NuHAG Numerical Harmonic Analysis Group

Alternative Approach to Distribution Theory for Engineers, motivated by Time-Frequency Analysis

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Different Motivations and Background I

Mathematicians, engineers and physicts have quite different background when they use Fourier Analysis!
Physicists often use a "continuous Dirac basis" in order to describe functions, while in the engineering literature often the *sifting property* of the Dirac delta FUNCTION (!?) is used:

$$f(t) = \int_{-\infty}^{\infty} \delta(t-s) f(s) ds$$

which is more or less equivalent (and equally vague).
Rarely there is a discussion about signals which are only defined "almost everywhere" (e.g. signals of finite energy).
Mathematicians do not care so much about the (real-world) applications of their results and are viewed as pedantic persons who ask for a "proof", even when something is evident.

Different Motivations and Background II

In applications heuristic (plausibility) considerations connected with applications play the key role. Facts are (considered to be) observed and mathematics is used to support (or formalize) such observations, maybe make them *quantifyable*. But who will ask for a *proof for (evident?) facts*?
When an engineer wants to do some computation the following argument is used in connection with the Fourier transform: Since the computer allows only the use of finite length vectors we have to replace the Fourier Transform (say for $\boldsymbol{L}^2$-functions) by the FFT/DFT (?? due to the similarity of names?).
In mathematics rigor and a strict logic derivation of results from assumptions or axioms is the basic requirement, while relevance of mathematical results (maybe for the real world) is considered as a question *outside of mathematics*, so not part of the task of a mathematical researcher!

Different Motivations and Background III

As an application-oriented mathematician I think that it is good to try to connect the different worlds as much as possible, keep eyes open, and look out for possible applications or interesting questions found in the applied field.
In fact, many problems in Gabor Analysis had their roots in engineering problems (such as mobile communication). Slowly varying channels are the kind of linear operators which are well approximated by Gabor multipiers. Music and audio signal processing are fields where time-frequency analysis plays an obvious role.
More recently quite interesting connections between time-frequency analysis or the Kohn-Nirenberg calculus for pseudo-differential operators and Quantum Harmonic Analysis (introduced 1984 by R. Werner) have been found.

Fourier Transform over the Real Line

The work of H.L. Lebesgue paved the way to a clean definition of the Fourier transform for "functions of a continuous variables" as an *integral transform* naturally defined on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$

$$\|f\|_1 := \int_{\mathbb{R}} |f(x)| dx, \quad f \in \boldsymbol{L}^1(\mathbb{R}). \tag{1}$$

The (continuous) Fourier transform for $f \in \boldsymbol{L}^1(\mathbb{R})$ is given by:

$$\hat{f}(s) := \int_{\mathbb{R}} f(x) e^{-2\pi i s x} dx, \quad s \in \mathbb{R}. \tag{2}$$

With this normalization the inverse Fourier transform looks similar, just with the conjugate exponent, and thus, *under the assumption that f is continuous and $\hat{f} \in \boldsymbol{L}^1(\mathbb{R})$* we have pointwise

$$f(t) = \int_{\mathbb{R}} \hat{f}(s) e^{2\pi i s t} ds. \tag{3}$$

Plancherel's Theorem: Unitarity Property of FT

Using the density of $\boldsymbol{L}^1(\mathbb{R}) \cap \boldsymbol{L}^2(\mathbb{R})$ in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ it can be shown that the Fourier transform extends an a natural and unique way to $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$:

Theorem

The Fourier (-Plancherel) transform establishes a unitary automorphism of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, i.e. one has

$$\|f\|_2 = \|\hat{f}\|_2, \quad f \in \boldsymbol{L}^2(\mathbb{R}),$$

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}).$$

In some sense *unitary* transformations of a Hilbert transform is like a change form one ONB to another ONB in $\mathbb{R}^n$.

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the "continuous representation" formula, using $\chi_s(x) = e^{2\pi isx}, \; x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{4}$$

This looks like a perfect orthogonal expansion, but unfortunately the "building blocks" $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

Convolution and the Fourier Transform

Another important fact about the Fourier transform is the so-called **convolution theorem**, i.e. the Fourier transform converts convolution into pointwise multiplication.
Again it is natural to define convolution on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g(x-y)f(y)dy \quad xa.e.; \quad (5)$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}).$$

For positive functions f, g one even has equality. This is relevant for the determination of probability distributions of a sum of *independent* random variables. Assume X has density f and Y has density g then the random variable $X + Y$ has probability density distribution $f * g = g * f$.

Banach algebras

Theorem

Endowed with the bilinear mapping $(f, g) \to f * g$ the Banach space $\left(\mathbf{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ becomes a commutative Banach algebra with respect to convolution.

The **convolution theorem**, usually formulated as the identity

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in \mathbf{L}^1(\mathbb{R}), \tag{6}$$

implies

Theorem

The Fourier algebra, defined as $\mathcal{F}\mathbf{L}^1(\mathbb{R}) := \{\hat{f} \mid f \in \mathbf{L}^1(\mathbb{R})\}$, with the norm $\|\widehat{f}\|_{\mathcal{F}\mathbf{L}^1} := \|f\|_1$ is a Banach algebra, closed under conjugation, and dense in $\left(\mathbf{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$ (continuous functions, vanishing at infinity).

Function space norms

Function spaces are typically infinite-dimensional, therefore we are interested to allow convergent series. In order to check on them we need norms and completeness (in the metric sense), i.e. Banach spaces!

The classical function space norms are

- $\|f\|_\infty := \sup_{t \in \mathbb{R}^d} |f(t)|$;
- $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$;
- $\|f\|_2 := (\int_{\mathbb{R}^d} |f(x)|^2 dx)^{1/2}$;
- $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} := \sup_{\|f\|_\infty \leq 1} |\mu(f)|$, or $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} = \int_{\mathbb{R}^d} 1 d|\mu|$.
- $\|h\|_{\mathcal{F}\boldsymbol{L}^1} = \|f\|_1, \;$ for $h = \hat{f}$.

The usual function spaces $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$

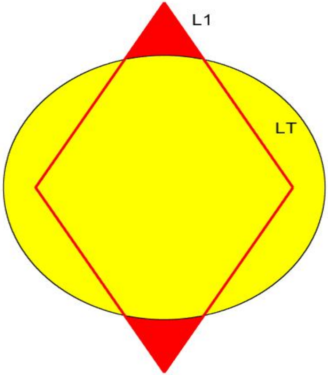


Figure: $\boldsymbol{L}^1(\mathbb{R}^d)$ (red) and $\boldsymbol{L}^2(\mathbb{R}^d)$ (yellow)

Typical windows or filters: Gauss, box, SINC

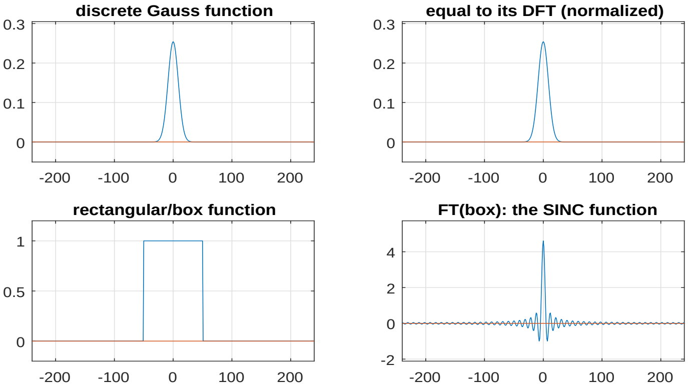
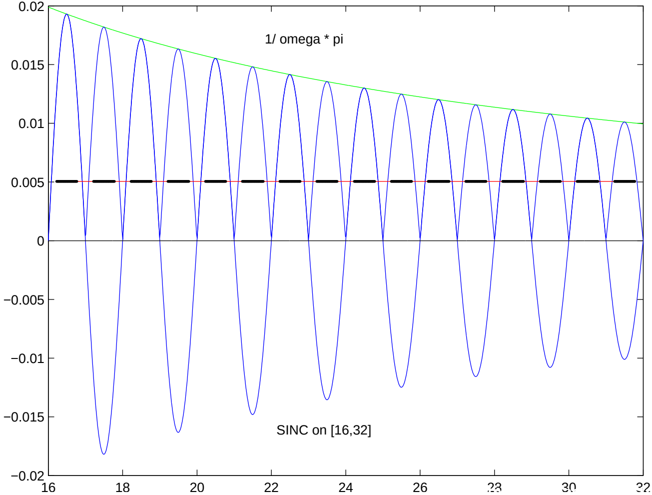


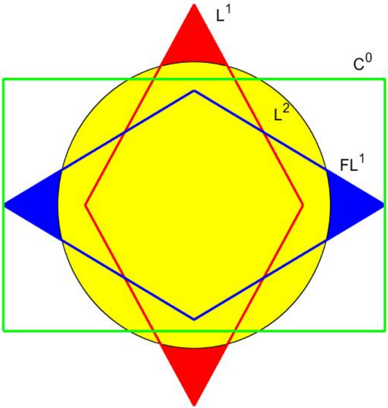
Figure: gausboxSINC1.eps

SINC is not integrable

As an illustration for the poor decay of the SINC-function (implying SINC $\notin \boldsymbol{L}^1(\mathbb{R})$!), let us have a look at the graph of this function:



$L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ and the Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$



The pairs $L^1(\mathbb{R}^d)$, $M(\mathbb{R}^d)$, $C_0(\mathbb{R}^d)$, $L^\infty(\mathbb{R}^d)$

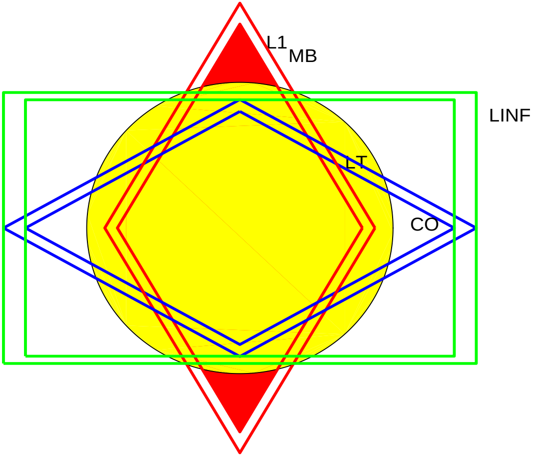


Figure: Pairs of spaces with the same norm and their duality (the outer is the dual of the inner of the other pair)!

The domain of the Fourier inversion theorem

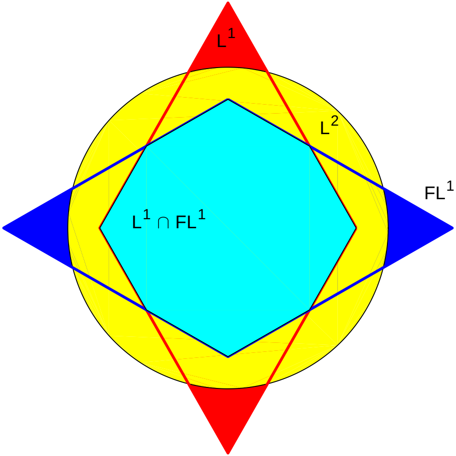


Figure: Light blue marks the domain of the Fourier inversion theorem (in a pointwise sense), still not Poisson!

Pictorial description of $W(C_0, \ell^1)$

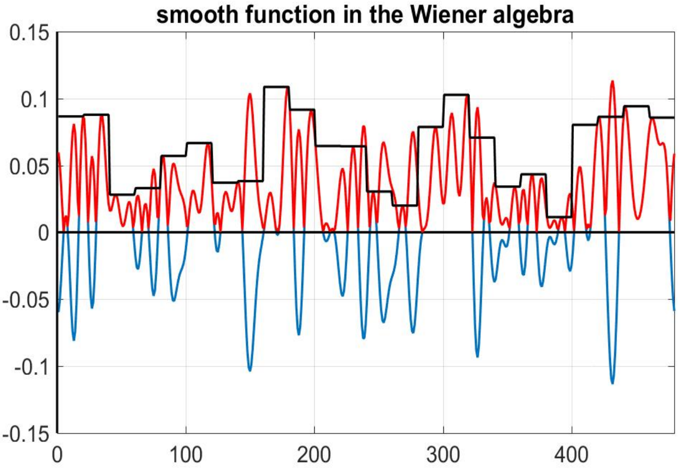
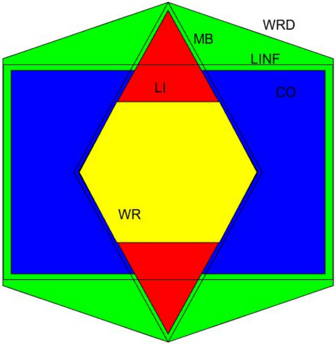


Figure: WienerSmo1.jpg

(

The Wiener algebra and its dual space);
The yellow part of $\boldsymbol{L}^1(\mathbb{R}^d)$ (red) depicts $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$. The green area outside represents the dual space: The space $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d)$ of *translation bounded* Borel measures.



Further function spaces

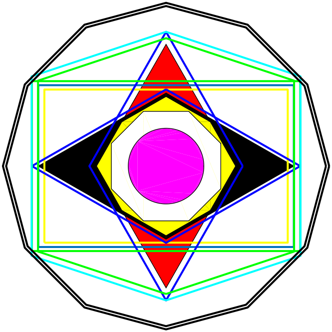


Figure: Inside is $\boldsymbol{S}_0(\mathbb{R}^d)$, at the extreme outside the space of all transformable measures (Gil de Lamadrid) with a FT which is also a transformable measure

Idea of Wiener amalgams: CUT and MEASURE

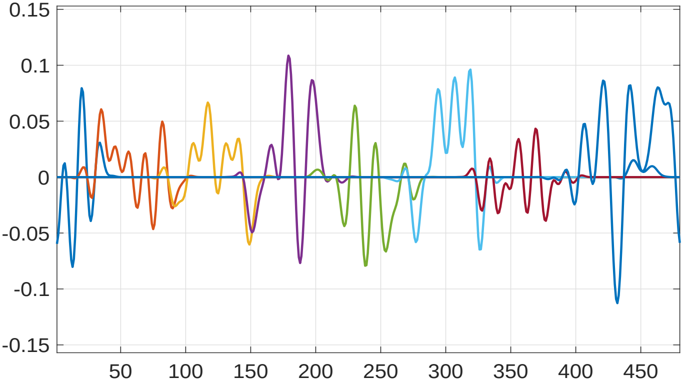
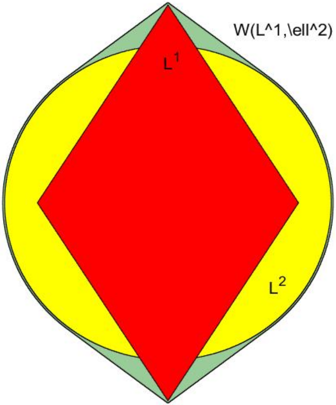


Figure: WFLIli-norm1.eps

A larger Wiener amalgam space $W(L^1, \ell^2)$

The maximal solid space with locally (square) integrable Fourier transforms.



A more detailed look at the various spaces

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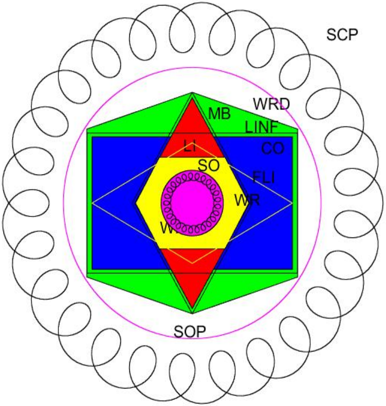
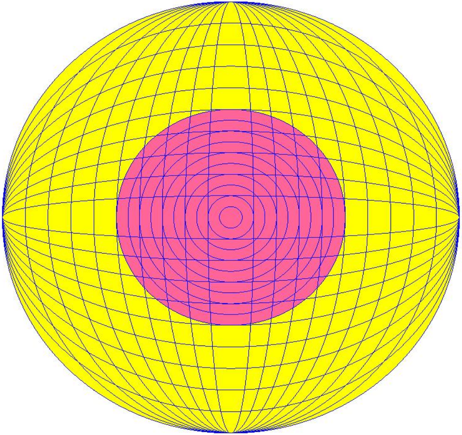


Sobolev spaces, weighted $L^2(\mathbb{R}^d)$, Shubin classes

Sobolev spaces and weighted L2 spaces and M^1_v spaces



Including the Banach spaces $S_0(\mathbb{R}^d)$, $S_0'(\mathbb{R}^d)$

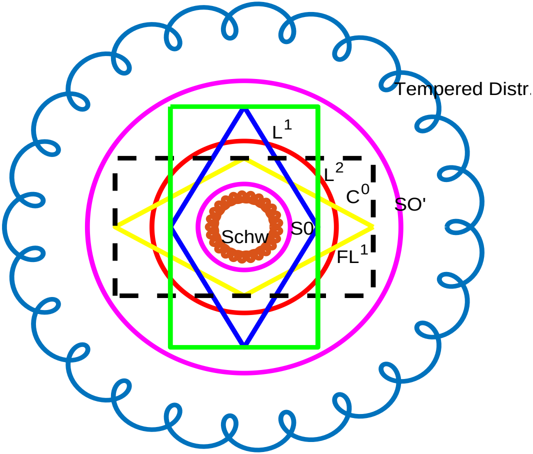
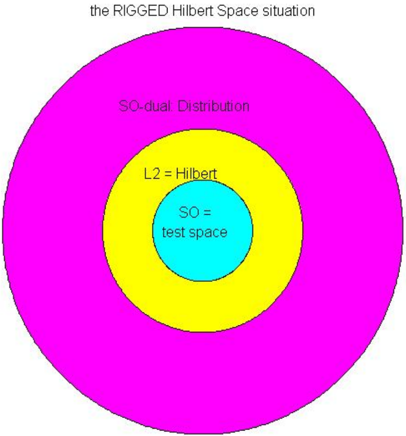


Figure: soplot2bf.eps

A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

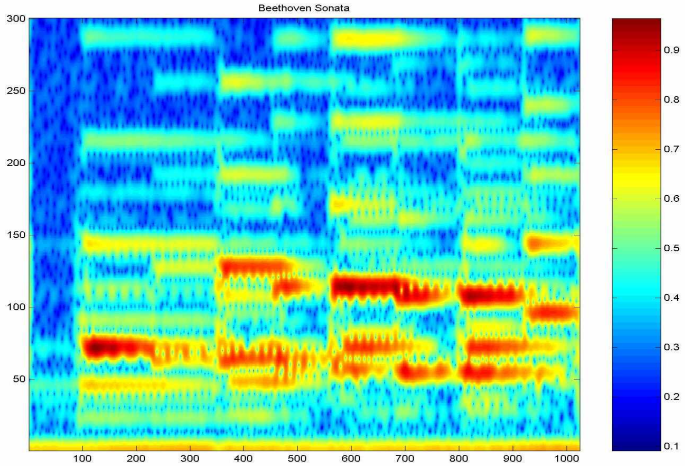
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

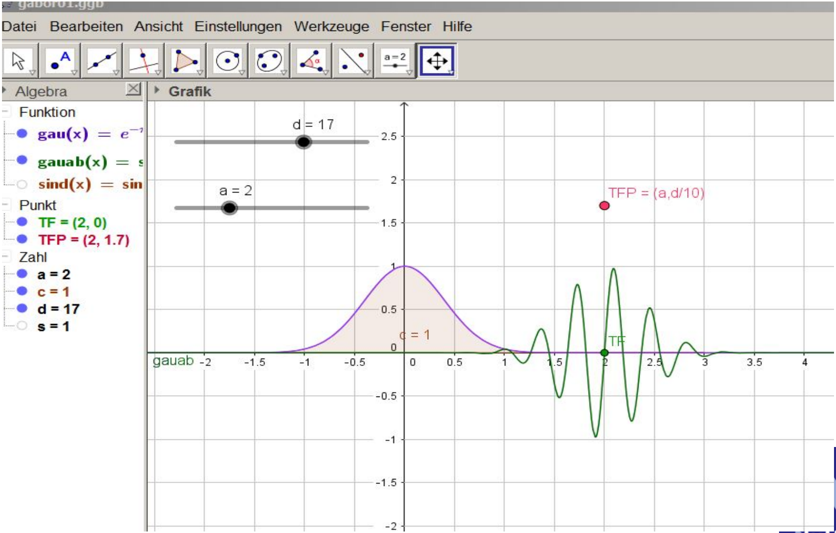
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t,\omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2 = 1$ makes $f \mapsto V_g(f)$ isometric, hence we request this from now on. Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^*: \mathcal{H}_2 \to \mathcal{H}_1$, simply because $\forall h \in \mathcal{H}_1$

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \tag{7}$$

and thus by the *polarization principle* $T^*T = Id$.
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \tag{8}$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \tag{9}$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \qquad (10)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \qquad (11)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \qquad (12)$$

with $\chi_s(t) = exp(2\pi i\langle s, t\rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

While the use of THE Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ is the goal we need intermediate steps in order to introduce/discuss "translation invariant channels" (linear operators), convolution (composition of such systems) and the Fourier transform, which turns the convolution operator (convolution of the input signal by the impulse response function) into a pointwise multiplication operator, the so-called transform function. In short, engineers get the explanation that any such linear operator is realized by convolution:

$$T(f) = \mu * f, \quad \text{or equv.} \quad \mathcal{F}(T(f)) = h \cdot \widehat{f}$$

where one may assume of course that $h = \widehat{\sigma}$, the FT of the impulse response μ (in order to use standard engineering terminology).

The concrete starting point I

We will first provide an approach to convolution of bounded measures (including discrete measures) which can be carried out without problems over any LCA group $\mathcal{G}$ (just for simplicity let us do it for $\mathcal{G} = \mathbb{R}^d$).
By the local compactness of $\mathcal{G}$ (e.g. closed unit balls are compact in $\mathbb{R}^d$) the space $\boldsymbol{C}_c(\mathbb{R}^d)$ of compactly supported, continuous, complex-valued functions is non-trivial (and translation invariant). Endowed with the sup-norm we can close it up inside $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and obtain $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, which turns out to be a nice Banach algebra with bouded approximate units (e.g. by taking $\mathrm{D}_\rho(h)(x) = h(\rho x)$, then $(\mathrm{D}_\rho h)_{\rho\to 0}$ is such a bounded approximate unit (BAE) in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ if $h(0) = 1$).

The concrete starting point II

Given any BUPU $(\psi_i)_{i\in I}$ we can show that

$$\sum_{i\in I} \|m\psi_i\|_{\mathbf{M}_b} = \|\mu\|_{\mathbf{M}_b}, \quad \mu \in \mathbf{M}_b(\mathbb{R}^d). \tag{13}$$

Note that finite discrete measures are good examples, with

$$\mu(f) = \sum_{i\in F} c_i \delta_{x_i}(f) = \sum_{i\in F} c_i f(x_i)$$

with F a finite index set, and satisfy $\mu \cdot h = \sum_{i\in F} h(x_i)\delta_{x_i}$. We also have convergence of such measures, meaning

$$\lim_{\alpha\to\infty} \mu_\alpha(f) = \mu_0(f), \quad \forall f \in \mathbf{C}_0(\mathbb{R}^d)$$

similar to convergence of Riemanian sums.

The concrete starting point III

The first main result is then the identification of the space of all bounded linear operators (TILS) on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ with this dual space, more or less via the principle of moving average.
More precisely one has to show:
Given such a linear operator one defines a linear functional μ by setting $\nu(f) := Tf(0)$. It is then easy to find that

$$Tf(x) = T_{-x}(Tf)(0) = T(T_{-x}f)(0) = \nu(T_{-x}f) := [T_x\nu](f).$$

We check that $T_x(\delta_z) = \delta_{z+x}$.
Using (13) one can also show the converse, namely define a moving average by setting $T_\nu(f)(x) = \nu(T_{-x}f)$ and finds that such an operator commutes with translations, and the norm of the operator equals the norm of the associated functional.

The concrete starting point IV

One can then go on to define to composition of two measures $\nu_1 * \nu_2$ to be the *unique* $\nu \in \boldsymbol{M}_b(\mathbb{R}^d)$ which represents the composite operator $T_{\nu_1} \circ T_{\nu_2}$. This approach transfers the composition law for linear operators to the new multiplication (called convolution) on $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.

We CALL it CONVOLUTION.

Clearly this gives an associative multiplication, satisfying the distributive law, but one has to check separately (via discrete measures) that it is also commutative!

The Fourier Stieltjes transform

The Fourier-Stieltjes transform for $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is defined as $\widehat{\mu}(s) := \mu(\chi_{-s})$, with

$$\chi_s(t) = \exp(2\pi i s t), \quad s, t \in \mathbb{R}^d.$$

Just not that μ extends to $\boldsymbol{C}_b(\mathbb{R}^d) \supset \boldsymbol{C}_0(\mathbb{R}^d)$ via

$$\mu(h) = \sum_{i \in I} \mu(\psi_i h) = \sum_{i \in I} (\mu \psi_i)(h).$$

Using the exponential law it is not difficult to verify then

$$\mathcal{F}(\mu_1 * \mu_2) = \mathcal{F}(\mu_1) \cdot \mathcal{F}(\mu_2).$$

Wiener's Algebra

Unfortunately there are now inclusion relations between $C_0(\mathbb{R}^d)$ and $M_b(\mathbb{R}^d)$, and thus it is better to introduce the *Wiener algebra* $\left(W(C_0,\ell^1)(\mathbb{R}^d),\ \|\cdot\|_W\right)$, again characterized via BUPUs:

$$W(\mathbb{R}^d) = W(C_0,\ell^1)(\mathbb{R}^d) := \{f \in C_0(\mathbb{R}^d), \sum_{i\in I} \|f\psi_i\|_\infty < \infty\}.$$

For the dual space (we could characterize it as the space of *translation-bounded (Radon) measures* $\left(W(M,\ell^\infty)(\mathbb{R}^d),\ \|\cdot\|_W\right)$) contains $M_b(\mathbb{R}^d)$ (because $W(C_0,\ell^1)(\mathbb{R}^d)$ is dense in $\left(C_0(\mathbb{R}^d),\ \|\cdot\|_\infty\right)$).
The Riemann integral (Haar measure) allows to identify $W(\mathbb{R}^d)$ (even $C_b(\mathbb{R}^d)$) as a subspace (of regular measures) via

$$\mu_h(f) = \int_{\mathbb{R}^d} f(x)h(x)dx.$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

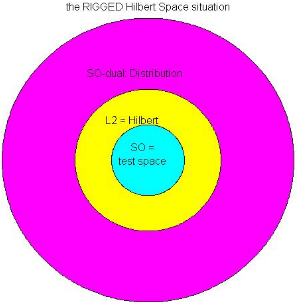
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$ for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Moreover $\mathcal{F}$ is *uniquely determined* (as BGTr isomorphism) by the property that $\widehat{\chi_s} = \delta_s$.

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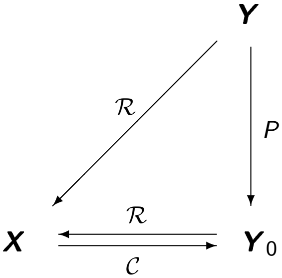
Frames and Riesz Bases: Commutative Diagrams

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$ in $\boldsymbol{Y}$:



Riesz Basis case: E.g. $\boldsymbol{X}_0 \subset \boldsymbol{X} = L^p$, and $\boldsymbol{Y} = \ell^p$:

Unconditional Banach Frames

A suggestion for "realistic Banach frames":

Definition

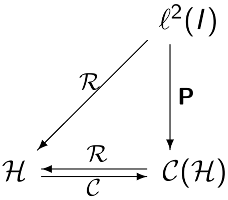
A mapping $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ defines an unconditional (or solid) Banach frame for $\mathbf{B}$ w.r.t. the sequence space $\mathbf{Y}$ if

1. $\exists \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, with $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,
2. $(\mathbf{Y}, \|\|_{\mathbf{Y}})$ is a solid Banach space of sequences over I, with $\mathbf{c} \mapsto c_i$ being continuous from $\mathbf{Y}$ to $\mathbf{C}$ and solid, i.e. satisfying $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : |x_i| \leq |z_i| \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$;
3. finite sequences are dense in $\mathbf{Y}$ (at least W^*).

Corollary

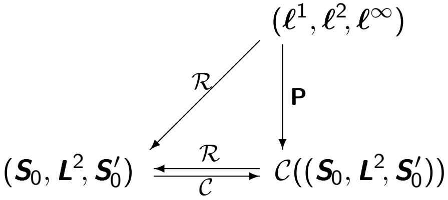
By setting $h_i := \mathcal{R} e_i$ we have $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ unconditional in $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, hence $f = \sum_{i \in I} T(f)_i h_i$ as unconditionally convergent series.

Similar to the situation for matrices of maximal rank (with row and column space, null-space of $\boldsymbol{A}$ and $\boldsymbol{A}'$) we have:
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Gelfand triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$:

A simple way to express the fact that good windows $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ allow to discretize properly the continuous reconstruction formula is the entry point to Gabor Analysis.



The KERNEL THEOREM for *ScRd*

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{14}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (14) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)

One of the corner stones for the kernel theorem is: One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert, [?], perhaps [?]) is the tensor-product factorization:

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \quad (15)$$

with equivalence of the corresponding norms.

The KERNEL THEOREM for S_0 II

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

In analogy to the matrix case, where the entries of the matrix

$$a_{k,,j} = T(\mathbf{e}_j)_k = \langle T(\mathbf{e}_j), \mathbf{e}_k\rangle$$

we have for $K \in \boldsymbol{S}_0$ the continuous version of this principle:

$$K(x,y) = \delta_x(T(\delta_y)), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $\big(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')\big)$*, where the first set is understood as the* w^* *to norm continuous operators from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\mathbb{R}^d)$*, the so-called regularizing operators.*

Spreading function and Kohn-Nirenberg symbol

1. For $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$$

The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}\,d\omega\Big)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta,u)M_\eta T_{-u}f(x)\,du\,d\eta.$$

$\widehat{\sigma}$ is called the spreading function of T_σ.

Further details concerning Kohn-Nirenberg symbol

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

The kernel $K(x,y)$ can be described as follows:

$$K(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x)$$
$$= \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y-x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

Kohn-Nirenberg symbol and spreading function II

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x), dt\,d\nu\,,$

My relevant tools

For me the most relevant tools to "produce mathematics" are:

- MATLAB (simulation, numerics, illustration);
- GEOGEBRA (geometry, trigonometry, dynamic);
- LATEX (for presentations);
- Didactically I compare the Banach Gelfand triple with

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{Q}$$

asking sometimes: How do you actually interpret $1/\pi^2$?

Summary

In this talk, aside from a picture book presentation I have tried to communicate various suggestions:

1. One needs to understand basic distribution theory (using Banach spaces only), no Lebesgue integration of topological vector spaces;
2. Computations, images, plots can help the understanding, not only illustrate results numerically;
3. Diagrams can provide a big help
4. Numerical simulations (e.g. MATLAB) can provide interesting experimental information

Using this approach one can SAVE statements such as...

The **sifting property**,

$$\int_{-\infty}^{\infty} f(x)\delta(x-\xi)dx = f(\xi), \tag{16}$$

The identity

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-l)x}dx = \delta(k-l) \tag{17}$$

The audio-engineer's work: Gabor multipliers

What else can you do with $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$?

1. It is a good replacement for $\mathcal{S}(\mathcal{G})$ (Schwartz-Bruhat) (except for applications in PDE);
2. It allows to demonstrate that there is "only one Fourier transform" (citation to Jens Fischer), including periodic and nonperiodic, discrete and continuous functions;
3. they can be mutually approximated in the w^*- topology;
4. $S_0(\mathbb{R}^d)$ contains all classical summability kernels,
5. we can use it for generalized stochastic processes
6. it is the basis of my concept of Conceptual Harmonic Analysis, a kind of marriage between abstract and numerical harmonic analysis.

Thanks!

THANK you for your attention

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