Feichtinger's Algebra and Mild Distributions: A Banach Gelfand Triple

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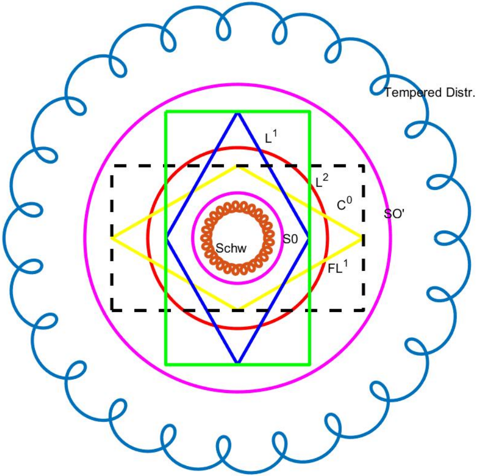
A Zoo of Banach Spaces for Fourier Analysis



The Banach Gelfand Triple for Fourier Analysis I

The historical development started from the theory of Fourier Series and has led to the development of functional analytic methods, mostly due to the fact that the Lebesgue integral allows to define Banach spaces which are suitable for the description of the Fourier Transform.

Letting the period tend to infinity the concept of the Fourier transform arose, even in the context of LCA groups, where the Lebesgue integral is based on the Haar measure. $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ is a Banach algebra with convolution, transferred into pointwise multiplication by the FT (*convolution theorem*), and appears also as the natural domain for the Fourier Transform, viewed as an *integral transform*. Hausdorff-Young tells us that $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is mapped into $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$ for $1 \leq p \leq 2$, with $1/p + 1/q = 1$.

The Banach Gelfand Triple for Fourier Analysis II

This situation has contributed a lot to the development of functional analysis and gave the Banach spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ a prominent role in the last century. Other function spaces (Banach Function Spaces in the sense of Luxemburg-Zaanen, or Function spaces in the sense of Hans Triebel) exist nowadays in abundance, and many fine estimates make use of them. Wavelet Theory gave a boost to their study, as they allow to provide discretization via unconditional bases for many of the smoothness spaces (in particular Besov-Triebel-Lizorkin spaces). Most of them can be viewed as subspaces of the space $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ of tempered distributions (or perhaps suitable spaces of ultra-distributions). By the Paley-Littlewood theory and the work of J. Peetre and subsequently Frazier/Jawerth their Fourier characterizations via dyadic decompositions plays a big role.

The Banach Gelfand Triple for Fourier Analysis III

Despite this richness none of these spaces very few of them are useful for engineers, who are dealing with time-variant (and not with periodic or decaying) signals. Signal Analysis courses often present the Dirac Delta as a mysterious generalized function (in fact bounded measure) or use Dirac combs for the derivation of the Shannon Sampling Theorem.

Gabor Theory discusses the expansion of tempered distributions or signals into double series of time-frequency shifted copies of a template, the so-called *Gabor atom*, typically a Gaussian! Regular Gabor expansions use TF-shifts along some lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, typically (for signals in $\boldsymbol{L}^2(\mathbb{R}^d)$) of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$.

For *Time-Frequency Analysis* (TFA) the STFT (Short Time Fourier Transform) plays the same role as the continuous wavelet transform, and the analogy of BTL-spaces are the so-called *Modulation Spaces* introduced around 1983 by the author.

The Banach Gelfand Triple for Fourier Analysis IV

The classical ones are denoted by $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$, with smoothness parameter $s \in \mathbb{R}$. The unweighted case (i.e. $s = 0$) has $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^{1,1}_0(\mathbb{R}^d)$ as the smallest element (also called Feichtinger's algebra, introduced in 1979) and the dual space, nowadays appearing as space of *mild distributions* as the biggest one: $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^{\infty}(\mathbb{R}^d) = \boldsymbol{M}^{\infty,\infty}(\mathbb{R}^d)$.
But it turned out that modulation spaces are not only a good tool in order to describe pseudo-differential operators, but also put classical questions into a modern light. Mild distributions can be seen as natural completion of some of the useful classical function spaces (including $\boldsymbol{S}_0(\mathbb{R}^d)$), in a TF-spirit. One can deal with **discrete and continuous, with periodic and non-periodic signals** in the realm of mild distributions. Finally, "general operators" can be described by their "kernels" in $\boldsymbol{S}_0(\mathbb{R}^{2d})$.

The Banach Gelfand Triple for Fourier Analysis V

This last statement is based on the so-called "Kernel Theorem", which is the analogue of "matrix representations" for linear mappings on $\mathbb{R}^n$. Physicists view it by interpreting the collection $(\delta_x)_{x \in \mathbb{R}^d}$ as a "continuous orthonormal basis" (although they do not belong to $\boldsymbol{L}^2(\mathbb{R}^d)$!).

In fact, we have a Banach Gelfand Triple of functions (signals, distributions) of the form $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, comparable to the situation of Sobolev spaces, also known as *rigged Hilbert spaces*. In the context of the Banach Gelfand Triple we can describe many operators (such as the Fourier Transform, the Gabor expansions, and so on), but also at the operator level their three layer viewpoint helps a lot. The center of this approach is the Kernel Theorem, which can be viewed as the extension of the characterization of Hilbert-Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$.

The Banach Gelfand Triple for Fourier Analysis VI

Fourier Analysis is an important branch of mathematical analysis, but also of our daily life, if we think of applications in signal processing or mobile communication. We transmit sound and images in a digital format, and compute Fourier transforms (at least approximately) using the FFT (Fast Fourier Transform). In my article Ingredients for Applied Fourier Analysis published in **Sharda Conference Feb. 2018**, Taylor and Francis, (2020) I described the setting of the Banach Gelfand Triple, consisting of the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ (a Banach algebra of continuous and integrable functions), the Hilbert space $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space, the space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ of so-called “mild distributions, as a possible setting for the derivation of a modern course in Fourier Analysis.

In the last 5 years this topic has been further developed, and thus a summary of the new perspectives concerning this approach can

The Banach Gelfand Triple for Fourier Analysis VII

be given. Originally this setting turned out to be useful in the context of time-frequency or Gabor analysis, because it allows to describe well the boundedness of operators involved in this theory of local Fourier analysis for non-periodic functions. Meanwhile the speaker has given courses on this subject e.g. at ETH Zuerich (which are found on YouTube, the material can be found at www.nuhag.eu/ETH20) and can thus report on different concrete building blocks for this new approach, even to questions of classical Fourier Analysis (where typically summability methods play a big role). In the talk we will address such results, e.g. in the context of the theory of Fourier multipliers, or concerning a simplified (sequential) approach to mild distributions, inspired by the Lighthill (and Temple) approach to the theory of tempered distributions in the sense of Laurent Schwartz.

Global Orientation I

This talk is (another) PERSPECTIVE talk of mine, trying to contribute to a timely interpretation of mathematical tasks related to Fourier Analysis in the modern world.

Classical (or later Abstract) Harmonic Analysis have been dealing with purely mathematical questions, such as Fourier Analysis over LCA (locally compact Abelian) groups which provides an good, qualitative framework.
One of the key-person (with his book L'integration dans les Groupes Topologiques et ses Applications. Hermann and Cie, (1940), Paris) was **Andre Weil**, who actually was a chair of Aligarh Muslim University from 1931 to 1932.
Unfortunately Lebesgue integration, or almost everywhere convergence of Fourier series (Carleson, 1972) so not play a role for engineering applications.

Abbildung: Andre Weil, Aligarh Muslim University, 1931-1932

Lack of Connection

Very unfortunately the classical tools as such are insufficient in order to deal with the problems that have to be addressed in the world of applications. Of course they form crucial building blocks for an introduction to a modern view on harmonic analysis, e.g. for *wavelet theory* of *time frequency analysis* and *Gabor Analysis*, for non-periodic and not decaying signals, like a piece of music.

I think it is in the very spirit of this conference to indicate that modern mathematical concepts are needed and provide important opportunities for relevant research work.

In my paper `Ingredients for Applied Fourier Analysis` published in the Proceedings of the Sharda Conference of Feb. 2018 , published with Taylor and Francis in 2020 p.1-22, I outline an alternative approach to Fourier Analysis, which does NOT require to first learn about Lebesgue integration or topological vector spaces leading to *tempered distributions*.

Modern Applications

There is a large variety of real-world applications of Fourier Analysis, and in fact the FFT (the Fast Fourier transform, implementing the DFT in an efficient way) is one of the backbones of the modern digital world.

In everyday life we make (mostly unconsciously) use of the FFT:

- making phone calls, exchanging messages;
- streaming music (MP3) or movies;
- taking pictures, face recognition;
- editing and filtering images;
- Scanners and MR-imaging in medicine;
- bar-codes and QR-codes, communication,...
- online conferences such as this one!

Official Abstract (for later reading)

It is the purpose of this presentation to explain certain aspects of Classical Fourier Analysis from the point of view of *distribution theory*. The setting of the so-called *Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ starts from a particular Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ of continuous and Riemann integrable functions. It is Fourier invariant and thus an extended Fourier transform can be defined for $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ the space of so-called **mild distributions**. Any of the $\boldsymbol{L}^p$-spaces contains $\boldsymbol{S}_0(\mathbb{R}^d)$ and is embedded into $\boldsymbol{S}_0'(\mathbb{R}^d)$, for $p \in [1, \infty]$.
We will show how this setting of *Banach Gelfand triples* resp. *rigged Hilbert spaces* allows to provide a conceptual appealing approach to most classical parts of Fourier analysis. In contrast to the Schwartz theory of tempered distributions it is expected that the mathematical tools can be also explained in more detail to engineers and physicists.

Function space norms

Function spaces are typically infinite-dimensional, therefore we are interested to allow convergent series. In order to check on them we need norms and completeness (in the metric sense), i.e. Banach spaces!

The classical function space norms are

- $\|f\|_\infty := \sup_{t \in \mathbb{R}^d} |f(t)|$;
- $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$;
- $\|f\|_2 := (\int_{\mathbb{R}^d} |f(x)|^2 dx)^{1/2}$;
- $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} := \sup_{\|f\|_\infty \leq 1} |\mu(f)|$, or $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} = \int_{\mathbb{R}^d} 1 d|\mu|$.
- $\|h\|_{\mathcal{F}\boldsymbol{L}^1} = \|f\|_1, \;$ for $h = \hat{f}$.

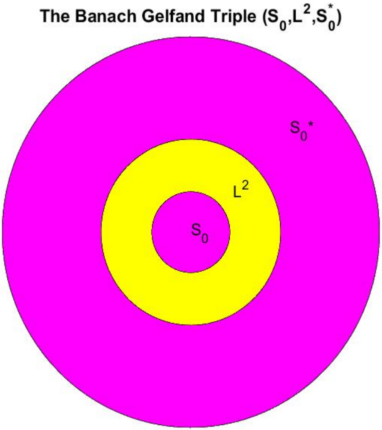


Abbildung: THE Banach Gelfand Triple

Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

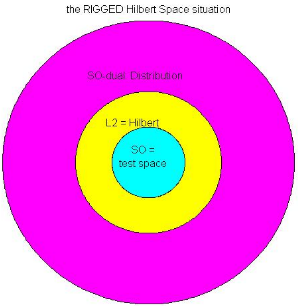


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$,

known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (lo83-1).
For $\mathbb{R}^d$ the most elegant way (which is describe in gr01 or ja18) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}dy$$

and the corresponding norm

$$\|f\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem

1. $\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow \big(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$;
2. $\mathcal{F}(\mathbf{S}_0(\mathbb{R}^d)) = \mathbf{S}_0(\mathbb{R}^d)$ *(isometrically);*
3. *Isometrically invariant under TF-shifts*

$$\|\pi(\lambda)(f)\|_{\mathbf{S}_0} = \|M_s T_t f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

4. $\big(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\big)$ *is an essential double module (convolution and multiplication)*

$$\boldsymbol{L}^1(\mathbb{R}^d) * \mathbf{S}_0(\mathbb{R}^d) \subseteq \mathbf{S}_0(\mathbb{R}^d) \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \mathbf{S}_0(\mathbb{R}^d) \subseteq \mathbf{S}_0(\mathbb{R}^d),$$

   in fact a Banach ideal and hence a double Banach algebra.

5. *Tensor product property* $\mathbf{S}_0(\mathbb{R}^d)\widehat{\otimes}\mathbf{S}_0(\mathbb{R}^d) \approx \mathbf{S}_0(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(\mathbf{S}_0(G)) = \mathbf{S}_0(H)$.

1. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ has various equivalent descriptions, e.g.
   - as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$;
   - via *atomic decompositions* of the form

$$f = \sum_{i \in I} c_i \pi(\lambda_i) g \text{ with } (c_i)_{i \in I} \in \ell^1(I).$$

2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under group automorphism;
3. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under the *metaplectic group*, and thus under the *Fractional Fourier transform* as well as the multiplication with *chirp signals*: $t \mapsto exp(-i\alpha t^2)$, for $\alpha \geq 0$.

In addition $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is quite universally useful in Classical Fourier Analysis and of course for *Time-Frequency Analysis* and *Gabor Analysis*, and as I am going to show also for QHA: Quantum Harmonic Analysis. In short, it is easier to handle than the Schwartz-Bruhat space or even the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, and since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ it is (much) bigger.

Theorem

1. $\mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$ *via* $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0'$.
2. *Identification of TLIS:* $\boldsymbol{H}_G(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$, $(\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \to \boldsymbol{S}_0(H) \hookrightarrow \boldsymbol{S}_0(G)$ *via* $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \boldsymbol{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \boldsymbol{S}_0(G) \,|\, \mathrm{supp}(\tau) \subset H\}$.

Theorem

1. $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}) = (\mathbf{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^\infty})$, *with* $V_g(\sigma)$ *and* $\|\sigma\|_{\mathbf{S}_0'} = \|V_g(\sigma)\|_\infty$, *hence norm convergence corresponds to uniform convergence on pahse space. Also* w^*-*convergence is uniform convergence over compact subsets of phase space.*
2. $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}) \hookrightarrow (\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$, *with density for* $1 \leq p < \infty$, *and* w^*-*density in* $\mathbf{S}_0'$. *Hence, facts valid for* $\mathbf{S}_0$ *can be extended to* $\mathbf{S}_0'$ *via* w^*-*limits.*
3. *Periodic elements* $(T_h\sigma = \sigma, h \in H)$ *correspond exactly to those with* $\tau = \mathcal{F}(\sigma)$ *having* $\operatorname{supp}(\tau) \subseteq H^\perp$.
4. *The (unique) spreading representation* $T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda) d\lambda$, $F \in \mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *for* $T \in \mathcal{B}$ *extends to the isomorphism* $T \leftrightarrow \eta(T)$ $\eta : \mathcal{B} \approx \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$, *uniquely determined by the correspondence with*
$$\eta(\pi(\lambda)) = \delta_\lambda, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Some conventions

Scalar product in $\mathcal{HS}$:

$$\langle T, S \rangle_{\mathcal{HS}} = \text{trace}(T * S^*)$$

In feko98 the notation

$$\alpha(\lambda)(T) = [\pi \otimes \pi^*(\lambda)](T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

and the covariance of the KNS-symbol is decisive:

$$\sigma(\pi \otimes \pi^*(\lambda)(T)) = T_\lambda(\sigma(T)), \quad T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

;

AND $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, or more generally $(\boldsymbol{H}_w^1, \boldsymbol{H}, \boldsymbol{H}_w^{1\,\prime})$ (in the context of *coorbit spaces*.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

Behavior under Fourier transform

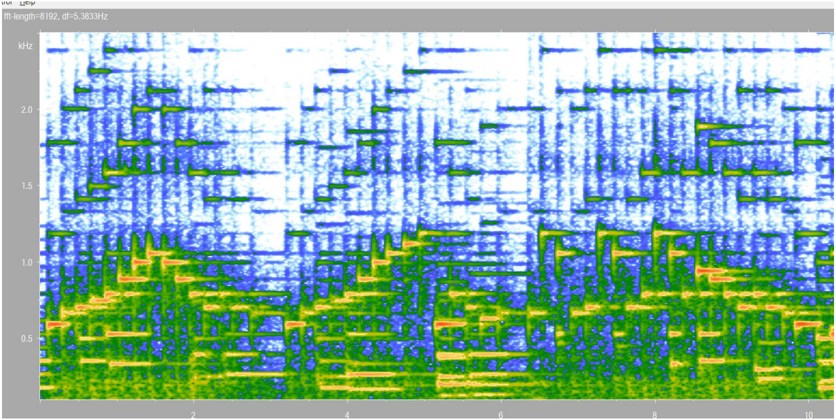
$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

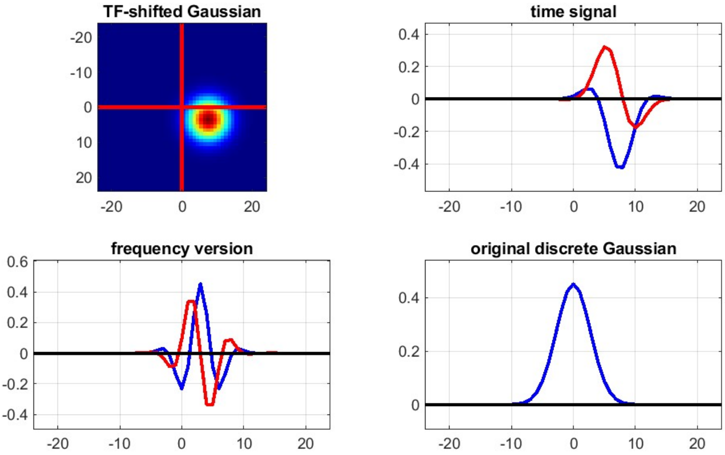
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

A typical piano spectrogram (Mozart), from recording





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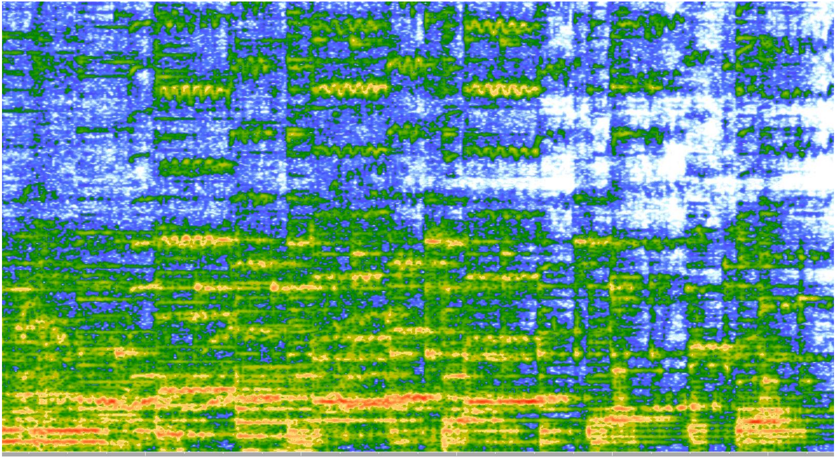
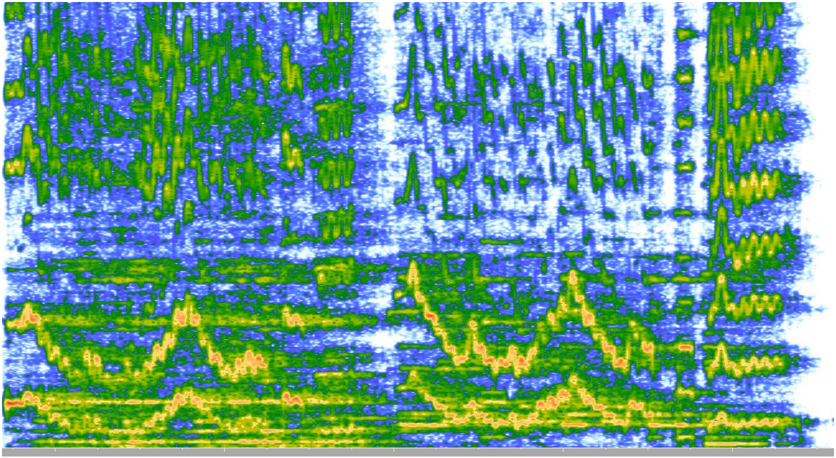
A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of V_g is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$. Assuming $\|g\|_2 = 1$ we have the *reconstruction formula*:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda)g,$$

which can be approximated in $\boldsymbol{L}^2$ by Riemannian sums.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

Our Aim: Popularizing Banach Gelfand Triples

According to the title I have to first explain what **Banach Gelfand Triples** are, with the specific emphasis on the BGTr $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, arising from the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (as a space of test-functions), alias the *modulation spaces* $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$.

Hence I will describe them, provide a selection of different characterizations (there are *many of them!*) and properties.

Finally I will come to the main part, namely applications or *use of this* (!natural) concept in the framework of **classical analysis**.

Mild Distributions: Convergence I

The abstract definition of a dual space means that one looks at all the bounded, linear mappings from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into the field of complex numbers, i.e., $\sigma \in \boldsymbol{S}_0'$, or σ is a mild distribution means: There exists $C = C(\sigma) > 0$ such that

$$f \mapsto \sigma(f), \quad \text{with} \quad |\sigma(f)| \le C\|f\|_{\boldsymbol{S}_0}, \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

While this gives the impression that mild distributions are "linear measurements" applied to functions one should better view them as "signals which can be measured", not by taking point values, but arising from profiles found in the vector space $\boldsymbol{S}_0(\mathbb{R}^d)$ to them. So we should look at the (equally) linear mapping induced by $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ on the space of "signals" (called mild distributions):

$$\sigma \mapsto \sigma(f) = M_f(\sigma) \quad \text{and} \quad |M_f(\sigma)| \le \|f\|_{\boldsymbol{S}_0}\|\sigma\|_{\boldsymbol{S}_0'}.$$

Mild Distributions: Convergence II

Since it is easy to show that any smooth function with compact support, or even any (product of) piecewise linear functions which is continuous belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$, and also that multiplication of $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with a *pure frequency* leaves the norm invariant, we can form for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$
Simple examples of elements in $\boldsymbol{S}_0'(\mathbb{R}^d)$ are the following ones

1. Dirac measures: $\delta_x : f \mapsto f(x)$;
2. regular distributions: $h \in \boldsymbol{C}_b(\mathbb{R}^d), \sigma_h(f) = \int_{\mathbb{R}^d} f(x)h(x)dx$;
3. Dirac combs: $⊔⊔_\alpha = \sum_{k\in\mathbb{Z}^d} f(\alpha k)$; for some $\alpha > 0$;
$(⊔⊔ = ⊔⊔_1 = \sum_{k\in\mathbb{Z}^d} \delta_k)$.

The mapping $f \mapsto \sigma_k$ defines an embedding of $\boldsymbol{C}_b(\mathbb{R}^d)$ (hence of $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$) which is continuous, thus ordinary functions can be viewed as "generalized functions" in the above sense.

Mild Distributions: Convergence III

Obviously δ_X or $\sqcup\!\sqcup_\alpha$ is *not a regular distribution*, but can we approximate a mild distribution $\sigma \in \boldsymbol{S}_0'$ by functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ (which are after all Riemann integrable!)?
The answer is given using the concept of w^*-convergence:

Definition

A sequence $(\sigma_n)_{n\geq 1}$ in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is w^*-convergent with limit $\sigma_0 \in \boldsymbol{S}_0'(\mathbb{R}^d)$ if one has for any $f \in$

$$\lim_{n\to\infty} \sigma_n(f) = \sigma_0(f), \quad f \in \boldsymbol{S}_0.$$

The so-called *Banach-Steinhaus Theorem* (from Functional Analysis) implies: it is enough to know that the limit on the left hand side exists for any (!) $f \in \boldsymbol{S}_0$, then it *defines already* a unique mild distribution $\sigma_0 \in \boldsymbol{S}_0'(\mathbb{R}^d)$.

Mild Distributions: Convergence IV

Theorem

$S_0'(\mathbb{R}^d)$ is w^*-dense in $S_0'(\mathbb{R}^d)$, i.e. for any σ_0 there exist sequences $(f_n)_{n\geq 1}$ in $S_0(\mathbb{R}^d)$ such that

$$\sigma_0 = w^*\text{-}\lim{}_{n\to\infty}\sigma_{f_n}.$$

Rather argument!

One possible proof is obtained by applying a so-called regularization process. Let us just consider the Dirac-comb $\sqcup\!\sqcup$, which is neither decaying at infinity nor represented by any continuous or even smooth function. Hence one has to "localize" it (first) and the "smooth it out" (by convolution with some Dirac-like, compressed Gaussian, for example). □

Mild Distributions: Convergence V

This fact opens the way to introduce $\boldsymbol{S}_0'(\mathbb{R}^d)$ in a different (not functional analytic) way. We can show that provides an alternative description of the same vector space of "generalized function" (mild distributions), following the general scheme of *completion*, but now in the sense of *mild convergence*.

One may compare this with the idea that an image is not function defined almost everywhere constituting an element of $\boldsymbol{L}^2(\mathbb{R}^2)$ but rather that one should identify it with the collection of all possible pixel images taken with whatsoever digital camera!

Of course in practice it is enough to go up to a certain resolution, very much as we are happy to know that $\pi \equiv 3.14....$.

Mild Distributions: Convergence VI

Lemma

One can identify elements σ of the vector space $\boldsymbol{S}_0'(\mathbb{R}^d)$ (containing $\boldsymbol{S}_0'(\mathbb{R}^d)$ as a subspace of regular distributions) with the corresponding sequence (or net) of regularized version of $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$.
Conversely, one can form the set of equivalence classes of (equiconvergent) limits of w^*-convergent sequences from $\boldsymbol{S}_0(\mathbb{R}^d)$. Even with the natural (inf-)norm this defines an isomorphism of normed spaces[a]

[a] This can be compared with the idea that the real numbers of the same as the equivalence classes of *Cauchy-sequences* of rational numbers. For details see the paper: "A sequential approach to mild distributions", Axioms, Vol.9 No.1, (2020) p.1-25.

Typical Heuristic Arguments I

Typical examples of w^*-convergence are often related to *heuristic arguments* in the engineering applications (or in physics).

1. We have $\delta_0 = w^*\text{-}\lim_{\alpha\to\infty} ⧢_\alpha$.
2. Correspondingly we for any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ one has

$$f = w^*\text{-}\lim_{\alpha\to\infty} ⧢_\alpha * f = \sum_{n\in\mathbb{Z}^d} f(x - n\alpha),$$

which is often used to "derive the validity of the Fourier Inversion Theorem for functions in $\boldsymbol{L}^1(\mathbb{R}^d)$ or $\boldsymbol{L}^2(\mathbb{R}^d)$
from the use of Fourier series.

Typical Heuristic Arguments II

3. We also have for any $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ (or even $f \in \boldsymbol{C}_b(\mathbb{R}^d)$): the regular distribution σ_h is can be approximated by discrete measures (weighted Dirac combs)

$$\sigma_h(f) = w^*\text{-}\lim_{\alpha \to 0} [\alpha^d \cdot ⊔⊔_\alpha] \cdot h(f) = \lim_{\alpha \to 0} \alpha^d \sum_{k \in \mathbb{Z}^d} h(\alpha k) f(\alpha_k).$$

Checking the technical details reveals that this is nothing but the claim that for any $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ the pointwise product is a Riemann integrable function!

Specifically for $h(x) \equiv 1$ and writing $\beta := 1/\alpha$ we get

$$\mathbf{1} = w^*\text{-}\lim_{\beta \to 0} \beta^{-d} ⊔⊔_{1/\beta}$$

Another intuitive description of w^*−convergence is to comes.

Why should be work with Banach spaces? I

It is fair to say that the purpose of function spaces is to allow a description of operators or approximation procedures. Living in a world of "signals", more precisely working with a vector space of functions or (mild) distributions over $\mathbb{R}^d$, which is to large to be spanned by a finite collection of *basis vectors*, we often talk in a sloppy way of an *infinite dimensional vector space*. In this situation it is unavoidable to measure the size of vectors (i.e., to introduce norms) in order to define convergence and allow infinite sums (series). Therefor, and for many reasons it is quite useful to ask for completeness of such normed spaces, meaning to work with *Banach spaces* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, sometimes with *Hilbert spaces* endowed with some scalar product.

The generalized Fourier transform I

The setting of THE Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ allows to describe the Fourier transform and other linear mappings of relevance in an elegant, functional analytic way. For example one can DEFINE

$$\widehat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \boldsymbol{S}_0.$$

This is justified by observing that for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the *Fundamental Identity for the Fourier Transform* we have

$$\int_{\mathbb{R}^d} \widehat{f}(y) g(y) dy = \int_{\mathbb{R}^d} \widehat{g}(x) f(x) dx,$$

(justified by Fubini's Theorem), or in short (!compatibility)

$$\widehat{\sigma_g} = \sigma_{\widehat{g}}, \quad g \in \boldsymbol{S}_0(\mathbb{R}^d).$$

The generalized Fourier transform II

The sequential approach to mild distributions (in the style of Lighthills treatment of tempered distributions, as described in the work of Laurent Schwartz) allows an alternative description of the extended version of the Fourier transform (as introduced above), avoiding the existence of integrals, and widening the scope of Fourier Analysis (away from spaces of locally Lebesgue-integrable functions and pointwise considerations!).

One can show (not done here) that the standard properties of the ordinary Fourier transform defined as usual (but only requiring Riemann integrals) as

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle s,x \rangle} dx, \quad f \in \boldsymbol{S}_0$$

The generalized Fourier transform III

maps *equivalence classes of Cauchy sequences* into equivalence classes, and since it coincides with the ordinary FT for "constant sequences" (in $\boldsymbol{S}_0(\mathbb{R}^d)$) it is a natural extension of the classical FT! Abstractly speaking, we can say: it is the unique extension of the ordinary FT, which respects w^*-convergence of sequences.

As an application one can show that there are two very important elements in $\boldsymbol{S}_0'(\mathbb{R}^d)$ which are Fourier *invariant*, namely

1. The Dirac comb, noting that $\widehat{\sqcup\!\sqcup} = \sqcup\!\sqcup$ is just the same as the validity of Poisson's formula for all $f \in \boldsymbol{S}_0$;
(> Shannon Sampling Theorem > CD-player!)
2. The chirp signal $\chi(t) = \exp(-i\pi t^2)$

Both examples describe cases of *double transformable measures*.

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ I

As already indicated, the space $\mathbf{S}_0'(\mathbb{R}^d)$ can embedded into $\mathcal{S}'(\mathbb{R}^d)$, the space of *tempered distributions*, because $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$, as a dense subspace.
Defining the STFT (Short-Time Fourier Transform) of a distribution, using a *window* $g \in \mathcal{S}(\mathbb{R}^d)$ by

$$V_g(\sigma)(t,s) := \sigma(\overline{M_s T_t g}), (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

One can show: A distribution is a *mild distribution* if and only if $V_g(\sigma)$ is not only continuous (with polynomial growth), but also bounded! In fact, this is then true for any $g \in \mathbf{S}_0(\mathbb{R}^d)$.
The abstract norm on $\mathbf{S}_0'(\mathbb{R}^d)$ coincides with the supremum of $V_g(\sigma)$ over the TF-plane, and norm convergence for $(\sigma_n)_{n\geq 1}$ in $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is thus just uniform convergence of the corresponding spectrograms $V_g(\sigma_n)$ to $V_g(\sigma_0)$.

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ II

Note that in this context different non-zero functions in $S_0(\mathbb{R}^d)$ define one of the many equivalent norms. Often one takes the Gaussian function $g_0(t) = exp(-\pi|t|^2)$, because it satisfies $\widehat{g_0} = g_0$, and thus allows to argue that $|V_g(\widehat{\sigma})|$ is just a rotated version of $|V_g(\sigma)|$. We fix one such non-zero g for the rest.
It is thus plausible, or a practical description of w^*-convergence for mild distributions to make use of the following simple-minded characterization:

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ III

Lemma

A sequence $(\sigma_n)_{n\geq 1}$ in $\mathbf{S}_0'(\mathbb{R}^d)$ is convergent in the w^*-sense if and only if one can observe uniform convergence of $V_g(\sigma_n)$ over compact set, i.e.
Given $R > 0$ and $\varepsilon > 0$ there exists n_0 such that

$$|V_g(\sigma_n)(t,s) - V_g(\sigma_0)(t,s)| < \varepsilon, \quad |t|,|s| < R.$$

Usually I tell the story about what kind of information is stored on a CD (at the rate of 44100 samples per second): It is just a (very reasonable) w^*-approximation of the signal which occurred!
Another more mathematical observation is the following one:
Since we have $\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0'(\mathbb{R}^d)$ we can consider finite dimensional vector spaces $\mathbf{V} \subset \mathbf{S}_0(\mathbb{R}^d) \subset \mathbf{S}_0'(\mathbb{R}^d)$ which are

Coming back to w^*-convergence in $S_0'(\mathbb{R}^d)$ IV

generated by a finite collection of (linear independent) elements in $\boldsymbol{S}_0(\mathbb{R}^d)$. We can say, that this are the subspaces that we can really control, which are not too big, and which can be described using methods from linear algebra.
Functional Analysis helps us to demonstrate that for any such space $\boldsymbol{V}$ there exists some projection $P_{\boldsymbol{V}}$ from $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ onto $\boldsymbol{V}$. Hence it is nice to realize that one has:

Lemma

A sequence $(\sigma_n)_{n\geq 1}$ in $\boldsymbol{S}_0'(\mathbb{R}^d)$ is convergent in the w^*-sense if and only on has norm convergence in $\boldsymbol{S}_0(\mathbb{R}^d)$ (or equivalently in $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ or in $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$) for any projection, i.e.

$$\lim_{n\to\infty} \|P_{\boldsymbol{V}}(\sigma_n) - P_{\boldsymbol{V}}(\sigma_0)\|_{\boldsymbol{S}_0} = 0.$$

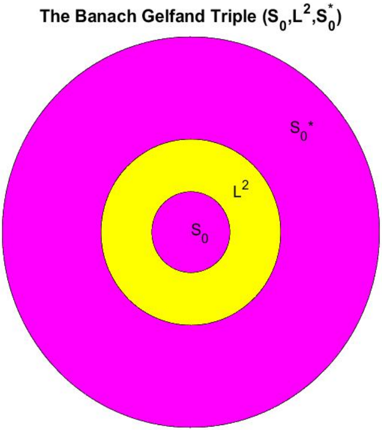


Abbildung: THE Banach Gelfand Triple

Usefulness of the Three Layer Situation

In contrast to the usual setting, where $\mathcal{L}(\mathcal{H})$, the space of bounded linear operators on a given Hilbert space appears to be the universe the Three Layer Situation allows to differentiate.

Important operators are often bounded on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and extend to the dual space, hence the are bounded (maybe even unitary or at least isometric) on the *Hilbert space* in the middle (by complex interpolation).

But we have *quality improving operators* which map the outer layer into the inner one, and constitute $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$. Among them the $w^*-$to-norm continuous ones and even better the w^*-nuclear ones are of particular interest.

On the other hand (complementary viewpoint) we have operators which might be not well defined on $\boldsymbol{L}^2(\mathbb{R}^d)$ but still map $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$ (of course then also $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, but with some disadvantages!).

Examples of non-trivial operators

Even in the classical setting there are many operator which are not bounded on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but still reasonable, say Fourier multipliers from $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ to $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$, for some pair of parameters with $1 \leq p < q < \infty$.
Gabor Analysis provides another setting, where this observation was made first: While the STFT is well defined (and even isometric, if suitably normalized) for windows g and signals f in $\boldsymbol{L}^2(\mathbb{R}^d)$ and also continuous, the restriction to a lattice (say $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \in \mathbb{R} \times \widehat{\mathbb{R}}$) need not be bounded mapping (for fixed g) from $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ to $(\mathbb{Z}^2)$!
There are two ways out: Either one assume that the window is in $\boldsymbol{S}_0(\mathbb{R})$ (so not the box-car function, but any classical summability kernel works) then it is a continuous mapping of into $\ell^2(\Lambda)$ of the form $f \mapsto (V_g f(\lambda))_{\lambda\in\Lambda}$. Or we view it as a mapping from $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $\ell^2(\Lambda)$ (for general $g \in \boldsymbol{L}^2(\mathbb{R})$).

Periodization and Sampling

Probably more important are the operators related to Shannon's Sampling Theorem. Via mild distributions it is easy to demonstrate that the sampling operator (multiplication with a Dirac comb) is a bounded operator mapping $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, which can be approximately inverted (only on $\boldsymbol{S}_0(\mathbb{R}^d)$!) by making the sampling dense enough, e.g. by piecewise linear interpolation or cubic spline quasi-interpolation ($>>$ BUPUs).
The proof is showing that sampling on the time side corresponds to periodization on the frequency side, and vice versa. Their combination provides a sampled and periodized in $\boldsymbol{S}_0'(\mathbb{R}^d)$. Of course sampling and period have to match well!Approximate reconstruction then is at the basis of a very practical question, such as: How can we approximate the FT (on $\boldsymbol{S}_0(\mathbb{R}^d)$!) by using FFT methods and subsequent interpolation?

The usefulness of $(S_0, L^2, S_0')(\mathbb{R}^d)$ in Gabor Analysis I

Although it is not surprising that $S_0(\mathbb{R}^d)$ and its dual are useful in Time-frequency Analysis, in particular for Gabor Analysis, the number of results where this setting is appropriate, is astonishing. Recall that Gabor Analysis deals with the exact reconstruction of a signal from the spectrogram, resp. the representation of a mild distribution as a (in fact w^*-convergent) double series of TF-shifted copies of some Gabor atom (from $S_0(\mathbb{R}^d)$, of course). Any classical summability kernel belongs to $S_0(\mathbb{R}^d)$!
Given a certain Gabor series expansion (often one requires that it is a tight Gabor frame) of the form

$$f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda .$$

The usefulness of $(S_0, L^2, S_0')(\mathbb{R}^d)$ in Gabor Analysis II

A so-called *Gabor multiplier* is then a linear operator which arises simply by multiplying the coefficients with some numbers $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$, the *upper symbol* of GM_m, given by

$$GM_m(f) = \sum_{\lambda\in\Lambda} m_\lambda \langle f, g_\lambda\rangle g_\lambda.$$

It is not hard to show that any bounded sequence $\mathbf{m} \in \ell^\infty(\Lambda)$ defines a bounded linear operator not only on $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but also on $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ and $\left(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'}\right)$, which is also w^*-w^*-continuous. Similar more involved methods allow to describe *pseudo-differential operators*, and in particular so-called Anti-Wick operators (continuous STFT-multipliers) in this setting.

Operating on the audio signal: filter banks

There are many more applications

The list of topics which we could not discuss here is long, we just mention very few of them:

1. A kernel theorem, allowing a kind of "continuous matrix representation" of linear operators from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.
2. Correspondingly description of operators using the spreading representation or equivalently the Kohn-Nirenberg calculus;
3. Using the FFT to compute $\widehat{f}$ from samples of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, followed by e.g., piecewise linear interpolation;
4. more general **modulation spaces**, such as , where $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ arises for $p = 1 = q$, and $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ corresponds to the case of $p = \infty = q$.
5. Wilson bases, coefficients in mixed norm sequence spaces.

Physics Nobel Prize 2017 (Jarnick Lecture in Prague)

Time-Frequency Analysis and Black Holes

Breaking News

Today, Oct. 3rd, 2017, the **Nobel Prize in Physics** was awarded

to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!

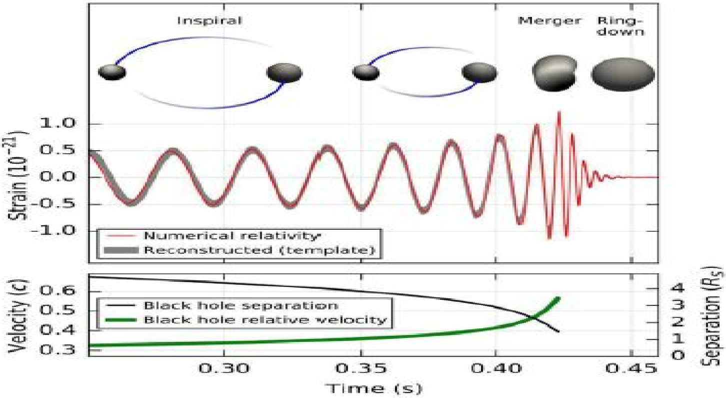
The Prize-Winners are

Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".
There had been two strategies:

- Searching for 2500 explicitly determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.
NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).