \mathbf{L}^2(\mathbb{R}^d))$ is continuous from $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ into the space of bounded linear operators on $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. In particular (due to the continuity of inversion of operators in this Banach algebra of operators) similar windows g in $\mathbf{S}_0(\mathbb{R}^d)$ have similar dual windows (for fixed Λ).

The dependence of $S_{g,\Lambda}$ on the lattice (parameters) is *not continuous* in the operator norm sense, but in the sense of the strong operator topology, i.e.

$$\Lambda_n \to \Lambda_0 \quad \Rightarrow \quad S_{g,\Lambda_n}(f) \to S_{g,\Lambda_0}(f), \forall f \in \boldsymbol{S}_0.$$

Hence it is not obvious (but a valid) statement that one has according to a joint paper with N. Kaiblinger

$$\widetilde{g}_n = S^{-1}_{g,\Lambda_n}(g) \quad \to \quad S^{-1}_{g,\Lambda_0}(g) = \widetilde{g}_0.$$

see: H. G. Feichtinger and N. Kaiblinger.

Varying the time-frequency lattice of Gabor frames
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.

Articles concerning $S_0(\mathbb{R}^d)$

1-2 H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

a18 M. S. Jakobsen
On a (no longer) New Segal Algebra: a review of the Feichtinger algebra.
J. Fourier Anal. Appl., 24(6):1579–1660, 2018.

a20 H. G. Feichtinger and M. S. Jakobsen.
Distribution theory by Riemann integrals.
Mathematical Modelling, Optimization, Analytic and Numerical Solutions, pages 33–76, 2020.

a18 H. G. Feichtinger and M. S. Jakobsen.
The inner kernel theorem for a certain Segal algebra.
2018.

r01 K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

Poisson's Formula

One of the key results in Fourier Analysis is Poisson's formula, usually presented for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$, but

onR

Theorem

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{n\in\mathbb{Z}} \widehat{f}(n), \quad \forall f \in S_0 \tag{6}$$

Poi

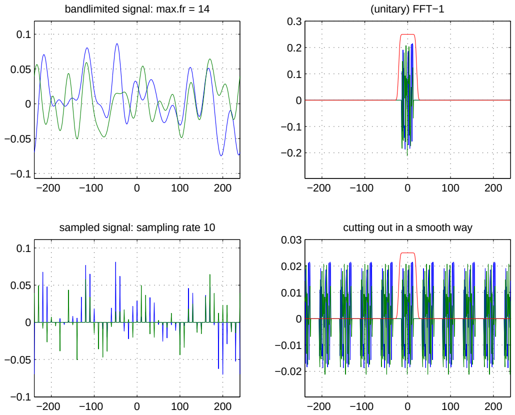
Using the standard conventions to write $⧢ = \sum_{k\in\mathbb{Z}^d} \delta_k$ and the definition of the extended Fourier transform on $S_0'(\mathbb{R}^d)$, via $\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in S_0$, this is equivalent to the statement

$$\mathcal{F}(⧢) = ⧢. \tag{7}$$

FTD

Using dilations one obtains corresponding formulas for Dirac combs over general lattices $\Lambda \lhd \mathbb{R}^d$.

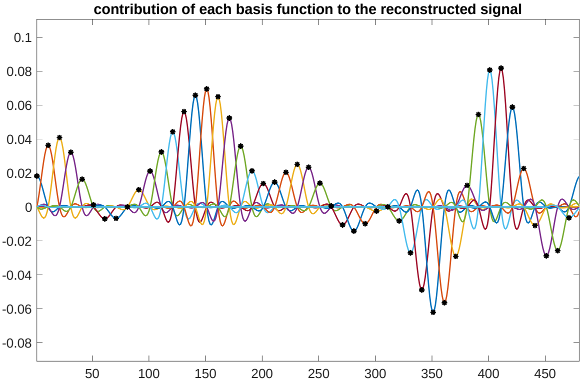
ShannonLinAlg1



shanndem4.pdf

Abbildung: If there is a bit of oversampling, i.e. the s Dirac comb used for sampling is a bit more fine than the minimal requirement (Nyquist criterion) then one has more freedom in the choice of $\widehat{g}$.

Shannlocal1



ondem2C.pdf

Abbildung: shannondem2C.pdf:
Showing the individual contributions of the well-localized version for the reconstruction of the **real part** of the signal.

Shannon Theorem: L^1-version

S01

Theorem

Given a compact set $\Omega \subset \widehat{\mathbb{R}^d}$, and some lattice $\Lambda \lhd \mathbb{R}^d$ with the property that the $\Lambda^\perp$-translates of Ω are pairwise disjoint, then one can recover any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\mathrm{supp}(\widehat{f}) \subset \Omega$ (thus in fact $f \in \boldsymbol{S}_0(\mathbb{R}^d)$) from the Λ-samples of f by the series expansion

$$f(t) = \sum_{\lambda \in \Lambda} f(\lambda) g(t - \lambda), \tag{8}$$

Sha

with convergence in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, in particular absolutely in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, hence absolutely pointwise for every $t \in \mathbb{R}^d$ and uniformly, for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\widehat{g}(\omega) \equiv 1$ for $\omega \in \Omega$ and $\mathrm{supp}(\widehat{g}) \cap \lambda^\perp + \Omega$ for any $\lambda^\perp \in \Lambda^\perp$, $\lambda^\perp \neq 0$.

Shann2dim

The general situation is described by the following picture:

periodOmega2a.pdf

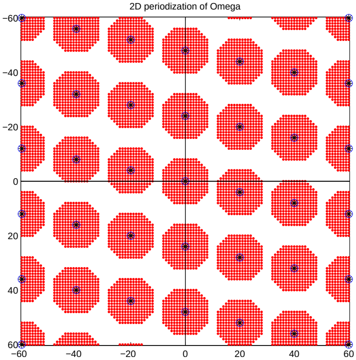


Abbildung: periodOmega2a.pdf: disjoint subsets of the fundamental domain along some lattice Λ.

The DFT/FFT as a special case

The extended Fourier transform $S_0'(\mathbb{R}^d) \to S_0'(\mathbb{R}^d)$ can also be applied to *periodic and discrete* signals (engineering terminology), which coincides with the DFT/FFT:

racCombs2.pdf

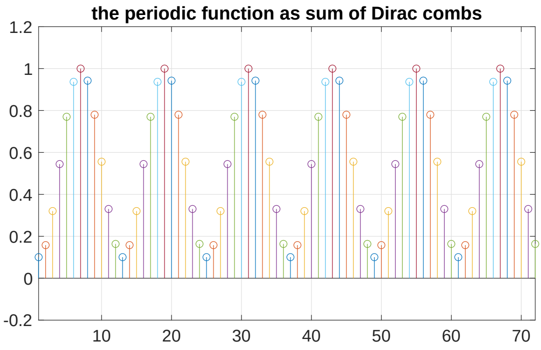


Abbildung: A periodic, discrete measure as a sum of Dirac combs.

Regularization of mild distributions

Note that $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)(\mathbb{R}^d)$ is the space of pointwise multipliers of $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, and hence the corollary implies

$$(\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d), \quad (9)$$

SOP

and the same relationship, on the Fourier transform side reads:

$$(\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d). \quad (10)$$

SOP

In this way one can show that $\boldsymbol{S}_0(\mathbb{R}^d)$ is w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$, which means, that the spectrogram of $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ can be pointwise (uniformly over compact sets) be approximated by corresponding spectrograms of approximating test functions.

Approximation by discrete, periodic measures

The following theorem implies that a function $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be approximately recovered from regular samples:

ORd

Theorem

Assume that $\Psi = (T_k\psi)_{k\in\mathbb{Z}^d}$ *defines a BUPU in* $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ *and write* $\mathrm{D}_\rho\Psi$ *for the family* $\mathrm{D}_\rho(T_k\psi) = (T_{\alpha k}\mathrm{D}_\rho\Delta)_{k\in\mathbb{Z}^d}$, *with* $\alpha = 1/\rho \to 0$. *Then* $|\mathrm{D}_\rho\Psi| \leq r\alpha \to 0$ *for* $\alpha \to 0$, *and*

$$\|f - \alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\,\mathrm{St}_\alpha\,\psi\|_{\boldsymbol{S}_0} \to 0, \qquad \text{for } \alpha \to 0, \forall f \in \boldsymbol{S}_0. \tag{11}$$

SOD

discrete approximation of FT I

rFT

Lemma

Given $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *we have* $\nu := f \cdot ⊔⊔ = \sum_{k\in\mathbb{Z}} h(k)\delta_k \in \boldsymbol{M}_d(\mathbb{R})$. *It has a Fourier transform which is a* $\mathbb{Z}$*-periodic function, as it can be described as* $⊔⊔ * \widehat{f}$:

$$\mathcal{F}(f \cdot ⊔⊔) = \widehat{f} * ⊔⊔, \quad f \in \boldsymbol{S}_0. \tag{12}$$

sha

In a similar way the periodized version of f is a periodic function whose Fourier coefficients are just the samples of $\widehat{f}$:

$$\mathcal{F}(f * ⊔⊔) = \widehat{f} \cdot ⊔⊔, \quad f \in \boldsymbol{S}_0. \tag{13}$$

sha

For positive values a, b, with $a = N \cdot b$ and $c = 1/b, d = 1/a$ we have, up to the constant $C = C_{a,b,c,d}$, for every $f \in \boldsymbol{S}_0$:

$$\mathcal{F}\left[⊔⊔_a * (⊔⊔_b \cdot f)\right] = C \cdot \left[⊔⊔_d \cdot (⊔⊔_c * f)\right] = C \cdot \left[⊔⊔_c * (⊔⊔_d \cdot \widehat{f})\right]. \tag{14}$$

sam

But for sufficiently large values of a and sufficiently small $b = a/N$ one can recover f approximately from these versions. In this way one can demonstrate that $\widehat{f}$ can be approximately computed via FFTs.r

Kernel Theorems I

The so-called Kernel Theorem for $\boldsymbol{S}_0$-spaces allows to establish a number of further unitary BGTr-isomorphism. It involves certain types of operators, their (integral) kernels, but also their representation as *pseudo-differential* operators, via the Weyl or Kohn-Nirenberg symbol, or (important for applications in mobile communication) their spreading distribution.
These situations allow to make use of the general principles. In order to understand the transformation one can start from the core spaces, where the analogy with the finite dimensional case is valid in a very natural sense, e.g. if one has $K(x,y) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, then red $K(x,y) = T(\delta_y)(x)$, in analogy to the matrices

$$a_{n,k} = [T(\mathbf{e}_k)]_n.$$

The Hilbert space case of the well-known characterization

Kernel Theorems II

hm1

Theorem

There is a unitary BGTr isomorphism between $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ *and* $(\mathcal{N}_{w*}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))(\mathbb{R}^d)$ *which is a unitary mapping between* $(\mathbf{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2)$ *and* $(\mathcal{HS}, \|\cdot\|_{\mathcal{HS}})$*, with*

$$\langle T_1, T_2\rangle_{\mathcal{HS}} = \mathrm{trace}(T_1 \circ T_2^*).$$

Alternative BGTr descriptions use the *Kohn-Nirenberg* symbol or the spreading representation : $T \in \mathcal{N}_{w*}(\mathbf{S}_0', \mathbf{S}_0)$ iff

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} H(\lambda)\pi(\lambda)d\lambda, \quad H \in \mathbf{S}_0(\mathbb{R}^{2d}).$$

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
Gabor Book 1998.

Wilson Bases

For the case $G = \mathbb{R}^d$ one can derive the kernel theorem also from the description of operators mapping ℓ^1 to ℓ^∞ or vice versa (in a $w^*-$to-norm continuous way).
The key is the fact, that local Fourier basis, but in particular the so-called Wilson bases are suitable for modulation spaces. In our situation we can formulate the following:

GTr

Theorem

Any Wilson ONB (obtained by a smart pairwise recombination of the elements of a tight Gabor frame of redundancy 2) establishes a unitary BGTr between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ *and* $(\ell^1, \ell^2, \ell^\infty)$.

H. G. Feichtinger, K. Gröchenig, and D. F. Walnut.
Wilson bases and modulation spaces.
Math. Nachr., 155:7–17, 1992.

Multiplier problems I

In engineering the consideration of TLIS (Translation invariant linear systems) is of great importance. Using manipulations involving the Dirac measure it is "derived" that any such operator T is of the form $T(f) = \sigma * f$, with $\sigma = T(\delta_0)$, the so-called impulse response, respectively $\mathcal{F}(T(f)) = \widehat{\sigma} \cdot \widehat{f}$, where $\widehat{\sigma}$ is called the transfer function.

But except for case of BIBOS systems (bounded linear operators on $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$) one may require more than just bounded measures in order to obtain such a representation. For the case of bounded linear mappings on $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ it suffices to work with pseudo-measures $\sigma \in \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$.

Multiplier problems II

For multipliers from $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ to $(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q)$ the so-called *quasi-measures* are required, which turn out to be continuous linear functionals $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_c(\mathbb{R}^d)$. But this space is too big to allow a Fourier transform. Moreover it is not clear whether δ_0 is in the domain of the operator.
The fact that all the interesting spaces, e.g. $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ satisfy

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$$

allows to make use of the following theorem:

Multiplier problems III

ult

Theorem

Given any translation invariant operator $T : (\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}) \to (\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ *one can show first that* $T(\mathbf{S}_0(\mathbb{R}^d)) \subset \mathbf{C}_b(\mathbb{R}^d)$*, and from this it follows:*
Any such operator is of the form

$$[T(f)](y) = \sigma * f(y) = \sigma(T_y(f^{\checkmark})), \quad y \in \mathbb{R}^d, f \in \mathbf{S}_0.$$

Moreover, the operator norm of the convolution operator is equivalent to the $\mathbf{S}_0'$*-norm of the convolution kernel* σ*. The equivalent description via the transfer function reads:*

$$\widehat{Tf} = \widehat{\sigma} \cdot \widehat{f}, \quad f \in \mathbf{S}_0.$$

Thank you and further links

Thank for your attention!

HAPPY BIRTHDAY Lars-Erik!!

A number of talks by the speaker are found at

www.nuhag.eu/feitalks

Details on the course: see www.nuhag.eu/ETH20
Direct question to: hans.feichtinger@univie.ac.at