NuHAG Numerical Harmonic Analysis Group

Discrete versus Continuous Time-Frequency (and Gabor) Analysis

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at **www.nuhag.eu**

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A contribution to
Conceptual Harmonic Analysis

ABSTRACT I

The regular sampling problem is usually related to classical work of Shannon, who has shown that band-limited functions in $L^2(R)$ can be reconstructed from their regular samples, if the Nyquist criterion is satisfied. There are many variations to this theme, taking other function spaces into account, which allow to grasp better the locality of the reconstruction, or even reconstruct from irregular samples, using iterative methods. For me *Wiener amalgam spaces* are the natural setting, because they provide methods to control the convergence in different function spaces or control jitter errors. The early work on sampling by Paul Butzer clearly shows the close connection to Poisson's formula. It results in the statement that the Dirac comb (viewed as a mild distribution) has the Dirac comb of the dual lattice as its Fourier transform. Thus sampling on the time side corresponds to periodization on the frequency side. For

ABSTRACT II

this way of looking at sampling we may use the Banach Gelfand triple $(SO, L2, SO^*)$, which arose in time-frequency analysis. The talk will discuss a few topics closely connected to the question whether a function in $SO(R)$ can be approximately reconstructed from regular samples, if they are taken over a sufficiently fine grid and over a sufficiently large interval. The approach takes a combined periodization sampling operator as the starting point and starts the analysis by looking at its spreading representation. Obviously it can be viewed as a mapping from $SO(R)$ into C^N (for sufficiently large N), or equivalently as a bounded linear mapping from $SO(R)$ into $SO^*(R)$. A corresponding paper is under preparation.

Shannon Sampling for Band-limited Functions

Shannon's Theorem states that one can reconstruct any band-limited function f with $\mathrm{supp}(\widehat{f}) \subseteq I = [-1/2, 1/2]$ from regular samples taken at the Nyquist rate (equal to $1 = 1/|I|$ with our normalization), using the well-known SINC-series:

$$f = \sum_{k \in \mathbb{Z}} f(k) T_k \, \mathrm{SINC}\,.$$

For better locality, or even just to have convergence in $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ for band-limited functions in $\boldsymbol{L}^1(\mathbb{R})$ one better chooses some $a < 1$ and $\varphi \in \boldsymbol{L}^1(\mathbb{R})$ with $\widehat{\varphi}(y) \equiv 1$ on I and $\mathrm{supp}(\widehat{\varphi}) \subseteq [-1/2a, 1/2a]$, thus giving

$$f = \sum_{k \in \mathbb{Z}} f(ak) T_{ak} \varphi,$$

with convergence taking place in $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ and uniformly.

Sampling corresponds to periodization on the FT side

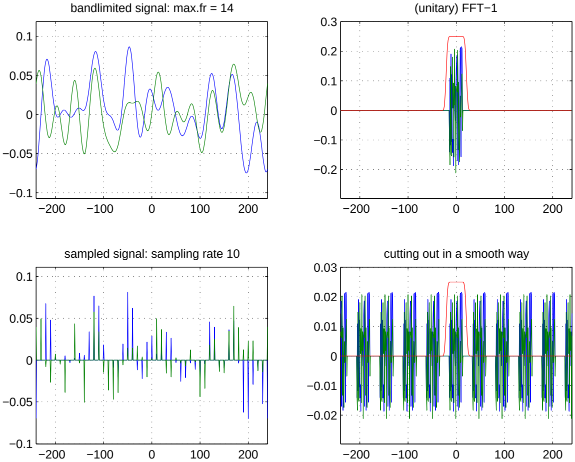


Figure: If there is a bit of oversampling, one can choose a better localized reconstruction atom (than SINC).

Distributional FT and FFT

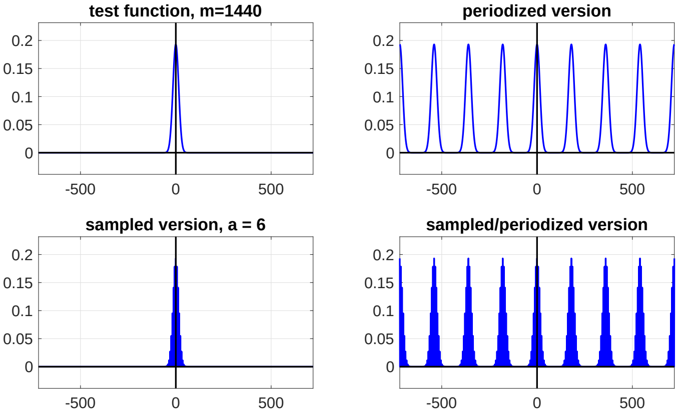


Figure: original(top left), periodize (top right),
or sample (left lower corner), or both (right lower corner).

The Spline Quasi-interpolation operators I

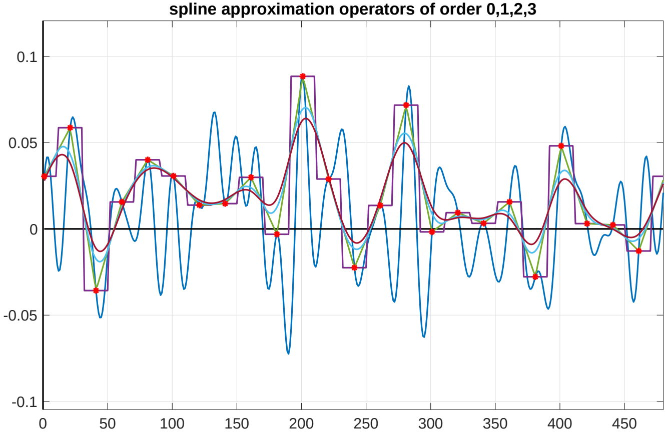


Figure: Approximation by spline functions of order 1, 2, 3, 4

Recovery from samples I

The information contained in the samples of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ is getting more and more. We know, that for any $f \in \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ one has norm convergence of $\mathrm{Sp}_\Psi f$ to f with respect to the sup-norm, as $|\Psi| \to 0$ (think of piecewise linear interpolation over $\mathbb{R}$).
Since the sequence of compressed triangular functions $\mathrm{St}_\rho\Delta, \rho \to 0$ forms a Dirac family one may expect that

$$\mathrm{St}_\rho\Delta * [\alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k)\delta_{\alpha k}] \to \delta_0 * (\mathbf{1}\cdot f) \to f, \quad f \in \boldsymbol{S}_0. \qquad (1)$$

We have to make two observations: First of all this is in fact an alternative description of piecewise linear interpolation, since $\mathrm{D}_{1\alpha}\Delta$ is just a triangular function with basis $[-\alpha, \alpha]$.

$$\mathrm{St}_\rho\Delta * [\alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k)\delta_{\alpha k}] = \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\mathrm{D}_\rho(\Delta). \qquad (2)$$

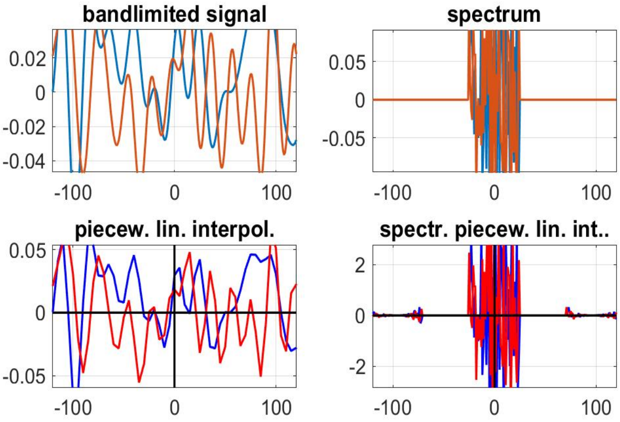


Figure: piecewise linear interpolation of a smooth, i.e. here band-limited signal (partial zoom in, to show the lack of smoothness).

Theorem

Assume that $\Psi = (T_k\psi)_{k\in\mathbb{Z}^d}$ *defines a BUPU in* $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$[a] *and write* $\mathrm{D}_\rho\Psi$ *for the family* $\mathrm{D}_\rho(T_k\psi) = (T_{\alpha k}\mathrm{D}_\rho\Delta)_{k\in\mathbb{Z}^d}$, *with* $\alpha = 1/\rho \to 0$. *Then* $|\mathrm{D}_\rho\Psi| \leq r\alpha \to 0$ *for* $\alpha \to 0$, *and*

$$\|f - \alpha^d \sum_{k\in\mathbb{Z}^d} f(\alpha k) T_{\alpha k}\,\mathrm{St}_\alpha\,\psi\|_{\mathbf{S}_0} \to 0, \quad \text{for } \alpha \to 0, \forall f \in \mathbf{S}_0. \quad (3)$$

[a] As it is required for the definition of $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$, for some $\psi \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ with $\mathrm{supp}(\psi) \subset B_r(0)$.

This result was the cornerstone for the subsequent result published by Norbert Kaiblinger, concerning the approximate computation of the Fourier transform of a "nice function" $f \in \mathbf{S}_0(\mathbb{R}^d)$ via an FFT routine, applied to a sequence of samples of the original function! (a claim which is "obvious" to engineers).

Historical Aspects of Fourier Analysis

The standard approach to Fourier Analysis reads like this:

1. classical Fourier series (periodic functions)
2. Fourier transform (Euclidean case);
3. tempered distributions (L.Schwartz);
4. computational aspects (FFT);

The tools required are:

1. Riemann and Lebesgue integral;
2. Hilbert spaces, Lebesgue spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$;
3. topological vector spaces, distributions
4. Besov-Triebel-Lizorkin spaces, etc.

Official Abstract I

The Regular Sampling Problem
seen from a Time-Frequency Perspective
We are going to study the periodization-sampling operators via the use of Gabor expansions. In this way we can guarantee the quality of the reconstruction of a function, but also of its Fourier transform, via the consideration of their samples, taken in a regular fashion.

TRIPLE OF OBJECTS

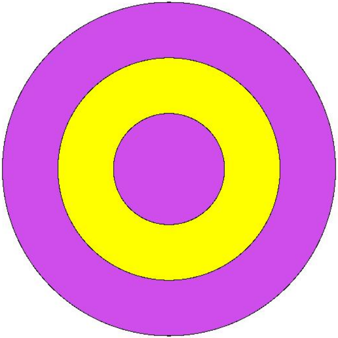


Figure: Note: Could be $\mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$

THE Banach Gelfand Triple

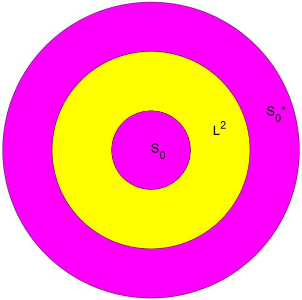


Figure: Banach Gelfand Triple, starting from $\boldsymbol{S}_0(\mathbb{R}^d)$, with Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and outer space $\boldsymbol{S}_0'(\mathbb{R}^d)$, of mild distribution.

The Schwartz Gelfand Triple

The usual triple is the *Schwartz Gelfand Triple*, consisting of the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ of *rapidly decreasing functions*, the same Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and the dual space, the space of *tempered distributions*.

In this case the underlying algebra of test functions is a *nuclear Frechet spaces*. Generally speaking it is believed that this enables the proof of a so-called *kernel theorem*, but this is not true in the literal sense, only the construction using projective and injective tensor products requires such a property.

Generally speaking *rigged Hilbert spaces* work with such a type of (topological) Gelfand Triples, e.g. for the construction of spaces of ultra-distributions (Beurling or Roumieu-type, according to the work of S. Pilipovic and coauthors).

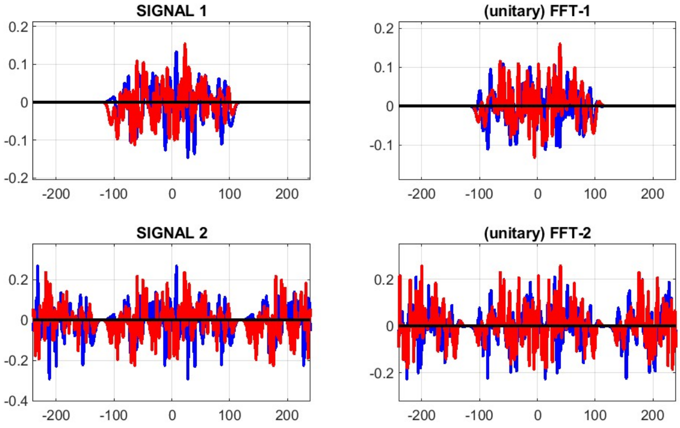


Figure: persampf2spA.jpg

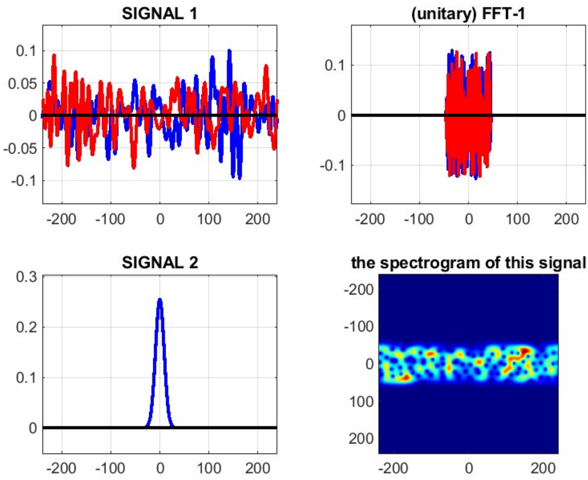


Figure: A low pass signal, with spectrogram

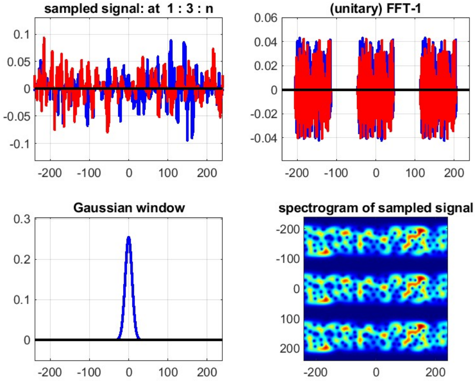


Figure: Effect of sampling in the spectrogram

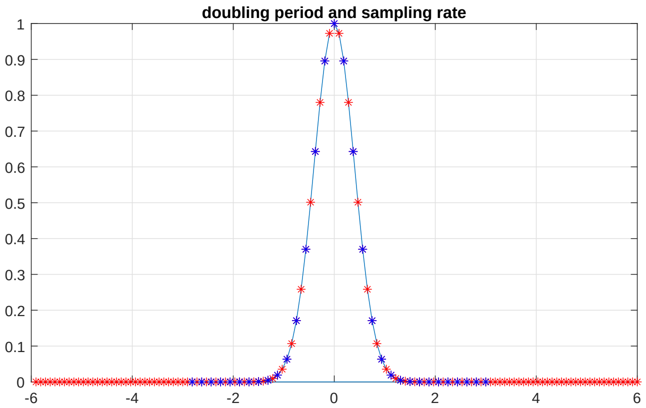


Figure: Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ I

It is a harmless but important observation that the squares of the elements of $\mathbb{Z}_n$ (rotation by multiples of $2\pi/n$) are just the elements of $\mathbb{Z}_{n/2}$ (only for n is even!), repeated twice.
Thus for us the operator which replaces a given function (or matrix) by its 2-periodic and 2-sampled version will be of big relevance. Also, since all the information comes twice (for matrices in both the world of column AND the world of rows) we have to understand how to extract properly the subsequence of indices "most representative" for such a reduction (turning vectors of length n into vectors of length $n/2$) or just of length $2n$ into vectors of length n and matrices of size $2n \times 2n$ into matrices of size n, in a compatible way.
We will illustrate this by some plots and also verify that this procedure is well compatible with many of the representations of functions of operators.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ II

As a basic example let us take a function with small support, then produce its p-periodic version, and then sample at the rate of $1/p$, $p \in \mathbb{N}$. Then you will find that the "representing sequence" of the Fourier version of such a function, treated in the same way, will be just (suitable normalized) the FFT of the finite vector (of length p^2, of course) of the vector in $\mathbb{C}^{p^2}$ representing the discrete and periodic signal on $\mathbb{R}$.

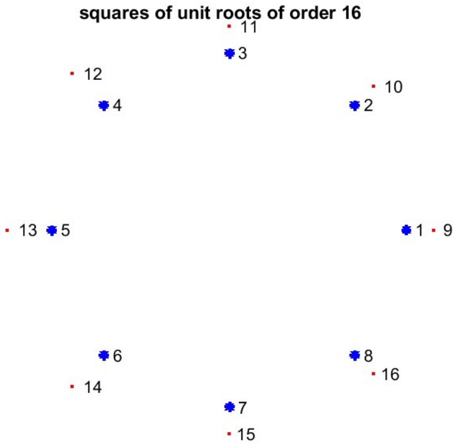


Figure: twiceZn16A.jpg

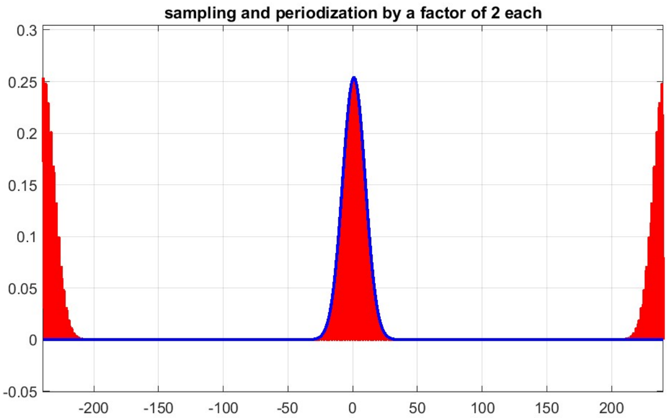


Figure: The reduction from the original curves (in blue) to the red curve is by sampling. Since every second value is zero the graph looks filles, plus periodic repetition.

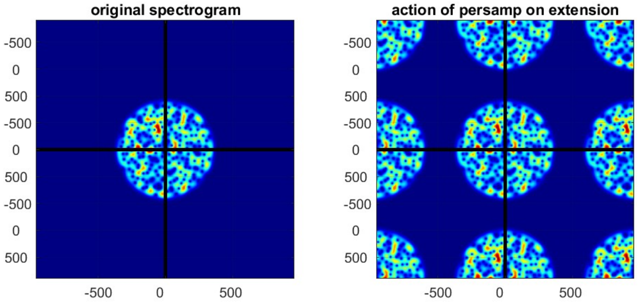


Figure: The picture in the STFT domain. Actually, it is the (twisted) convolution with a 2D Dirac comb with 4 points, so it is a 2D-periodization of the spectrogram.

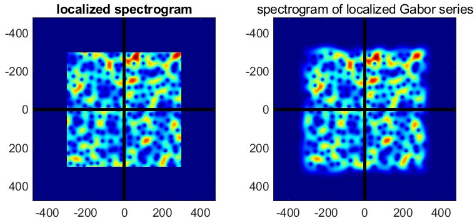


Figure: localGabSpec25A.jpg

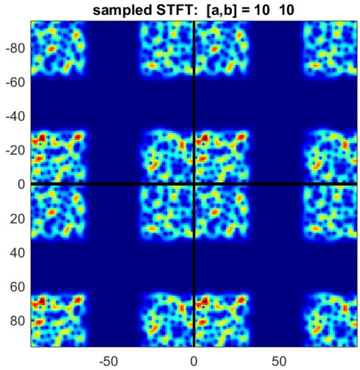


Figure: localGabSpec25B.jpg

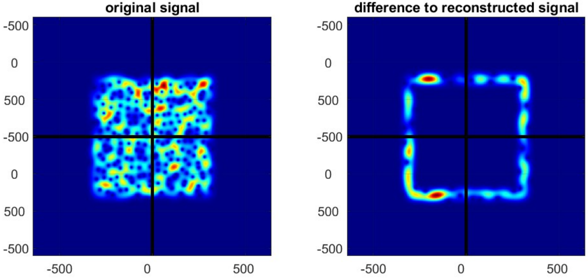


Figure: localGabSpec25C.jpg

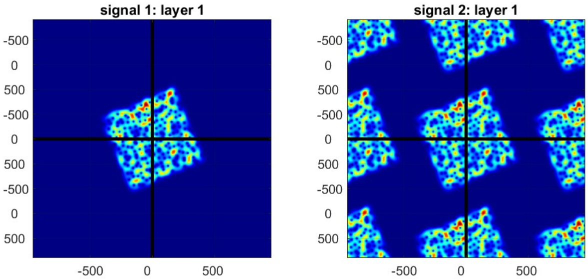


Figure: localGabSpec25D.jpg

The Key-players for Time-Frequency Analysis (TFA)

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

–

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,,$$

with $x, \omega, t \in \mathbb{R}^d$, compatible with the Fourier transform

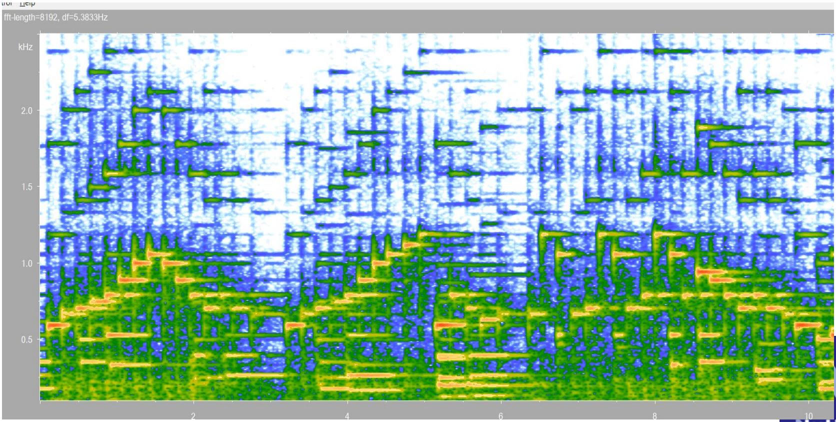
$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

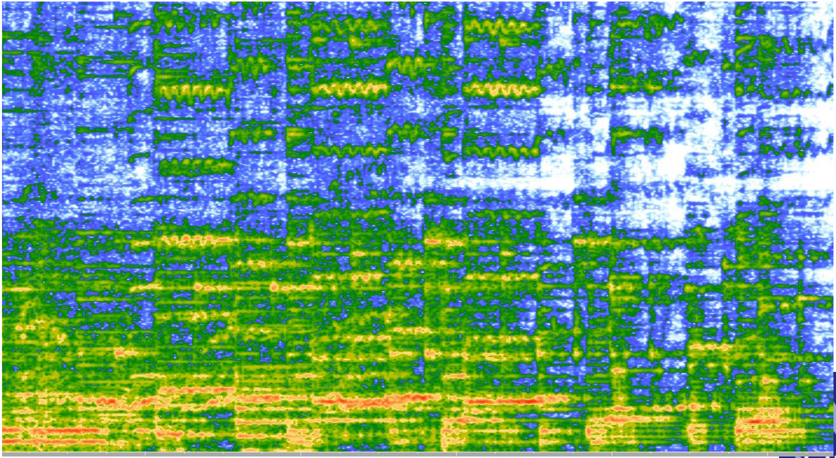
$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t,\omega);$$

A Typical Musical STFT

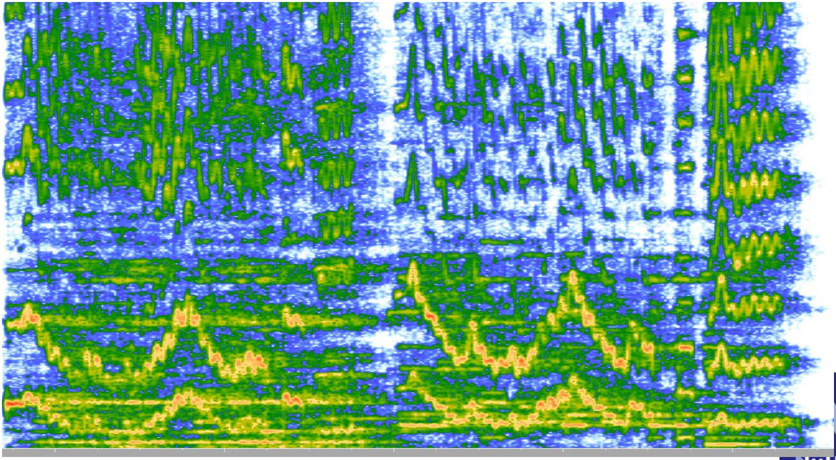
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

1. $S_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

Making this Intuition Work I

First of all we have to start with a *suitable* function class. For many applications in Classical Fourier Analysis (all the summability kernels), or applied Fourier Analysis (B-splines, Schwartz functions) as well as (of course) time-frequency analysis the Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ (which coincides with the modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$ is a suitable Banach algebra of continuous and (absolute Riemann) integrable functions. For simplicity let us assume that the STFT $V_g f$ is obtained from the *signal* f using a fixed Gaussian (and thus Fourier invariant) window g.
In such a case it is easy to obtain a tight Gabor window (most similar to the original Gaussian window) for a TF-shift of decent redundancy. One my use TF-translates along the lattice $\Lambda_2 = 1/2 * \mathbb{Z}^{2d}$. In such a case one can characterized the membership of an $\boldsymbol{L}^2$-function in $\boldsymbol{S}_0(\mathbb{R}^d)$ by the equivalent $\ell^1(\Lambda)$-norm of the restriction of $V_g f$ to the lattice Λ_2.

Making this Intuition Work II

This has to do with the fact that one has automatically for $f \in \mathbf{S}_0(\mathbb{R}^d)$ not only $V_g f \in \mathbf{L}^1(\mathbb{R}^{2d})$, but also $V_g f \in \mathbf{S}_0(\mathbb{R}^{2d})$!
The next step makes use of the fact that the combined periodization and sampling operator (at the level of the signal f) corresponds to a classical periodization in the $2D$ sense of the STFT of g, as indicated by the illustrations.
IF THE STFT was fully concentrated on a compact set (square, circle etc.) then in analogy to the usual proof of Shannon's Sampling Theorem should be possible, otherwise one has to control the tails. Unfortunately this does not happen with the representation chosen, but the good TF-localization of the Gabor atoms allows to have strong control over the tails.

Approximate Reconstruction Arguments

Using the discrete Gabor characterization we observe that the procedure of localizing the TF-periodized STFT allows to approximate the Gabor coefficients of the original function $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ better and better, because the inner part is reproduced better and better (with more and more coarse periodization in the TF-plane) while to total contribution of the other (outside parts) corresponds roughly to the ℓ^1-norm of the outer part and thus is vanishing as the TF-periodization (via Λ_2 type lattices) is getting more and more coarse.
A similar effect appears on the time-side, where the periodization can be ignored at the signal level, and one can also start as well from the samples of f, taken at the positions

$$f(0), f(\beta), ..f(k\beta), f((-k+1)\beta), ..f(-\beta)$$

for $\beta = 1/\sqrt{n}$ roughly, and k approx. $1/\beta$.

A Problem arising from Phase Factors

Looking closer one has to observe that the periodization view-point is only justified with the sampling rate and period are compatible with the lattice used for the Gabor family. In other words, TF-shifts used for the Gabor family and the periodization lattice $\Lambda_p = p \cdot \mathbb{Z}^d$ have to commute, or in other words the lattice Λ_p should be contained in $(1/a)\mathbb{Z}^2$ (if we choose $a = b$ for the Gabor lattice, e.g. $a = 1/2 = b$, and $\beta = 1/p$).

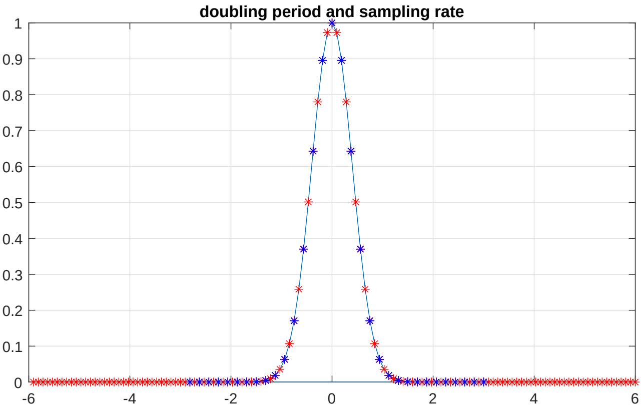


Figure: Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.

Illustration of the BGTr: Discrete Approximations

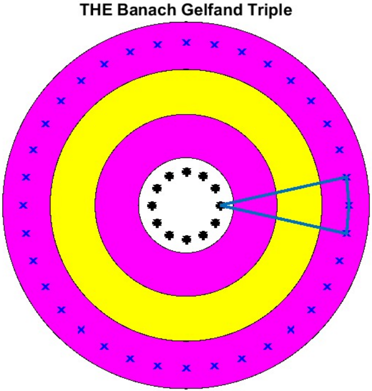


Figure: SOhann24D.jpg: inner and outer discrete case

Rates of Convergence I

The setting described so far uses the fact that - due to the integrability of $V_g f$ "most of the information in the signal" $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ is contained in a sufficiently large fundamental domain of what we could call the *Periodization-Sampling Lattice* $\Lambda = p\mathbb{Z}^{2d}$.

In Gabor Analysis and the theory of pseudo-differential operators weighted subclasses of $\boldsymbol{M}_0^{1,1}(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$ are in good use, among them quite a few Fourier invariant Banach spaces (and many of them Banach algebras), making use of the radial symmetric polynomial weight (Japanese bracket), for $\lambda = (t, s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, described as follows:

$$v_s(\lambda) = (1 + |\lambda|^2)^{s/2} = \langle \lambda \rangle^s.$$

Rates of Convergence II

Definition

For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$ we set:

$$\boldsymbol{M}^p_{v_s}(\mathbb{R}^d) := \{f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \mid V_g f \in \boldsymbol{L}^p_{v_s}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)\}$$

with the corresponding natural norm:

$$\|f\|_{\boldsymbol{M}^p_{v_s}(\mathbb{R}^d)} := \|V_g f \cdot v_s\|_{\boldsymbol{L}^p(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}.$$

For any p the intersection of these spaces over $s \geq 0$ gives the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, and their union (over $s \leq 0$) is just $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$. For $p = 2$ they are Hilbert spaces, known as *Shubin classes*. They arise in the study of the *harmonic oscillator* and coincide with tempered distributions with Hermite coefficients in a suitable weighted ℓ^2-space (of order $s/2$, roughly).

It is easy to verify that these classes are closed under complex interpolation and duality (taking the w^*-topology for the case $p = \infty$) in a natural way (using the indices $1/q$ and geometric means of the corresponding weight parameters).
A simple Cauchy-Schwartz argument shows that $\boldsymbol{L}^2_{v_s} \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^{2d})$ if (and only if) $s > d$, thus we have among others
$\boldsymbol{Q}_s(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d)$in this case. In a similar way we have
$\boldsymbol{M}^\infty_{v_r} \hookrightarrow \boldsymbol{M}^1(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$ if (and only if) $r > d$.
For fixed s one has the chain of continuous (and mostly proper and dense) embeddings $\boldsymbol{M}^r_{v_s} \hookrightarrow \boldsymbol{M}^t_{v_s}$ whenever $r < t$.
For our application the following characterization of these spaces using Gabor expansions is relevant. Here we assume that the norm on the spaces $\boldsymbol{M}^p_{v_s}(\mathbb{R}^d)$ are given, using the definition for a fixed window g.

Basic Observation concerning Gabor Coefficients

Theorem

Starting from a Gaussian window $g \in \mathbf{S}_0(\mathbb{R}^d)$ we have: Given the interval $J = [1/4, 3/4]$ and $s_0 \geq 0$ there exists a constant $C = C(s_0, J) > 0$ such that one has for any pair (a, b) with $a, b \in J$ and $|s| \leq s_0$:

$$C^{-1}\|f\|_{M^p_{v_s}(\mathbb{R}^d)} \leq \left(\sum_{(n,k)\in\mathbb{Z}^d\times\mathbb{Z}^d} |V_g f(an, bk)|^p v_s(an, bk)^p \right)^{1/p}$$

$$\leq C\|f\|_{M^p_{v_s}(\mathbb{R}^d)} .$$

We will only make use of this fact for the case $p = 1$.

The Overall Procedure

- Given that signal $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we can determine a region $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ of high concentration of $V_g f$;
- First we fix a good Gabor system, such as a Gaussian window and $a = 1/2 = b$ and determine the dual or tight Gabor atom;
- Depending on Q we choose some period $p \in 2\mathbb{N}$ and sampling rate $r = 1/p$ in order to make the sampling and periodization Fourier invariant, compatible with the given Gabor family;
- For the discrete periodic signal (in $\boldsymbol{S}_0(\mathbb{R}^d)$!) we have exact knowledge of the Gabor coefficients, namely the coarsely periodized Gabor coefficients of original f.
- A (controlled) error occurs of f is just sampled, but Gabor coefficients can be computed discretely;
- Using the localized Gabor coefficients we reconstruct f.

We will only make use of this fact for the case $p = 1$.

M. Ehler, K. Gröchenig, and A. Klotz.
Quantitative estimates: How well does the discrete Fourier transform approximate the Fourier transform on R.
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THANK you for your attention

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Email: hans.feichtinger@univie.ac.at

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