NuHAG & ARI/OEAW

Wigner, Gabor and Time-Frequency Analysis

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Key aspects of my talk I

- Some personal memories
- Some historical comments
- Mutual inspiration (examples)
- Current status
- Outlook 2024

Starting with my personal background

- My education at the University of Vienna started as a *teacher student* in Mathematics and Physics;
- "Imprecision" in physics and a BOURBAKI-style introduction to analysis pulled me towards (pure) mathematics;
- In my third year (1971) H. Reiter became professor in Vienna, so I became a "Harmonic Analyst";
- After my habilitation (1979) I learned about the connection between Fourier Analysis and signal processing in Heidelberg;
- Since that time I tried to look out for *real world applications* and call myself an *application oriented* mathematician;
- For several years I try to promote what I call CONCEPTUAL HARMONIC ANALYSIS.

A personal background story

At some point in the early 80th I tried to connect with the applied people at TU Vienna, and (fortunately) I ended up contacting Franz Hlawatsch (Communication Theory Dept., TU Vienna). *During our first meeting* he explained to me, that he was working on the so-called *Pseudo-Wigner distribution*, which is a kind of smoothed version of the Wigner distribution, with the idea of reducing the so-called interference terms in a Wigner distribution. I told him that I was studying a certain function space (I called it $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, because it was a special *Segal algebra*), given by:

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{f \mid f = \sum_{n\geq 1} c_n M_{\omega_n} T_{x_n} g_0, \text{ with } \sum_{n\geq 1} |c_n| < \infty\}, \qquad (1)$$

where $g_0(x) = exp(-\pi|x|^2)$ is the usual Gauss function.

The relevance for my future work

At our first meeting I told Franz about my space $S_0(\mathbb{R})$ and its atomic representation. He then asked **two important questions**:

1. Do you really need all TF-shifts??
   D. Gabor has suggested to use only $x_n, \omega_n \in \mathbb{Z}^d$!
2. and: How do you compute the coefficients?

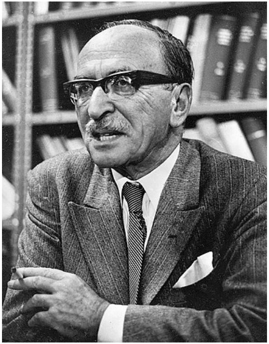
According to the claim made in D. Gabor's paper of 1946 one *should expect* that an optimally concentrated representation of any function should be possible, using integer TF-shifts of the Gauss function (achieving equality for *Heisenberg's Uncertainty*!).

The studies of Franz Hlawatsch were based on the work of W. Mecklenbräuker who had brought the Wigner distribution to signal processing. My scientific background were the publications of my advisor Hans Reiter (Classical Harmonic Analysis and Locally Compact Groups/LN on Segal algebras).

Eugene Wigner

Eugene Paul Wigner (November 17, 1902 to January 1, 1995), was a Hungarian-American theoretical physicist, engineer, and mathematician. Nobel Prize in Physics in 1963 *for his contributions to the theory of the atomic nucleus and the elementary particles and symmetry principles.*

Dennis Gabor



Dennis Gabor (5 June 1900 to 9 February 1979) was a Hungarian-British electrical engineer and physicist, most notable for inventing holography, for which he later received the 1971 Nobel Prize in Physics.

Feichtinger's Algebra $S_0(\mathbb{R}^d)$ via the Wigner transform

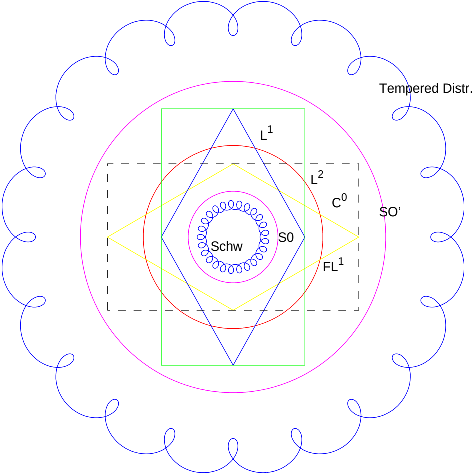
$$S_0(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^{2d}} \|M_\omega f * f\|_1 \, d\omega < \infty \right\}. \tag{2}$$

Condition (2) is equivalent to the integrability of the Wigner functions over phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
Here $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$, $t \in \mathbb{R}^d, \omega \in \mathbb{R}^d$, is the modulation operator and $*$ is the usual convolution of $L^1(\mathbb{R}^d)$-functions. Any non-zero function $g \in S_0(\mathbb{R}^d)$ defines a norm on $S_0(\mathbb{R}^d)$ via

$$\|f\|_{S_0,g} = \int_{\widehat{g}} \|M_\omega g * f\|_1 \, d\omega, \tag{3}$$

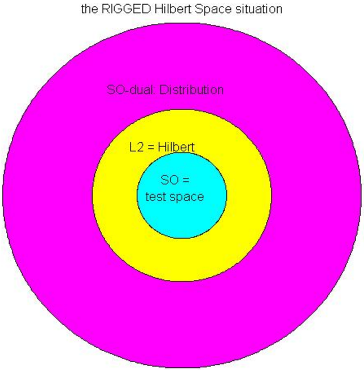
that turns $S_0(\mathbb{R}^d)$ into a Banach space. These norms are pairwise equivalent and we therefore allow ourselves to simply write $\|\cdot\|_{S_0}$ without specifying g. (e.g., $g =$ Gaussian).

A schematic description: all the spaces



Banach Gelfand Triples: the simplified setting

Test-functions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

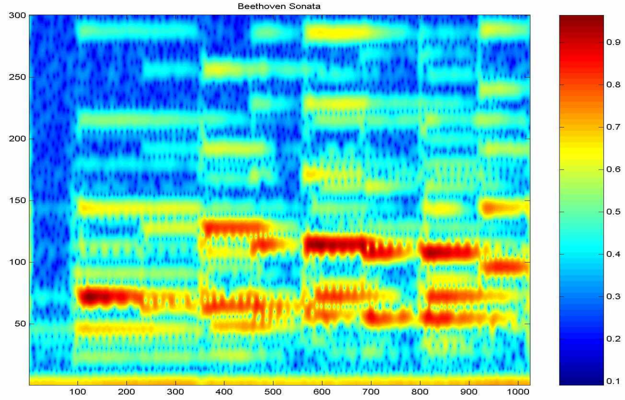
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle,\ \ \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

Various Function Spaces

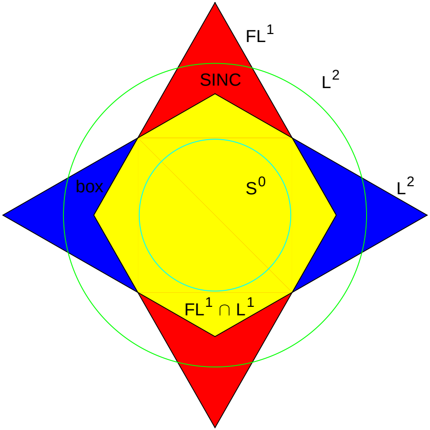


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

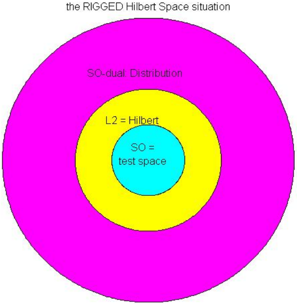
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{4}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Diving into Gabor's World

Once I got aware of the work of D. Gabor it was natural to check the literature (I was a librarian at that time) for mathematical work in this direction. This was really a game-changer!
I got into contact with A.J.E.M. Janssen and M. Bastiaans in Eijndhoven, who had done work in this area. E.g. Janssen has shown that distributional convergence does not save the convergence problem for the critical lattice case (the Neumann lattice, with Gaussian window).

A. J. E. M. Janssen.
Gabor representation of generalized functions.
J. Math. Anal. Appl., 83:377–394, October 1981.

Opening up for Applications I

Already during my first PostDoc semester (with M. Leinert, 1980) in Heidelberg I developed the foundations of a theory of *Wiener amalgam spaces* $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$ with general local and global behaviour. They can be characterized using BUPUs, or by equivalent norms using a continuous *control function*, a *sliding window*. This approach is more elegant, but technically more demanding. In this period I also learned about the STFT (Sliding Window Fourier Transform) through contacts with Werner Deutsch from OEAW. I developed a theory of function spaces involving the STFT, now known as *modulation spaces*, giving in the (long unpublished technical report of 1983) the continuous description. This manuscript was rejected, and finally published in 2003, even after their appearance in Charly's book on TF-analysis (from 2001): Foundations of Time-Frequency Analysis.

Opening up for Applications II

The period from 1985 to 1992 was a period of intensive work with Karlheinz Gröchenig. I would say that a summer school of 1985 in Germany, organized by D. Poguntke, and with R. Howe an E. Stein as key scientists, opened our eyes for the connections between Gabor Analysis and the Schrödinger representation of the Heisenberg group.
Wavelet Theory appeared around 1985/86, and Yves Meyer was surprised that there was no Balian-Low obstacle in the case of the continuous wavelet transform. In other words, he found an ONB of wavelets, with band-limited atoms of good decay.
Our result was the creation of *Coorbit Theory*, resulting in the publication of a series of joint papers, also developing the concept of *Banach frames* in "Describing Functions.." (K.Gr.).

Yves Meyer - Construction of Orthonormal Wavelets

Y. Meyer.
De la recherche petrolière la géometrie des espaces de Banach en passant par les paraproduite. (From petroleum research to Banach space geometry by way of paraproducts).
In *Sém. sur les équations aux dérivées partielles, 1985 - 1986, Exp. No. I*, page 11. Ecole Polytechnique, 1986.

I. Daubechies, A. Grossmann, and Y. Meyer.
Painless nonorthogonal expansions.
J. Math. Phys., 27(5):1271–1283, May 1986.

P. G. Lemarié and Y. Meyer.
Ondelettes et bases hilbertiennes.
Rev. Mat. Iberoam., 2:1–18, 1986. submitted Dec. 1985!

The Age of Coorbit Theory: 1986-1991 I

The analogy between the properties of the continuous wavelet transform, the construction of orthonormal wavelet bases, the Frazier-Jawerth atomic characterization of Besov spaces and the corresponding properties of modulation spaces was calling for the investigations of the common ground. Since there are no orthonormal Gabor basis (due to Balian-Low) with smooth and well concentrated atoms one has to deal frames, in fact with Banach frames, where the properties of the function (distribution) under consideration is reflected by the decay and summability properties of the corresponding collection of (canonical or other) coefficients in an appropriate "non-orthogonal, coherent atomic representation". There was a series of joint papers, mostly 1988-1991 on this topic (our most cited papers).

Consequences of Coorbit Theory for Gabor Analysis I

Let us just not that this abstract approach opens the path to the discussion of *irregular Gabor frames* for atoms of "of good quality" (meaning $g \in \boldsymbol{S}_0(\mathbb{R}^d)$).
The fact that dual Gabor frames (regular, i.e. generated by lattice Λ in phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, or irregular) have also good, meaning well concentrated in the TF-sense dual frames allows to characterize the properties of a function or distribution by looking at the corresponding Gabor coefficients. Despite their non-uniqueness the "canonical choice" (based on a MNLSQ-idea, i.e. referring to the pseudo-inverse of the analysis operator) any such system, once fixed, serves this purpose.
Since in such a situation (unlike Gabor's original setting) both the determination of coefficients and the synthesis operator are well localized in the TF-sense one can expect that *Gabor multipliers* show the expected behaviour (studies around 2003).

Computational Gabor Analysis

In the mid 90th we began to do "Computational Gabor Analysis". It was mostly my PhD student Sigang QIU (Phd 2000) who studied discrete Gabor Analysis, starting around 1995. Also at that time Helmut Bölcskei (PhD 1997) attended our seminar regularly and I was pushing for publication in IEEE, SPIE, and Opt. Eng. journals and conferences. The "NuHAG spirit" was created. Around 1996/97 (my daughter was born) T. Strohmer suggested to do a Gabor book. At that time my team had very three strong members, Werner Kozek, Thomas Strohmer and Georg Zimmermann. Also Peter Prinz was working on fast algorithms and Norbert Kaiblinger studied the metaplectic group (master thesis). T. Strohmer suggest to put together a *GABOR BOOK*.

Preparing the First Gabor Book I

H. G. Feichtinger and T. Strohmer.
Gabor Analysis and Algorithms. Theory and Applications.
Birkhäuser, Boston, 1998.

H. G. Feichtinger and T. Strohmer.
Advances in Gabor Analysis. Birkhäuser, Basel, 2003.

These two books and the original contributions (unfortunately not published in the journal literature) are still the basis for current work on Fourier and TF-Analysis, as it contains the basic observations concerning what is now called the *Banach Gelfand Triple*, consisting of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, the space of *"mild distributions"*.

Most recently I have come to the conviction that

Signals ARE Mild Distributions!

Operators ARE Integral Operators using Mild Distributions!

The Kernel Theorem I

We all know that linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be describe by matrix multiplication with some $m \times n$-matrix $\boldsymbol{A}$.
$\boldsymbol{A} * \mathbf{x}$ gives the linear combination $\sum_{k=1}^{n} x_k \mathbf{a}_k$.
The corresponding "kernel theorem" is known in two versions. One is the characterization of (compact) $\mathcal{HS}$-operators on a separable Hilbert space $\mathcal{H}$ via infinite matrices with ℓ^2-entries. For the case $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$ one can thus have a kernel $K(x,y) \in \boldsymbol{L}^2(\mathbb{R}^{2d})$.

The theory of L. Schwartz allows to characterize continuous linear operators T from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ to $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ via (unique) "kernels" $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^{2d})$, by writing $f \otimes g(x,y) = f(x)g(y)$:

$$Tf(g) = \sigma(f \otimes g), \quad f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

The Kernel Theorem II

Recall that $\mathcal{S}(\mathbb{R}^d)$ is a so-called "nuclear Frechet-space", already indicating, that this property (not shared by any infinite dimensional Banach space, such as $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}))$ has something to do with a kernel-theorem (kernel = nucleus). Nevertheless one has

Theorem

There is an isomorphism between $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$, the Banach space of bounded linear operators from $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ to $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$, with mild distributions $\sigma \in \mathbf{S}_0'(\mathbb{R}^{2d})$ via

$$Tf(g) = \sigma(f \otimes g), \quad f, g \in \mathbf{S}_0(\mathbb{R}^d).$$

Operators with $\sigma \in \mathbf{S}_0(\mathbb{R}^{2d}) \subset \mathbf{S}_0'(\mathbb{R}^{2d})$ have particularly good properties ("inner kernel theorem"). $K(x, y) = T(\delta_y)(x)$!

Consequences for Gabor Analysis I

The relevant articles in our book [fest98] are:

[fezi98] H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In [fest98], pages 123–170. Birkhäuser Boston, 1998.

[feko98] H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In In [fest98], pages 233–266. Birkhäuser, Boston, MA, 1998.

Using the results there, one has:
for any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and any $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ one has:

$$f \mapsto S_{g,\Lambda}(f) := \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d)$$

defines a bounded linear operator from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, commuting with TF-shifts from Λ.

Consequences for Gabor Analysis II

One of the basic facts of regular Gabor Analysis are the observation that the dual frame is generated by the *dual Gabor atom* $\widetilde{g} := S_{g,\Lambda}^{-1}(g)$, thus creating (at least for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$) the representations

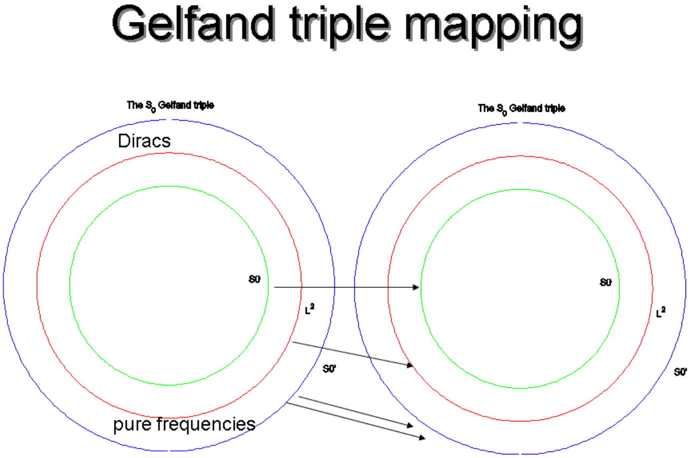
$$f = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{5}$$

By a non-trivial results due to Gröchenig/Leinert $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ implies $\widetilde{g} = S_{g,\Lambda}^{-1} g \in \boldsymbol{S}_0(\mathbb{R}^d)$ in this case, and consequently the coefficients for f in (5) are in $\ell^2(\Lambda)$ for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, and in $\ell^1(\Lambda)$ for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ (by [fezi98]), and so on.

Consequences for Gabor Analysis III

By a "change of basis" one can describe operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ also by a "time-variant transfer function" (as I learned from W. Kozek) $\kappa(T)$, known as KNS, or *Kohn-Nirenberg symbol of T*, or by (its *symplectic Fourier transform*) the spreading function $\eta(T)$. These identifications create alternative description of operators. One can characterize the spreading identification as the BGT-isomorphism (automorphism over $\mathbb{R}^{2d}$) determined by the fact that one has for $\lambda = (t, s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$: $\pi(\lambda) = M_s T_t$ and $\eta(\pi(\lambda)) = \delta_\lambda$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
The commutation rule observed above corresponds to periodicity of $\kappa(S_{g,\Lambda})$ with respect to $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ which in turn implies that $\eta(T) = \mathcal{F}(\kappa(T))$ is a sum of Dirac measures supported on the adjoint lattice Λ°. This leads to the *Janssen representation* of $S_{g,\Lambda}$ and the *Wexler-Raz* biorthogonality condition.

A pictorial presentation of the BGTr morphism



Interpretation of key properties of the Fourier transform

Engineers and theoretical physicists tend to think of the Fourier transform as a change of basis, from the continuous, orthonormal system of Dirac measures $(\delta_x)_{x\in\mathbb{R}^d}$ to the CONB $(\chi_s)_{s\in\mathbb{R}^d}$. Books on quantum mechanics use such a terminology, admitting that these elements are "slightly outside the usual Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$", calling them "elements of the *physical Hilbert space*" (see e.g. R. Shankar's book on Quantum Physics). Within the context of BGTs we can give such formal expressions a meaning: The Fourier transform maps pure frequencies to Dirac measures:

$$\widehat{\chi_s} = \delta_s \quad \text{and} \quad \widehat{\delta_x} = \chi_{-x}.$$

Given the w^*-totality if both of these systems within $\boldsymbol{S}_0'(\mathbb{R}^d)$ we can now claim: The Fourier transform is the *unique* BGT-automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ with this property!

M. de Gosson: Book on the Wigner Transform

For $\phi \in \mathcal{S}(\mathbb{R}^n)$ the short-time Fourier transform (STFT) V_ϕ with window ϕ is defined, for $\psi \in \mathcal{S}'(\mathbb{R}^n)$, by

$$V_\phi \psi(z) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x'} \overline{\phi(x'-x)} \psi(x') \mathrm{d}x'. \tag{6}$$

The STFT is related to a well-known object from quantum mechanics, the cross-Wigner transform $W(\psi, \phi)$, defined by

$$W(\psi, \phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p \cdot y} \psi(x + \tfrac{1}{2}y) \overline{\phi(x - \tfrac{1}{2}y)} \mathrm{d}y. \tag{7}$$

In fact, a tedious but straightforward calculation shows that

$$W(\psi, \phi)(z) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} e^{\frac{2i}{\hbar} p \cdot x} V_{\phi^{\vee}_{\sqrt{2\pi\hbar}}} \psi_{\sqrt{2\pi\hbar}}\left(z \sqrt{\tfrac{2}{\pi\hbar}}\right) \tag{8}$$

where $\psi_{\sqrt{2\pi\hbar}}(x) = \psi(x\sqrt{2\pi\hbar})$ and $\phi^{\vee}(x) = \phi(-x)$;

This formula can be reversed to yield:

$$V_\phi\psi(z) = \left(\tfrac{2}{\pi\hbar}\right)^{-n/2} e^{-i\pi p\cdot x} W(\psi_{1/\sqrt{2\pi\hbar}}, \phi^\vee_{1/\sqrt{2\pi\hbar}})(z\sqrt{\tfrac{\pi\hbar}{2}}). \quad (9)$$

In particular, taking $\psi = \phi$ one gets the following formula for the usual Wigner transform:

$$W\psi(z) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} e^{\frac{2i}{\hbar}p\cdot x} V_{\psi_1}(\psi_2)(z\sqrt{\tfrac{2}{\pi\hbar}}).$$

with $\psi_1 = \psi^\vee{}_{\sqrt{2\pi\hbar}}$ and $\psi_2 = \psi_{\sqrt{2\pi\hbar}}$.

M. A. de Gosson. The Wigner Transform. World Sci. Publishing Co. Pte. Ltd., Hackensack, 2017.

Another reference is the book of K. Gröchenig [gr01], which contains (in the terminology used there) in Lemma 4.3.1 the following formula, using the convention $g^{\checkmark}(x) = g(-x)$:

$$W(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} V_{g^{\checkmark}} f(2x, 2\omega). \quad (10)$$

Charly (in [gr01]) also provides the following *covariance properties*

$$W(T_u M_\eta f) = Wf(x-u, \omega-\eta). \quad W(\hat{f}, \hat{g})(x,\omega) = W(f,g)(-\omega, x).$$

Gabor Analysis can be nicely transferred to the discrete domain by replacing $\mathbb{R}^d$ by e.g. a cyclic group $\mathbb{Z}_N$. However, if one tries to define the Wigner transform of a signal (the adjoint of the Weyl quantization) the mapping $x \mapsto 2x$ creates a PROBLEM. This is why W. Kozek suggested to use the KNS-calculus!

The Period of Joint Projects

The close scientific interaction started - according to my records and memories - with my hiring of Werner Kozek, who was waiting for the evaluation of his PhD thesis. His work with Franz Hlawatsch (published around 1990 - 1994) had great influence on my future work and cooperation. See e.g.

W. Kozek.
On the generalized Weyl correspondence and its application to time-frequency analysis of linear time-varying systems.
In *IEEE Int. Symp. on Time–Frequency and Time–Scale Analysis*, pages 167–170, Victoria, Canada, October 1992.

It was then intensified by work with Helmut Boelcskei:

H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger.
Oversampled FIR and IIR DFT filter banks and Weyl-Heisenberg frames.
Proc. IEEE ICASSP-96, vol. 3, p. 1391–1394, Atlanta (GA), May 1996.

The WWTF-program

One of the boosters for our formal cooperation was the WWTF program "MATHEMATICS PLUS .... With K. Gröchenig we had the chance to run the MOHAWI project (call 2004): Modern Harmonic Analysis Methods for Advanced Wireless Communications, running from 08/2005 to 07/2009. Another WWTF project was NOWIRE: Noncoherent Wireless Communications over Doubly Selective Channels (2011-2013).

https://www.wwtf.at/funding/programmes/past/ma/MA04-044/

We even were able jointly to obtain two patents for Mobile Communication with Saptarshi Das as key person.

Fractional FT and Discrete Hermite I

A generalization of the Fourier transform is the *Fractional Fourier Transform* $\mathcal{F}_\alpha$, with $\alpha \in \mathbb{R}$. These unitary transformation form an Abelian subgroup of the *metaplectic group*, and can be realized as Hermite multipliers, with a phase change according to $\chi_\alpha(n) = exp(2\pi isn)$. Although $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$ cannot be characterized easily by Hermite functions it is nevertheless true that this group of operators is uniformly bounded on $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and $f \mapsto \mathcal{F}_\alpha(f)$ depends continuously on α, because the linear span of the Hermite functions is dense in $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

There is vast recent literature on discrete Fractional FTs and what is called the "linear canonical transformations" (also sliding window versions, etc.). It is natural to obtain such transformation via a discrete substitute of Hermite Functions and (discrete) pure frequencies as multipliers, thus obtaining similar properties.

Fractional FT and Discrete Hermite II

The concrete method developed on the basis of the discrete Gabor analysis is to use the following fact, going back to I. Daubechies:

Lemma

The Hermite ONB for $\mathbf{L}^2(\mathbb{R})$ is the joint eigen-basis for all the Anti-Wick operators (or STFT-multipliers with Gaussian window) with radial symmetric weights on $\mathbb{R} \times \widehat{\mathbb{R}}$.

Since we can implement Gabor multipliers (and for $a = 1 = b$ this are just STFT-multipliers) efficiently using the KNS-connection it is enough to compute (discrete) radial symmetric weight functions on the discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$ (with a peak at $(0,0)$) and from this the corresponding Gabor multiplier. One may expect (and this has been verified systematically in the last years) that this system of eigenvectors forms an ONB for $\mathbb{C}^N = \ell^2(\mathbb{Z}_N)$.

Fractional FT and Discrete Hermite III

It is not difficult to show that in these "radial Anti-Wick operators" commute with the DFT (as in the continuous case over $\mathbb{R}$) and thus this basis of eigenvectors has to belong to the eigenspaces of the (unitary version) of the DFT, with possible eigenvalues $1, i, -1, -i$ respectively.
Inspection of the numerical results is very promising, and there is much better similarity to the continuous Hermite functions of the same order, in fact we expect even convergence in a suitable sense (piecewise linear interpolation, say) in the sense of $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
So finally one would like to verify the covariance property for the corresponding *discrete Wigner transform*.
So how should one define the discrete Wigner transform??

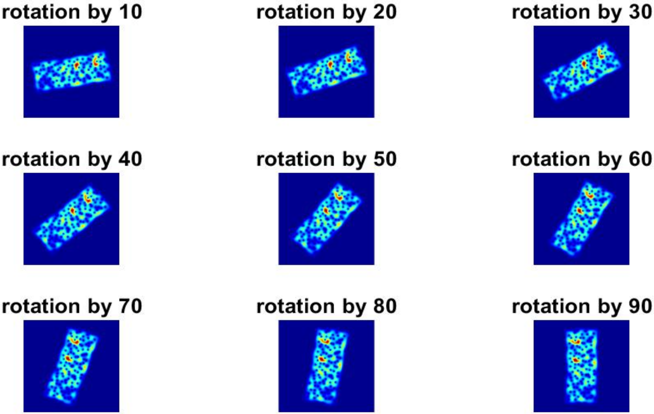
MATLAB Code I

```
function [HERM,hermeig,GMW,W] = hermf(n);

RW = radwgh(n); MRW = max(RW(:)); W = 1+MRW - RW;
[HH, hermeig] = eigsort(gabmulhf(W,g,1,1));
HERM = twtoreal(HH.').';    removes comput. noise
if norm(imag(HH(:))) < 1000*eps;
HERM = real(HH); end;
for jj=1:n; if sum(HERM(jj,1:round(n/2))) < 100*eps;
HERM(jj,:)=-HERM(jj,:); end; end;

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
function radM = radwgh(m,n); dm=min(0:m-1,m:-1:1);
dn=min(0:n-1,n:-1:1);
radM= 1+sqrt((ones(m,1)*dn).^2+((dm(:)*ones(1,n)).^2));
```

The localized Fourier transform (spectrogram)



Discrete Hermite Functions

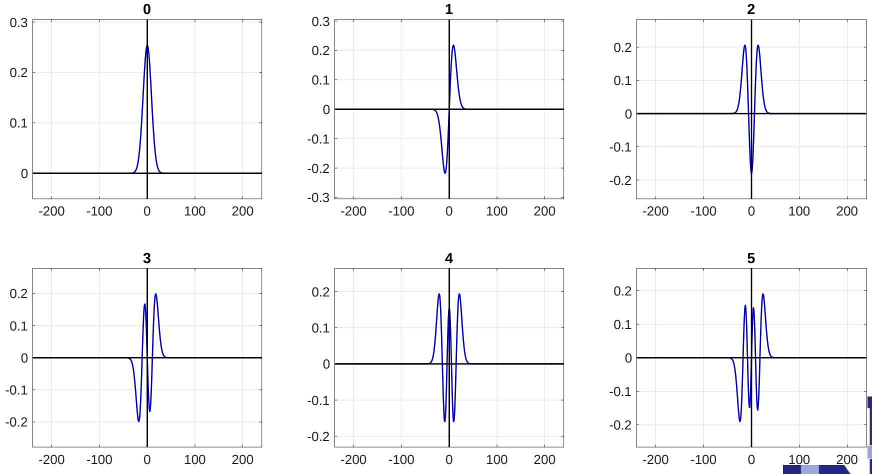


Figure: herm0to5.eps

Wigner in the Discrete Setting

In order to verify the covariance property of the Wigner transform under the fractional Fourier transform (namely rotation in phase space!) one should have a discrete variant of the cross Wigner transform for any signal length N, not only for odd N or even more restrictive for $N = p^k$, some power of a prime number, as one often finds.
For $N = p$ (prime) the cyclic group $\mathbb{Z}_p$ can be viewed as a finite *field*, with $\mathbb{Z}_N \setminus \{0\}$ is a commutative group (under multiplication)!

Only recently I have found a way that allows to **define the cross Wigner transform** or in fact a transform which maps kernels (here $N \times N$-matrices) into a Wigner domain which is isometric with the natural Euclidean norm on matrices (Frobenius norm), and which is valid for any N, including even N!

How this can be achieved

There is no time here to explain the details, but in essence the result (after so many years of considering this as a very difficult question) was based on some naive consideration based on the following puzzle:

- What are engineers doing, literature on discrete Wigner;
- What would the abstract view-point suggest;
- Which considerations can be transferred (reflection operator);
- How can one implement it, using MATLAB.

The experience with the decomposition of a matrix in the sense of the spreading matrix (superposition of TF-shifts, or Fourier decomposition of cyclic side-diagonals!) suggests to decompose a matrix into shifted "anti-diagonals". By choosing suitable phase factors (!) one can assure that a positive matrix (in particular a projection operators) have a *real-valued transform*!

Congratulation and Thanks

So let me conclude with the following summary:

- Without the contact to Franz Hlawatsch NuHAG would not exist and (after EUCETIFA) be seen as an international center for Time-Frequency and Gabor Analysis!
- Thanks for the many joint projects which have been hard work AND inspiration for both sides!
- Our cooperation was a good model for younger scientists, which can make us proud as academic teachers.
- ALL THE BEST for a healthy, enjoyable and long period as a retired person.