NuHAG Numerical Harmonic Analysis Group

Wiener Amalgam Spaces for Gabor Analysis

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Introduction for talk 20.06.2024, 14:30 HGFei I

Wiener Amalgam Spaces are an extremely useful and versatile tool, especially for Fourier Analysis, Time-Frequency Analysis and Gabor Analysis. Although developed already 45 years ago they have not found the attention which they might deserve, as a tool which allows to describe local and global properties of a function or distribution better than the usual $\boldsymbol{L}^p$-spaces.
It is the purpose of this talk to demonstrate that they are *easy to understand* and *easy to use*, given a few basic principles. Their study paved the way to *modulation spaces*, which are nowadays an important family of function spaces.
From the many application areas where they turn out to be useful we choose those which are connected with Fourier or Gabor Analysis, also in the wider context of *Coorbit Theory*.

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OUTLINE I

Gabor Analysis goes back to the claim of D. Gabor in his famous 1946 paper: Every function has a unique representation as a double series, with building blocks (Gabor atoms) begin TF-shifted Gaussians (vague citation!).

1. Gabor Analysis in the finite setting (Linear Algebra)
2. Gabor Analysis in the continuous setting
3. Functional Analytic Foundations
4. Wiener Amalgam Spaces and BUPUs (quick review)
5. Wiener Amalgams and Gabor Analysis
6. State of the Art of Gabor Analysis (outlook)

Gabor Analysis using MATLAB I

In this finite-dimensional setting we have *cyclic* shifts, a DFT/FFT (we choose the unitary version), and a cyclic rotation on the Fourier transform side, which corresponds to a pointwise multiplication by a "character" or "*pure frequency*", which is just one of the rows (or columns) of the DFT matrix.
Note that there are N cyclic shift operators $T_k, 0 \leq k \leq N-1$, and the same number of *modulation operators* $M_n = \mathcal{F}^{-1} \circ T_n \circ \mathcal{F}$, and thus N^2 TF-shifts (time-frequency shifts) $\pi(k,n) = M_n T_k$.
In fact, they form an orthonormal basis for the space of $N \times N$ matrices, called the *spreading representation*, using the Frobenius scalar product for matrices (Hilbert Schmidt norm), given by

$$\mathbf{A} = \frac{1}{N} \sum_{n,k} \operatorname{trace}(\mathbf{A} \circ \pi(n,k)^*) \pi(n,k).$$

Gabor Analysis using MATLAB II

Recall that one has

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_{\mathcal{HS}} = \sum_{k,n} a_{n,k} \overline{b_{n,k}} = \operatorname{trace}(\boldsymbol{A} * \boldsymbol{B}^*).$$

In other words one uses the Euclidean structure (like pixel images) here, or the Frobenius/Hilbert-Schmidt norm.

Gabor families are obtained by choosing for vector $g \in \mathbb{C}^N$ and applying a finite set of TF-shifts. Such a Gabor family is a *Gabor frame* if it is a generating family for $\mathbb{C}^N$, and a *Gabor Riesz sequence* if it is linear independent. Our main interest is in Gabor frames with low redundancy, i.e. using $M \geq N$ vectors only and thus red $= M/N \geq 1$ (red $= 1$ is critical density).
For the so-called *regular Gabor case* one just ask, whether a Gabor family is a generating system for $\mathbb{C}^N$, where the parameter set is a *subgroup* $\Lambda \lhd \mathbb{Z}_N \times \mathbb{Z}_N$.

Gabor Analysis using MATLAB III

Gabor Analysis in the finite $\mathbf{Z}_N$, the cyclic group of order $N \in \mathbb{N}$, is reducing the (functional analytic) problems arising in the continuous context (Euclidean context, for $\mathbb{R}^d$) context. Point evaluations are obtained using unit vectors and there is no need for infinite series (with potential convergence problems). In this way the algebraic structure becomes clear: [fekolu09]. We determine the subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ (this depends on the number of different divisors of N, e.g. $N = 480$) and so on.
On the other hand it is a special chapter of Linear Algebra. It tells us that the *Gabor frame question* just means (putting the vectors into a matrix) whether they form a *generating system* for $\mathbb{C}^N$, or alternatively (using the so-called adjoint group), whether they form a linear independent family, which in turn is equivalent to the invertibility of their Gram matrix (Wexler-Raz condition).

The ideas of D. Gabor from 1946

When one moves to the continuous setting many questions arise. When D. Gabor suggested in 1946 that "every function" has a signal expansion as a double series (with hopefully uniquely determined coefficients) using time-frequency shifted Gaussians (along the integer lattice), many natural questions have not even been formulated. In fact, until around 1980 his approach was mostly ignored by mathematicians, and **Gabor Analysis** as we call it now started in the 80th of the last century.
For a more precise formulation let us recall the key players:
We need time-frequency shifts $\pi(t,s) = M_s T_t$, and the Gauss function $g_0(t) = e^{-\pi|t|^2}$ (adjusted to our FT convention, with $\mathcal{F}(g_0) = g_0$).
D. Gabor has good arguments for choice of the Gauss function (Heisenberg uncertainty), and to suggest $a = 1 = b$ as lattice constants in the TF-plane (short verbal explanation).

A list of Gabor related questions

The first questions concern the coefficients. How can we establish a rule for the coefficients, in analogy to Fourier expansions. Note that we have a double series (so summability methods might help).

- How can they be obtained and are they unique?
- For which functions do they make sense? $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R})$?
- What about convergence, note the non-orthogonality!

In retrospect one might think that Gabor was hoping the the system he has suggested could be a Riesz basis (non-orthogonal, but stable basis) in the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.

Let us start with the Hilbert space I

It is natural to start the discussion with $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which is a *separable Hilbert space* with respect to the usual scalar product

$$\langle f, g\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(t)\overline{g(t)}\,dt \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

The hope is to establish an expansion of the form

$$f = \sum_{(k,n)\in\mathbb{Z}^2} \langle f, g_{k,n}\rangle_{\boldsymbol{L}^2} h_{k,n}, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d),$$

for suitable functions $g, h \in \boldsymbol{L}^2(\mathbb{R}^d)$ and $g_{k,n} = M_n T_k g$ (similarly $h_{k,n}$), with (unconditional norm convergence in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$), ideally with $g = h$ (constituting a tight Gabor frame).

Let us start with the Hilbert space II

In other words, one can hope for a Riesz basis for the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, of Gaborian type. Then the biorthogonal Riesz basis *has to be also* of the same form, and the roles of g and h can be interchanged.

In fact, this is not impossible, one can take as g the indicator function of the cube $[0,1)^d \subset \mathbb{R}^d$ and obtains an orthogonal system by expanding the restrictions of f to each of the cubes $k + [0,1)^d, k \in \mathbb{Z}^d$, into a d-dimensional Fourier series, but this is not in the sense of Gabor, because the Fourier transform of $\chi_{[0,1)^d}$ is not integrable, i.e. has pure decay. It is in fact a tensor product of SINC-functions (which do not belong to $\boldsymbol{L}^1(\mathbb{R}^d)$).

BUT one can show (this is known as the Balian-Low Principle) that one cannot choose such a function in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, not even in $\boldsymbol{S}_0(\mathbb{R}^d)$ (the corresponding function would be in $\boldsymbol{S}_0(\mathbb{R}^d)$ as well and this creates a contradiction to Balian-Low).

Let us start with the Hilbert space III

KEYPLAYERS of TF-ANALYSIS: TF-shifts and the STFT
Let us recall two basic facts from "continuous" TF-analysis:
The TF-shifts are unitary operators on the Hilbert space $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Their composition law (with phase factors showing up) implies that $\lambda = (t,s) \mapsto \pi(\lambda) = M_s T_t$ is a strongly continuous *projective representation* of phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on $\mathcal{H}$.
Thus for functions $f, g \in \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ the STFT, given by

$$V_g f(\lambda) = \langle f, \pi(\lambda) g \rangle_{\boldsymbol{L}^2}, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$$

defines a bounded continuous function in $\boldsymbol{C}_0(\mathbb{R}^{2d})$.
For normalized g with $\|g\|_2 = 1$ it defines even an isometric embedding of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ (**Moyal**).

Let us start with the Hilbert space IV

So it seems that we do not need an admissibility condition, and in fact the adjoint defines the inverse on the range. It is of the form

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\, \pi(\lambda) g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^{2d}),$$

thus implying a representation of $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ "in the weak sense"

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\, \pi(\lambda) g \, d\lambda = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle g_\lambda \, d\lambda,$$

showing analogy to the Fourier inversion formula for $f, \widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$

$$f(t) = \int_{\widehat{\mathbb{R}}^d} \widehat{f}(s) \chi_s(t) ds = \left[\int_{\widehat{\mathbb{R}}^d} \langle f, \chi_s \rangle \chi_s ds \right](t).$$

In contrast, $\chi_s \notin \boldsymbol{L}^2(\mathbb{R}^d)$ (hence also no Riemannian sum).

Bessel Condition for Gabor families I

We will restrict our discussion here to the regular case (irregular means: no group structure involved).
The expected expansion using *Gabor atoms* requires first to study for fixed $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ the *coefficient (or analysis) mapping*

$$f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$$

which is said to be a *Bessel family* if it is bounded from $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ to $(\ell^2(\Lambda), \|\cdot\|_{\ell^2(\Lambda)})$. Obviously this is the case if and only if the adjoint mapping is bounded, which is the so-called *synthesis mapping*

$$(c_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g.$$

The frame operator $S_{g,\Lambda}$ is the composition of these two.

Bessel Condition for Gabor families II

It is one of the fundamental insights of frame theory (which was greatly influenced by Gabor analysis, because there are not Gaborian orthonormal bases, unlike the wavelet case!) is the fact that one "just has to verify the invertibility" (on $\mathcal{H}$) of the frame operator in order to come up with a representation based on the simple identities $\mathrm{Id} = S_{g,\Lambda}^{-1} \circ S_{g,\Lambda} = S_{g,\Lambda} \circ S_{g,\Lambda}^{-1}$, which can be rewritten as (using Λ as index set)

$$f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g_\lambda} = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g_\lambda} \rangle g_\lambda, \quad f \in \mathcal{H}, \tag{1}$$

with unconditional convergence in $\mathcal{H}$. For the regular case one has $\widetilde{g_\lambda} = \pi(\lambda)(\tilde{g}) = \tilde{g}_\lambda$ with $\widetilde{g} = S_{g,\Lambda}^{-1}(g)$, or $S_{g,\Lambda}(\tilde{g}) = g$. $\widetilde{g}$ is called the *dual Gabor window* (given the pair (g, Λ)).

Classical Wiener Amalgam Spaces I

For the Euclidean case the classical Wiener amalgam (originally Wiener-type) spaces are defined making use of a decomposition of a given locally integrable function along shifts of a fundamental domain for a lattice Λ along that lattice. For $\Lambda = a\mathbb{Z}^d$ with $a > 0$ this means that up to dilation (choice of $a = 1$) one restricts $f \in \boldsymbol{L}^1_{loc}(\mathbb{R}^d)$ to the cubes $Q_k = k + [0,1]^d$.
Given any *solid BF-space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (nowadays called *ball function space*, of Banach function space) one obtains a sequence of finite norms for $f_k = f \cdot \mathbf{1}_{Q_k}$, and thus defines a function space by applying a sequence space norm (over Λ) to the (non-negative) "sequence" $(\|f_k\|_{\boldsymbol{B}})_{\lambda\in\Lambda}$.
For $\Lambda = \mathbb{Z} \lhd \mathbb{R}$, $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = (\boldsymbol{L}^p(\mathbb{R}), \|\cdot\|_p)$ and sequence space norms $(\ell^q, \|\cdot\|_q)$ one obtains the classical spaces already used by Wiener (e.g. $p = 2, q = 1$), written as $\ell^2(\boldsymbol{L}^1)(\mathbb{R})$.

Classical Wiener Amalgam Spaces II

Recalling the concept of solid BF-spaces:
A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ contained in $\boldsymbol{L}^1_{loc}(\mathbb{R}^d)$ is called a BF-space if convergence in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ implies $\boldsymbol{L}^1$-convergence over compact subsets $Q \subset \mathbb{R}^d$, i.e., for any compact set $Q \subset \mathbb{R}^d$ there exists a constant C_Q such that

$$\int_Q |f(x)|dx = \|f \cdot \mathbf{1}_Q\|_1 \leq C_Q \|f\|_{\boldsymbol{B}}, f \in \boldsymbol{B}.$$

It is called *solid* if it satisfies for any $f \in \boldsymbol{B}$ and $g \in \boldsymbol{L}^1_{loc}(\mathbb{R}^d)$:

$$|g(x)| \leq |f(x)| \ \ a.e. \ \ \Rightarrow g \in \boldsymbol{B} \text{ and } \|g\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}}.$$

Such spaces are translation invariant, i.e. satisfy $T_x\boldsymbol{B} \subseteq \boldsymbol{B}$ if and only if any of the translation operators T_x, $x \in \mathbb{R}^d$ defines a bounded and linear operator on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (This is a consequence of the closed graph theorem!).

Classical Wiener Amalgam Spaces III

In order to increase smoothness we can convolve any such family $(T_k\varphi)_{k\in\mathbb{Z}^d}$ by any probability measure, or $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\widehat{g}(0) = 1$ and it will still form a partition of unity.
For $d = 1$ the self-convolution of the box-car function gives the triangular function Δ, which is piecewise linear, and satisfies also $\Delta(0) = 1$, and $\Delta(k) = 0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$. W Taking further convolution powers we obtain in this way the family of B-splines of order k (piecewise linear corresponds to order 2, as it is continuous and piecewise a polynomial order 2, i.e. with two coefficients).
Since cubic B-splines are piecewise cubic polynomials, concatenate in a C^2-manner (twice continuous differentiable) they are of order 4 (their Fourier transform is $\mathrm{SINC}^4(s) \geq 0$ and with good decay for $|s| \to 0$).

Classical Wiener Amalgam Spaces IV

Anyway, we can say that (regular) **BUPUs** (or UCPUs) are families of the form $T_k\varphi, k \in \mathbb{Z}^d$, arising from a compactly supported function (of some *regularity*), with

$$\sum_{k\in\mathbb{Z}^d} \varphi(x-k) \equiv 1, \quad x \in \mathbb{R}^d.$$

Given a Banach space $\boldsymbol{B} \hookrightarrow \boldsymbol{L}^1_{loc}(\mathbb{R}^d)$ we assume that the action of the members of this family is uniformly bounded, i.e. there exists some $C > 0$ such that

$$\sup_{k\in\mathbb{Z}^d} \|T_k\varphi \cdot f\|_{\boldsymbol{B}} \leq C\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{2}$$

In this situation we call the family $(T_k\varphi)_{k\in\mathbb{Z}^d}$ a $\boldsymbol{B}$-BUPU, i.e. a *bounded uniform partition of unity FOR* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$

Introducing BUPUs I

These problems have been overcome by the introduction of the concept of BUPUs, which was inspired by the dyadic Fourier decomposition used for the characterization of Besov spaces in the work of Peetre, Triebel or Frazier-Jawerth. We will generalize the concept of Wiener amalgam spaces not by using "better decompositions" (allowing to measure locally with smoothness spaces), and the go on to show that most of the time there are equivalent "continuous norms".

We will not do the most general case of BUPUs, which was very useful in our subsequent work on the irregular sampling problem respectively for *coorbit theory* (where irregularities are unavoidable), but rather restrict our attention to the so-called *regular case*, i.e. BUPUs associated with some lattice $\Lambda \lhd \mathbb{R}^d$. Note however, the concept of BUPUs can be realized in over general locally compact groups.

Introducing BUPUs II

Arbitrary fine BUPUs have been constructed in a recent paper by the author in [fe22] in order to realize "integrated group representations" from isometric representations on Banach spaces, giving something like $\mathbf{L}^1(\mathbb{R}^d) * \mathbf{L}^p(\mathbb{R}^d) \subseteq \mathbf{L}^p(\mathbb{R}^d)$, but we will not go into this direction today.

Starting with the most simple and perhaps most important practical case let us look at the case $\Lambda = \mathbb{Z}^d \lhd \mathbb{R}^d$ (the most general lattice in $\mathbb{R}^d$ are obtained by applying suitable invertible matrices to the standard lattice $\mathbb{Z}^d$, thus this case shows all the relevant properties for fixed Λ).

We can start from the partition of unity over the cubes $Q_k = k + [0,1)^d$, or $Q_k = k + [-1/2, 1/2]^d$, with $k \in \mathbb{Z}^d$. Integration of a probability measure against this (obvious, discontinuous) partition of unity is known as *histogram*.

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In order to increase smoothness we can convolve any such family $(T_k\varphi)_{k\in\mathbb{Z}^d}$ by any probability measure, or $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\widehat{g}(0) = 1$ and it will still form a partition of unity.
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Introducing BUPUs IV

Anyway, we can say that BUPUs are families of the form $T_k\varphi, k \in \mathbb{Z}^d$, arising from a compactly supported function (of some *regularity*), with

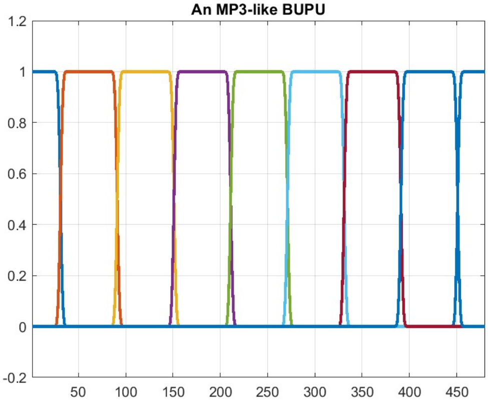
$$\sum_{k\in\mathbb{Z}^d} \varphi(x-k) \equiv 1, \quad x \in \mathbb{R}^d.$$

Given a Banach space $\boldsymbol{B} \hookrightarrow \boldsymbol{L}^1_{loc}(\mathbb{R}^d)$ we assume that the action of the members of this family is uniformly bounded, i.e. there exists some $C > 0$ such that

$$\sup_{k\in\mathbb{Z}^d} \|T_k\varphi \cdot f\|_{\boldsymbol{B}} \le C\|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{3}$$

In this situation we call the family $(T_k\varphi)_{k\in\mathbb{Z}^d}$ a $\boldsymbol{B}$-BUPU, i.e. a *bounded uniform partition of unity FOR* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$

b



MP3BUPU.jpg

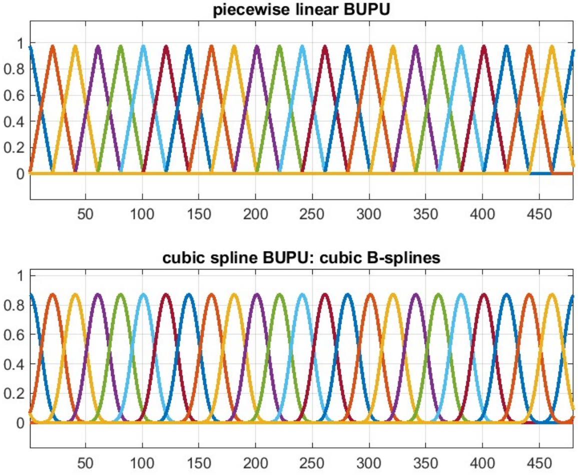


Figure: BUPUplncub00.jpg

b

Our notation for *Wiener Amalgam spaces is* $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{Y})$, where $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ is some discrete Banach function space (we call it solid BF-space) over Λ. This raises several questions:

1. Does the definition depend on the ingredients (especially on the lattice Λ or the(compact) fundamental domain)?
2. Even if both $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ are isometrically translation invariant, this is only true for a (complicated) equivalent norm, not the "*natural*" norm.
3. If one wants to measure local properties such as smoothness, say with respect to some Sobolev norm, sharp truncation is not a good idea, one would have to take the restriction norm which is cumbersome, as it requires to talk about representatives for the quotient norm.

For our discussion weighted ℓ^q-spaces are the prototypical examples, such as (for $s \in \mathbb{R}$, and $q \in [1, \infty)$):
$\ell^q_s := \{ (c_k)_{k \in \mathbb{Z}^d} \mid \left(\sum_{k \in \mathbb{Z}^d} |c_k|^q (1+|k|)^{sq}\right)^{1/q} < \infty \}$
endowed with the natural norm (similar for $q = \infty$).
The corresponding (!continuous) weighted $\boldsymbol{L}^q$-space is
$\boldsymbol{L}^q_s := \{ f \in \boldsymbol{L}^1_{loc},\text{with } f \langle \cdot \rangle^s \in \boldsymbol{L}^p(\mathbb{R}^d) \}$, with the natural norm
$\|f\|_{\boldsymbol{L}^q_s} = \|f \langle \cdot \rangle^s\|_{\boldsymbol{L}^p(\mathbb{R}^d)}$, where we use
the standard convention $\langle () \rangle x) = (1 + |x|^2)^{1/2}$.

Equivalence: BUPUs vs. continuous characterization I

While φ is moved only along the lattice and (better) satisfies the BUPU condition, any compactly supported (non-zero) function ϕ with the property that the family $T_x\phi, x \in \mathbb{R}^d$ defines a uniformly bounded family of pointwise multipliers on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ can be used to define an equivalent "continuous norm" using the idea of a *control function* $\kappa(f,\phi)(x) = \|f \cdot T_x\phi\|_{\boldsymbol{B}}, \quad x \in \mathbb{R}^d$.
Most of the time $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is isometrically invariant under translation and then this condition boils down to the simple assumption that ϕ is a pointwise multiplier of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, OR just $\phi \cdot f \in \boldsymbol{B}$, for all $f \in \boldsymbol{B}$. We have this **equivalence**:

Theorem

$$\boldsymbol{W}(\boldsymbol{B}, \ell^q_s) = \{f \in \boldsymbol{B}_{loc}, \kappa(f,\phi) \in \boldsymbol{L}^q_s(\mathbb{R}^d)\}.$$

Further key facts about Wiener Amalgams I

The discrete description using BUPUs is more convenient for proofs, while the continuous description is somehow more elegant! We can establish a couple of natural/expected facts:

1. Inclusions coordinatewise: e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^2)$;
2. If $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and finite sequences in $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $\left(\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{Y}), \|\cdot\|_{\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y})}\right)$;
3. In this situation $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{Y})' = \boldsymbol{W}(\boldsymbol{B}', \boldsymbol{Y}')$;
4. Complex interpolation applies "coordinatewise
5. Reflexivity of both $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ implies reflexivity of $\left(\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{Y}), \|\cdot\|_{\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y})}\right)$.
6. Multiplication also goes coordinatewise;
7. Convolution can also be done "coordinatewise".

The usefulness of $= S_0(\mathbb{R}^d)$, First Claims I

It is meanwhile established theory that $\mathbf{S}_0(\mathbb{R}^d) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d)$ is helping to provide an answer to such questions (arguments will follow later). We have to following results from the last 30 years of Gabor analysis. Recall the chain of inclusions

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}) \hookrightarrow (\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2).$$

Consequently all the statements to follow are valid for Schwartz functions or functions having sufficient smoothness and decay, but $\mathbf{S}_0(\mathbb{R}^d)$ is a much larger (Banach) space.
Note that for function $f \in \mathbf{L}^1(\mathbb{R})$ it is enough that a function satisfies $f', f" \in \mathbf{L}^1(\mathbb{R})$, or that it is piecewise linear, with a set of nodes of minimal distance (result of 2023). Any classical summability kernel belongs to $\mathbf{S}_0(\mathbb{R})$ (see F. Weisz).
Note also that these results are not restricted to the case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$.

The usefulness of $= S_0(\mathbb{R}^d)$, First Claims II

1. For $g \in S_0(\mathbb{R}^d)$ the family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Bessel family, hence both analysis and synthesis operators and thus the frame operator are bounded for each lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (with uniform estimates); [fezi98]
2. The dual window $\widetilde{g}$ belongs to $S_0(\mathbb{R}^d)$ by [grle04], and depends continuously on $g \in S_0(\mathbb{R}^d)$ and $\Lambda = A * \mathbb{Z}^d$. [feka04]
3. The Riemannian sums for the inversion formula converge in the sense of $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$ whenever $g \in S_0(\mathbb{R}^d)$. (F. Weisz, various papers).
4. Given $g \in S_0(\mathbb{R}^d)$ one can show that any sufficiently dense lattice Λ generates a Gabor frame. In fact [fegr89],[fezi98]

$$\lim_{\Lambda\to(0,0)} C_\Lambda S_{g,\Lambda} = \mathrm{Id}\,.$$

The Segal algebra $S_0(\mathbb{R}^d)$ as a prototype I

The definition (see [fe81-2]) of the Segal algebra $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ (also well defined for LCA groups) as Wiener Amalgam spaces as $(W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d), \|\cdot\|_{W(\mathcal{F}L^1,\ell^1)})$ is a good example of the usefulness of Wiener amalgams. We have a chain of continuous embeddings:

$$W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^1, \ell^\infty)(\mathbb{R}^d).$$

The largest space in this chain is in fact the space of pointwise multipliers of the algebra $S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$.
One inclusion follows from the pointwise relation

$$W(\mathcal{F}L^1, \ell^\infty) \cdot W(\mathcal{F}L^1, \ell^1) \subset W(\mathcal{F}L^1, \ell^1),$$

taking local and global components separately.

Wiener Amalgam in Action: Regularization

Convolution and pointwise multiplier results imply that

$$\boldsymbol{S}_0(\mathbb{R}^d) \cdot (\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad (4)$$

$$\boldsymbol{S}_0(\mathbb{R}^d) * (\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad (5)$$

Proof.

The key arguments for both of these regularization procedures, be it convolution followed by pointwise multiplication (a CP or product-convolution operator), or correspondingly PC operators, are based on the pointwise and convolutive behavior of generalized Wiener amalgam spaces, such as the relation $\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)$. Combined with the multiplier result of the last slide we are done. The second one is the Fourier version of the same claim. □

Fourier Invariance

We know that $(\boldsymbol{S_0}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S_0}}) = (\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1,\ell^1)})$ is Fourier invariant (using e.g. the characterization via STFT with Gaussian window), FT corresponds to rotation.
Obviously $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^2) = \boldsymbol{L}^2(\mathbb{R}^d)$ (with norm equivalence), and thus also is Fourier invariant by Plancherel's theorem. This implies that also $\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p,\ell^p)(\mathbb{R}^d)}\right)$ is Fourier invariant by complex interpolation for $1 \leq p \leq 2$ and subsequently by duality of $p \in [2, \infty]$.
Since $(\boldsymbol{M}^{p,p}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^{p,p}})$ is (by definition) the inverse FT of $\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p,\ell^p)(\mathbb{R}^d)}\right)$, hence equal, this implies that the spaces $(\boldsymbol{M}^p(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^p})$ form a scale of Fourier invariant Banach spaces. For $1 \leq p \leq \infty$ we have:

$$\boldsymbol{M}^1(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^p(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^\infty(\mathbb{R}^d).$$

Study of Gabor Multipliers I

One of the main reasons to study Gabor expansions is to use the signal expansion in order to manipulate signals in the sense of digital signal processing. The naive idea of just forming the spectrogram of a signal f with - say - some Gaussian window, then manipulate pixels of this (discrete) phase space "picture" of f does not work for various reasons:

- The Heisenberg uncertainty tells us that there it is not even meaningful to try to determine the level of energy at a given frequency and at a given time (cf. Dirac, Fourier basis etc.).
- Small errors (at worst on a set of measure zero) would not change the output under the given reconstruction regime.
- Actual digital sound processing should be real time!

The audio-engineer's work: Gabor multipliers

Study of Gabor Multipliers I

For the study of Gabor multipliers we are looking into expressions of the form

$$GM_m = GM_{m,g,\Lambda} = \sum_{\lambda \in \Lambda} m(\lambda) P_\lambda$$

where $P_\lambda(f) = \langle f, g_\lambda \rangle g_\lambda$ is the orthogonal projection of $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ into the one-dimensional subspace generated by $g_\lambda = \pi(\lambda)(g)$.
Anti-Wick operators are a continuous analogue of the form

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} m(\lambda) P_\lambda d\lambda.$$

One can interpret Gabor multipliers (with bounded symbols) as Anti-Wick operators, with *upper symbol* $m = \sum_{\lambda \in \Lambda} m(\lambda)\delta_\lambda$, i.e. a *weighted Dirac comb*.

Connection to Anti-Wick Operators I

The theory of STFT-multipliers, also known as Anti-Wick operators requires to make use of the Wiener amalgams. Among others we know that for $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ we have $V_g f \in \boldsymbol{L}^2(\mathbb{R}^{2d}) \cap \boldsymbol{C}_0(\mathbb{R}^{2d})$. For example, we have for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$:

$$V_g f \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^2)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \ell^2)(\mathbb{R}^{2d}).$$

In fact, not only bounded functions in $\boldsymbol{L}^\infty(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ define bounded STFT-multipliers on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but also more general multipliers define bounded operators, not only on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but also on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual.

What are Wiener Amalgams good for I

- They allow to separate local and global behaviour
- Product operators improve global behaviour
- Convolution operators improve local behaviour
- *Product-Convolution operators* improve both, and thus they are in fact compact operators
- *Shannon-type reconstruction formulas* are if the form $f \mapsto (f \cdot ⧢) * g$. For band-limited functions in $\boldsymbol{L}^p(\mathbb{R}^d)$ one has the equivalence of the usual $\boldsymbol{L}^p$-norm and that of . Hence

$$f \cdot ⧢ \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^p) \cdot \boldsymbol{W}(\boldsymbol{M}, \ell^\infty) \subset \boldsymbol{W}(\boldsymbol{M}, \ell^p).$$

Thus one has for $g \in \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$:

$$(f \cdot ⧢) * g \subset \boldsymbol{W}(\boldsymbol{M}, \ell^p) * \boldsymbol{W}(\boldsymbol{C}_0, \ell^1) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^p).$$

OUTLOOK on further Extensions I

- **Modulation spaces**, as $\mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d))$;
- The duality theory for Wiener amalgam spaces was actually realized in the context of the (more general) **decompositions spaces**, with P. Gröbner ([fegr85,fe87]), F. Voigtlaender.
- As a special case he discussed the so-called α-modulation spaces, which kind of geometrically interpolate between Besov and modulation spaces; (P.Gröbner, PhD, 1992).
- **Coorbit Theory** ([fegr88],[fegr89,fegr89-1], and [gr91]) develops the group theoretical aspect further, unifying among others also wavelet and STFT, resp. Besov and modulation spaces (different groups, different group representations, via function spaces on groups, in particular Wiener Amalgams on non-commutative groups, like $ax + b$ or Heisenberg).

Sobolev Algebras and Embedding

It is also not so difficult to show that $(\mathcal{H}_s(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_s})$, the usual Sobolev space satisfies $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{H}_s, \boldsymbol{\ell}^2)$ and to verify the Sobolev embedding theorem for $s > d/2$, making use of our Hausdorff-Young Theorem. In fact, by Cauchy-Schwarz we have $\ell^2_s(\mathbb{Z}^d) \hookrightarrow \boldsymbol{\ell}^1(\mathbb{Z}^d)$ for $s > d/2$ and thus we have

$$\mathcal{H}_s(\mathbb{R}^d) = \mathcal{F}^{-1}(\boldsymbol{W}(\boldsymbol{L}^2, \ell^2_s) \hookrightarrow \mathcal{F}^{-1}(\boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{\ell}^1)) \subseteq$$

$$\subseteq_{HY} \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^2) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^2) \subset \boldsymbol{L}^2(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d).$$

In a similar way ([fe90]) it is also not difficult to verify that the space of pointwise multipliers of $\mathcal{H}_s(\mathbb{R}^d)$ coincides with $\boldsymbol{W}(\mathcal{H}_s, \boldsymbol{\ell}^\infty)$.

Warning about Terminology I

The theory of then *Wiener-type spaces* in [fe83] (Budapest Conference of summer 1980) has been developed with the goal to produce smoothness spaces over LCA groups, where one cannot define easily dyadic partitions of unity used for the definition of Besov spaces. The idea was to decompose the Fourier transform not into *dyadic* but into *uniform* blocks. The method should apply for the full range of parameters, i.e. for $1 \leq p \leq \infty$. Hence the Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \hookrightarrow \boldsymbol{C}_0(\mathbb{R}^d)$ (by Riemann-Lebesgue) had to be a special case. Recalling the fact the smoothness of the function on the Fourier transform side (e.g. two derivatives in $\boldsymbol{L}^1(\mathbb{T})$) gives absolute summability of the Fourier coefficients) it was just a matter of abstraction to come up with the notion of BUPUs as developed in [fe83].

The control of overlaps of supports turned out to be the relevant assumptions, which is encountered even in much more general

Warning about Terminology II

situations, see "decomposition spaces" (typically over topological measure spaces, e.g. spaces of *homogeneous nature*).
Early on I had discussions about modulation spaces with Hans Triebel, but his emphasis was on $0 < p < 1$ while I found that the theory (restricted to Banach spaces, as a matter of personal choice) should be realized over LCA groups.
In recent years some authors have started to suggest to change the order in the mixed norm description of modulation spaces via the STFT (in analogy to the transition between Besov or Triebel-L. spaces), believing that they are describing new spaces.
Consequently they have to demonstrate that these spaces coincide with (very special!) Wiener amalgam spaces.

Warning about Terminology III

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Thanks!

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