Gabor Analysis, Wigner Transform and the Metaplectic Group

Hans G. Feichtinger, Univ. Vienna and ARI
hans.feichtinger@univie.ac.at
www.nuhag.eu TALKS at www.nuhag.eu/talks

Talk at ETH Zürich
May 19th, 2025

Abstract Provided I

My first contact, ca. 1980 with the Communication Theory Group at TU Vienna was Franz Hlawatsch, who explained to me that he was working on the Wigner distribution, following his advisor W. Mecklenbräuker. I told him about what is now called Feichtinger's algebra $S_0(R^d)$ and he pointed out to me that this was closely related to the claim of D. Gabor concerning the representation of signals as double sums using time-frequency shifted copies of the Gaussian. Already in 1989 H. Reiter published his work on the Metaplectic group using this Segal algebra. In the same year G. Folland's book on Phase Space Analysis appeared.
Meanwhile Gabor Analysis is a well-established field mathematical analysis and in particular the Banach Triple (S_0, L_2, S_0*), consisting of Feichtinger's algebra, the Hilbert space L_2 an the dual space, the space of mild distributions (which appears to be the correct ambient vector space of all objects, treated as signals in

Abstract Provided II

application areas) are established as key objects. Even classical Fourier analysis (DFT versus distributional Fourier transform) or the description of pseudo-differential operators (slowly varying systems in mobile communication) can be treated elegantly in this setting.

Chirp signals are important objects in many areas, including optics. Their Wigner transform is a Dirac measure concentrated along a line. In mathematics these functions of the form $\exp(2\pi t^2)$ (an their dilated versions) are known as characters of second degree. Together with the Fourier transform and the usual dilation operators the (obviously unitary) pointwise multipliers by such chirps define a group of unitary operators, the so-called metaplectic group (the group of linear canonical transformations in the engineering literature, or the special affine Fourier transform, with the Fractional Fourier transform as special case).

Abstract Provided III

Starting from the observation that these unitary operators extend naturally to Banach Gelfand Triple automorphism allows to put these operators into a natural context, going beyond the Hilbert space setting. Despite the fact that discrete Wigner transforms, discrete Hermite functions and in particular discrete version of the Fractional Fourier transform (a commutative subgroup of the metaplectic group) have been studied in many engineering papers the results available so far are usually based on heavy computations and do not reveal the structural properties. As time permits such aspects and preliminary answers to pending questions will be offered.

Coherent Expansions I

Time-Frequency Analysis (as we know it today, in short TFA) is sometimes characterized as the subdomain of mathematical analysis (from the foundations to the applications in the theory of pseudo-differential operators or in quantum harmonic analysis: QHA) which can be derived from so-called time-frequency shifts. Given a signal f (e.g. a bounded, continuous signal of time) the STFT provides a description of f by correlating f with TF-shifted versions of some template, typically a Gaussian function. The STFT can be thought to be isometric from $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$, but extends to much larger signal classes.

$$V_g f(t, \omega) = \int_{\mathbb{R}^d} f(y) M_{-\omega} T_t g(y) dy.$$

It is also possible to recover f from $V_g f$. **Gabor analysis** cares about the restriction of $V_g f$ to lattices $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Coherent Expansions II

For every decent g there is some density of Λ such that f can also be recovered from the restriction of $V_g f$ to Λ. In fact, one has a representation $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ using the *dual Gabor atom* $\widetilde{g}$:

$$f = \sum_{\lambda \in \Lambda} V_g f(\lambda) \widetilde{g}_\lambda .$$

In a similar way any (reasonable) operator T can be described as a (continuous) *spreading repesentation*, formally we have

$$T = \int_{\mathbb{R}^{2d}} \eta(\lambda) \pi(\lambda) d\lambda,$$

with $\pi(\lambda) = M_\omega T_t$ for $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. For the case of Hilbert-Schmidt operators with $\langle T, S \rangle_{\mathcal{HS}} := \operatorname{trace} TS^*$ we even have an isometry $T \to \eta(T)$ onto $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$.

Coherent Expansions III

The symplectic Fourier transform $\sigma(T)$ of the spreading symbol $\eta(T)$ is known as the Kohn-Nirenberg symbol $\sigma(T)$ of T. Again, there is a unitary isomorphism between the two representations. $\sigma(T)$ has an important covariance property: Conjugation of T corresponds to an ordinary shift of $\sigma(T)$ as a function over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. This also helps to interpret *Gabor multipliers* in the KSN-picture as a convolution products of $\sigma(P_g)$ (the projection operator $f \mapsto \langle f, g \rangle_{\boldsymbol{L}^2} g$) with a weighted Dirac comb of the form $\sum_{\lambda \in \Lambda} m_\lambda \delta_\lambda$.

Such convolution relations between operators are also at the core of what is called **quantum harmonic analysis**, somehow moving Fourier analysis from the level of functions (or distributions) to the level of operators. There is also a so-called operator Fourier transform which leads to the spreading symbol. As expected it transfers convolution into pointwise multiplication.

Coherent Expansions IV

What is often forgotten is the fact, that the theory of coherent states, nowadays synonymous with use of families of the form $\pi(\lambda)g$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, arose out of the context of the study of the pair of so-called **creation** and **annihilation** operators, by forming suitable exponential sums, with maximal concentration in phase-space at a given point λ (or $z \in \mathbb{C}$ for the case $d = 1$), which can be described as unilateral shifts in the Hermite-basis. In terms of coefficients with respect to the Hermite ONB for $\boldsymbol{L}^2(\mathbb{R})$ one has simply the operators $[c_0, c_1, \cdots] \mapsto [0, c_0, c_1, \cdot]$ respectively $[c_0, a_1, \cdots] \mapsto [c_1, c_2, \cdots]$.

This connection has not really been explored in TFA up to now!

It requires properly defined Hermite functions over finite Abelian groups, or the analogue of the HARMONIC OSCILLATOR.

Only then one can define fractional FTs and hope to verify the covariance of a properly defined WIGNER transform.

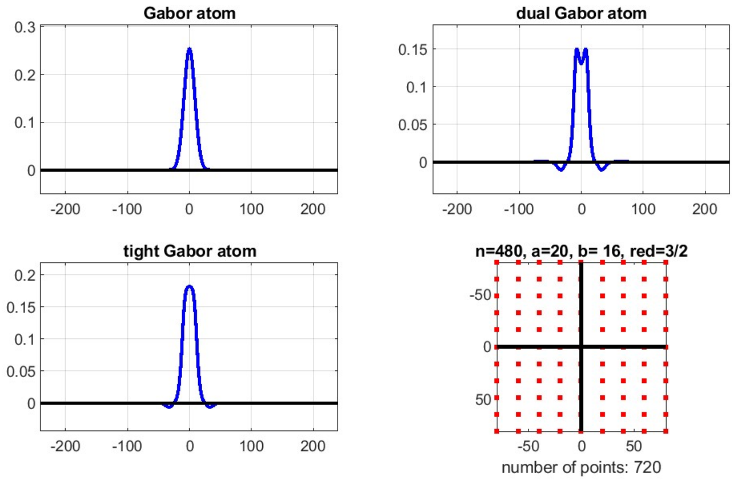


Abbildung: gabbasic01.jpg

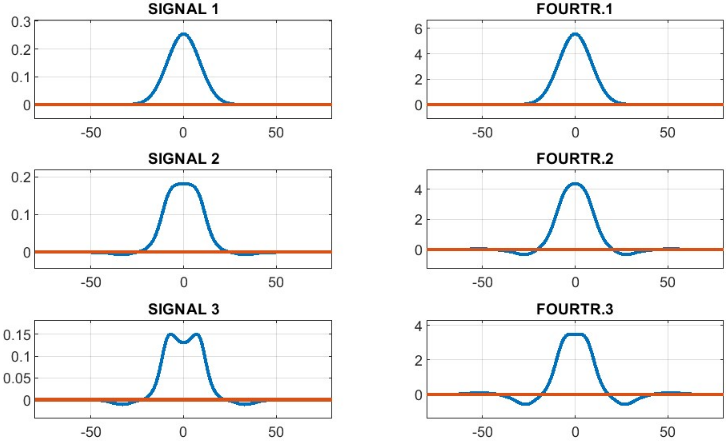


Abbildung: gabdualtgt01.jpg

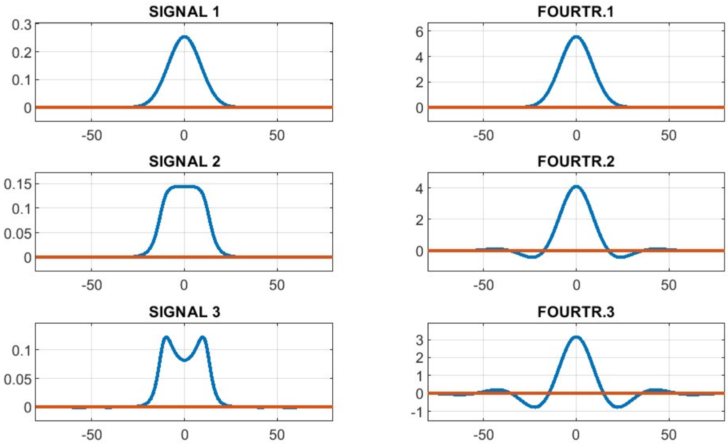


Abbildung: gabdualtgt02.jpg: n=480, a=24, b=10, red=2

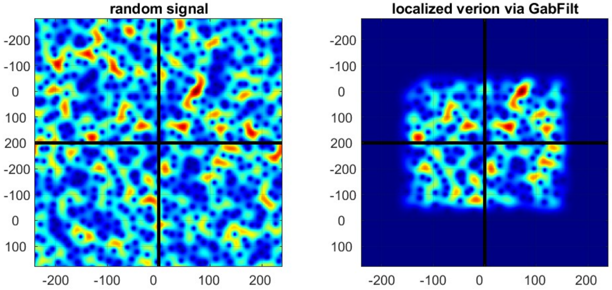


Abbildung: gabfiltrec01.jpg

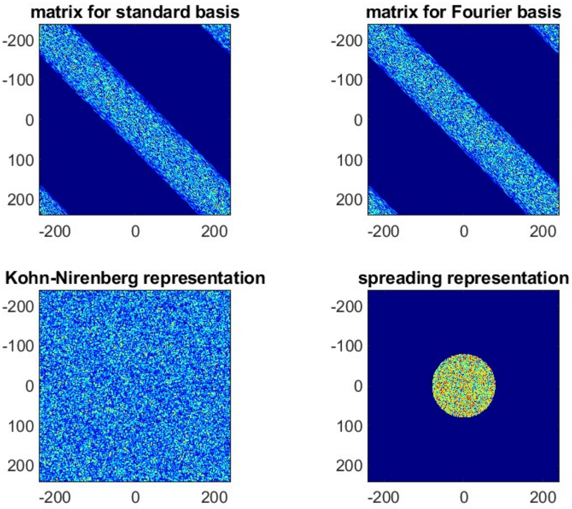


Abbildung: lowsprN80A.jpg

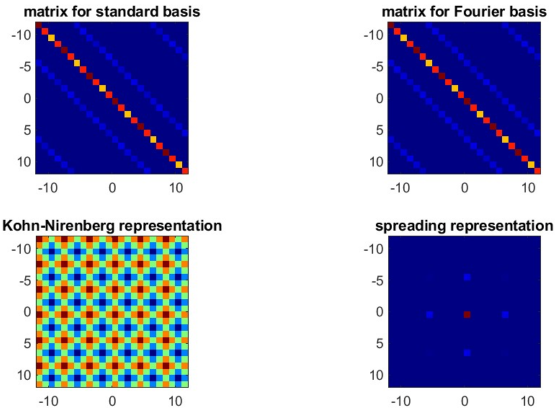


Abbildung: gabmatr2444a.jpg: Different representations of a typical Gabor frame operator.

On a discrete version of the ambiguity function

Nicolae Cotfas, 27 Feb. 2022

Def **Some definitions.** For $d=2s+1\in\{1,3,5,\ldots\}$ and $\mathcal{H}\equiv\mathbb{C}^d\equiv\{\,\psi:\{-s,-s+1,\ldots,s-1,s\}\to\mathbb{C}\,\}\equiv\{\,\psi:\mathbb{Z}_d\to\mathbb{C}\,\}$

Def1 $\langle\varphi|\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n)\equiv\sum_{n\in\mathbb{Z}_d}\overline{\varphi(n)}\,\psi(n).$

Def2 The *canonical orthonormal basis* is formed by $\delta_{-s},\delta_{-s+1},\ldots,\delta_s:\{-s,-s+1,\ldots,s-1,s\}\to\mathbb{C}$,

$$\delta_m(n)=\delta_{nm}=\begin{cases}1 & \text{for } n=m,\\ 0 & \text{for } n\neq m.\end{cases}$$

By using the notation $|m\rangle=|\delta_m\rangle$, we have

$$\boxed{\langle j|m\rangle=\delta_{jm}\quad,\quad\sum_{m=-s}^{s}|m\rangle\langle m|=\mathbb{I}}.$$

Alternatively, we can regard it as the set $\{\,\delta_m:\mathbb{Z}_d\to\mathbb{C}\mid m\in\mathbb{Z}_d\,\}$, where

$$\delta_m(n)=\delta_{nm}^{[d]}=\begin{cases}1 & \text{for } n=m \bmod(d),\\ 0 & \text{for } n\neq m \bmod(d).\end{cases}$$

Def3 *Finite Fourier transform* $F:\mathcal{H}\to\mathcal{H}$ is

$$\begin{array}{c}\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}\\ \downarrow\\ F[\psi]:\mathbb{Z}_d\longrightarrow\mathbb{C}\end{array}\qquad \boxed{F[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)}$$

Def4 *Position* operator $Q:\mathcal{H}\to\mathcal{H}$, $\boxed{Q\psi(n)=\tilde{n}\psi(n)}$

Def5 *Momentum* operator $P:\mathcal{H}\to\mathcal{H}$, $\boxed{P=F^\dagger QF}$

where $\tilde{n}$ is the representative modulo d of n satisfying $-s\leq\tilde{n}\leq s$.

Def6 *Translation* operators $(n,k\in\mathbb{Z}_d)$ $\mathcal{H}\to\mathcal{H}$, $\boxed{\begin{array}{l}\psi\mapsto\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi\\ \psi\mapsto\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi\end{array}}$

Def7 *Displacement* operators $(-s\leq n,k\leq s)$ $\boxed{D(n,k)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}}$

Def8 *Parity operators* $(n,k\in\mathbb{Z}_d)$ $\boxed{\Pi(n,k)=D(n,k)\,\Pi\,D(n,k)^\dagger}$ where $\Pi\psi(n)=\psi(-n)$.

Def9 *Wigner function* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is $\boxed{\mathcal{W}_A(n,k)\stackrel{\text{def}}{=}\frac{1}{d}\langle\Pi(n,k),A\rangle=\frac{1}{d}\mathrm{tr}(\Pi(n,k)A)}$

Def10 *Ambiguity function* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is $\boxed{\chi_A(n,k)\stackrel{\text{def}}{=}\frac{1}{d}\langle D(n,k),A\rangle=\frac{1}{d}\mathrm{tr}(D(n,k)^\dagger A)}$

R **Some remarks:**

Discrete Hermite Functions I

It is well known that the Hermite functions are (among others) joint eigenvectors of Anti-Wick operators arising from rotationally symmetric *upper symbols* (multipliers). These operators commute with the Fourier transform, and in fact with Fractional Fourier transform. Hence it is plausible that the corresponding discrete version (which can be realized in MATLAB) using the covariance of the Kohn-Nirenberg symbol of matrices. They commute with the DFT and are thus also eigenvectors of the (normalized) FFT. Such discrete Hermite functions (DHFs) can also be used to realize (a discrete variant of) the Fractional Fourier Transform (FrFT) by building the corresponding *Hermite Multipliers* (using pure frequencies).

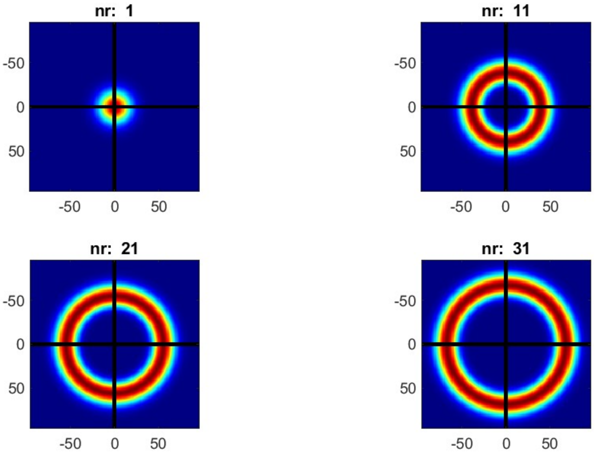


Abbildung: herm1to31.jpg

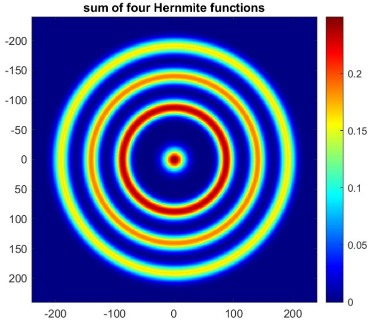


Abbildung: The spectrogram of a sum of four discrete Hermite functions

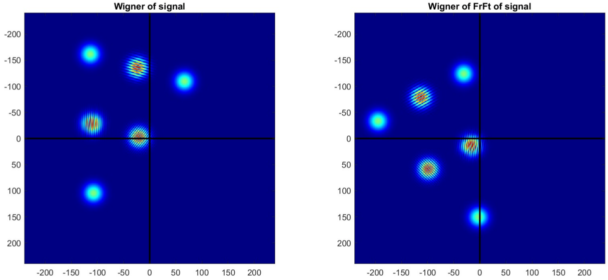


Abbildung: DemWigHerm20A.jpg: A discrete Wigner function of a signal and the signal after Fractional Fourier transform.

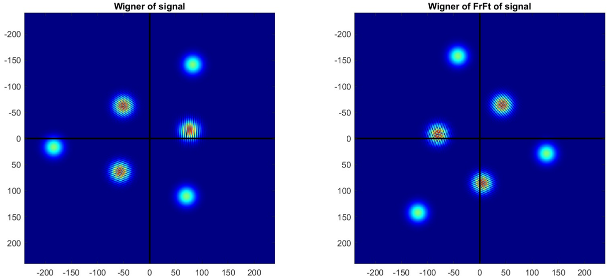


Abbildung: DemWigHerm20B.jpg: A discrete Wigner function of a signal and the signal after Fractional Fourier transform.

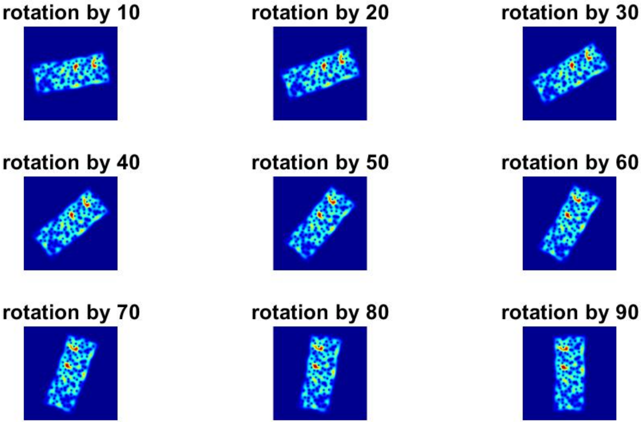


Abbildung: TEXT

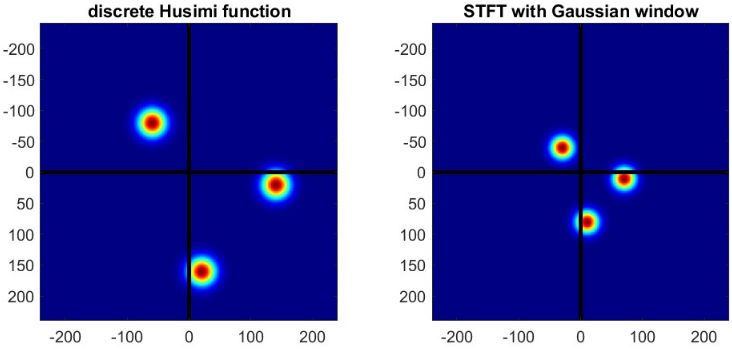


Abbildung: TEXT

How to form discrete Hermite functions I

First let us quickly recall the classical approach:
Hermite functions form a complete orthonormal system in $L^2(\mathbb{R})$ and play a central role in quantum mechanics and time-frequency analysis. They are defined in terms of the Hermite polynomials $H_n(x)$, which satisfy a well-known three-term recurrence relation. The *physicists' Hermite polynomials* $H_n(x)$ are defined by

$$H_n(x) = (-1)^n e^{x^2} [n] x e^{-x^2}, \quad n \in \mathbb{N}_0.$$

The *Hermite functions* $h_n(x)$ are then given by

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x).$$

These functions satisfy the three-term recurrence relation

$$\sqrt{n+1}\, h_{n+1}(x) = \sqrt{2}\, x\, h_n(x) - \sqrt{n}\, h_{n-1}(x), \quad n \geq 1,$$

How to form discrete Hermite functions II

with the initial terms

$$h_0(x) = \pi^{-1/4} e^{-x^2/2}, \quad h_1(x) = \sqrt{2}\, x\, h_0(x).$$

The Hermite functions are eigenfunctions of the Fourier transform:

$$\mathcal{F}[h_n](\xi) = (-i)^n h_n(\xi),$$

where $\mathcal{F}$ denotes the unitary Fourier transform on $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$. Alternatively one can characterize them as the eigenvectors of the so-called Harmonic Oscillator

The Hermite functions $h_n(x)$ are the eigenfunctions of the quantum harmonic oscillator, governed by the time-independent Schrödinger operator

$$H = -\frac{1}{2}[2]x + \frac{1}{2}x^2,$$

How to form discrete Hermite functions III

acting on $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$. In this context, H is the Hamiltonian of a particle in a quadratic potential well.
One has

$$Hh_n(x) = \left(n + \frac{1}{2}\right) h_n(x), \quad n \in \mathbb{N}_0.$$

Thus, the spectrum of H is purely discrete and equidistant:

$$\mathrm{spec}(H) = \left\{ \left(n + \frac{1}{2}\right) : n \in \mathbb{N}_0 \right\}.$$

Finally it has been found (going back to the work of I. Daubechies on localization operators) that the Hermite functions are also joint eigenvectors to the family of STFT multipliers with Gaussian window (Anti-Wick operators) for *radial symmetric weights*.

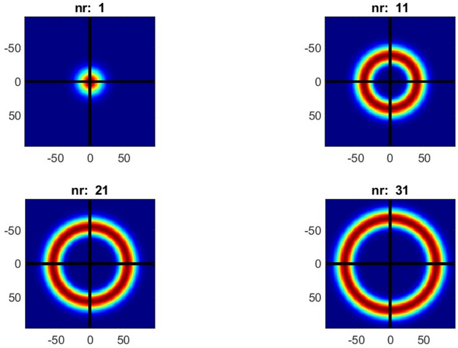


Abbildung: herm1to31.jpg

Cross Wigner Transform and STFT I

The *cross Wigner transform* and the *short-time Fourier transform (STFT)* are two fundamental tools in time-frequency analysis. Despite their different constructions—the Wigner transform being quadratic and the STFT being linear—they are closely connected via a symplectic Fourier transform. This document gives precise definitions and states the connection formula.
For $f, g \in L^2(\mathbb{R}^d)$ we define the *cross Wigner distribution* by:

$$W(f,g)(x,\xi) := \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \xi \cdot t} \, dt. \qquad (1)$$

If $f = g$, this reduces to the classical Wigner distribution. It is real-valued when $f = g \in L^2(\mathbb{R}^d)$ and is commonly used in quantum mechanics and signal analysis.

Cross Wigner Transform and STFT II

Given a window function $g \in L^2(\mathbb{R}^d)$, the *short-time Fourier transform* (STFT) of f with respect to g is

$$V_g f(x,\xi) := \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i \xi\cdot t}\, dt. \qquad (2)$$

The STFT represents the local frequency content of f near the point x in time, analyzed using the window g.
The *symplectic Fourier transform* of $F\colon \mathbb{R}^{2d} \to \mathbb{C}$ be given by

$$\mathcal{F}_\sigma F(x,\xi) := \int_{\mathbb{R}^{2d}} F(y,\eta)\, e^{-2\pi i(\xi\cdot y - x\cdot\eta)}\, dy\, d\eta. \qquad (3)$$

Then the cross Wigner transform is related to the STFT by the following identity:

$$W(f,g)(x,\xi) = 2^d \cdot \mathcal{F}_\sigma \left[V_g f\right](2x, 2\xi). \qquad (4)$$

Cross Wigner Transform and STFT III

An alternative, more elementary identity expressing the Wigner distribution directly in terms of the STFT is:

$$W(f,g)(x,\xi) = 2^d \, e^{4\pi i x\cdot\xi} \, V_{\check{g}} f(2x, 2\xi), \tag{5}$$

where $\check{g}(t) := g(-t)$ is the reflection of the window function.

Wigner

What are the general properties of WIGNER distributions?

Expected properties are:

1. $W(f) = W(f, f)$ is supposed to be real-valued;
2. The mapping $f \mapsto W(f, g)$ defines a unitary mapping in the Hilbert-Schmidt (Frobenius norm) sense;
3. Integration along time/frequency gives "correct marginals", namely $|f(t)|^2$ or $|\widehat{f}(y)|^2$.

Difficulties arising (especially for EVEN n) are:

- The mapping $k \mapsto 2k$ **is not an automorphism** of the cyclic group of order n!
- Flandrin and Co. have even shown that the above conditions cannot be satisfied for even n!

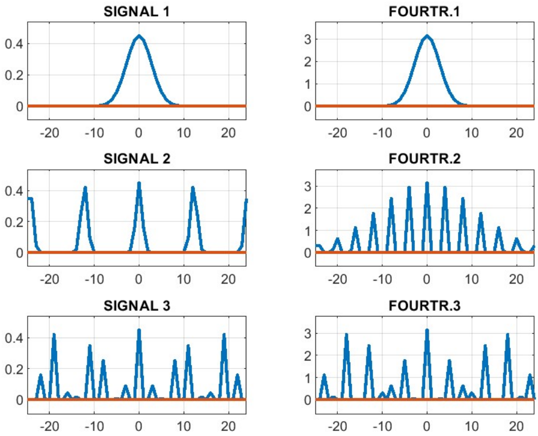


Abbildung: dil49iterA.jpg: iterating the ?dilation? operator $k \to 2k$ for $n = 49$

Grossman/Royer introduced the use of the parity operator

According to Grossman/Royer one can explain the Weyl-Wigner calculus, i.e. the transition from function on phase space to operators (so-called *Weyl-quantization*, or the inverse, which for the case of the projection operator P_g gives the *Wigner transform* $W(g)$ as follows: One has to conjugate (!) the parity operator $P = \mathcal{F}^2$, with $Pf(x) = f^{\checkmark}(x) := f(-x)$ with TF-shifts, but in order to fit the normalizations one has to use beforehand the mapping $(r,s) \to (r/2, s/2)$, for $(r,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$!
For the even case one can imitate this as well, but the linear span of all the (self-adjoint) matrices arising in this way has just dimension $n^2/4$!

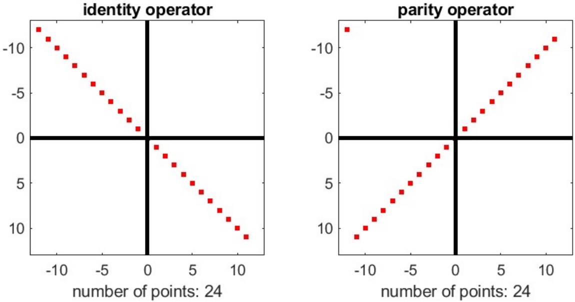


Abbildung: IdPar24.jpg: The corresponding matrices are constant along the main or the anti-diagonal over $Z_n \times Z_n$

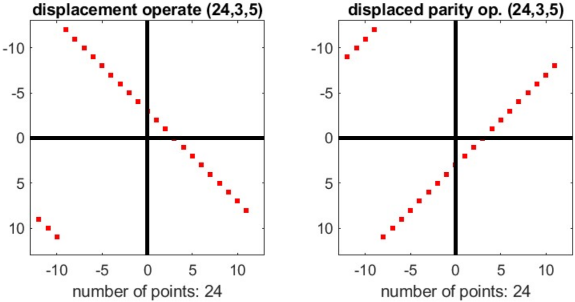


Abbildung: IdPar24d35.jpg: Concentration of TF-shifts and displaced parity operators.

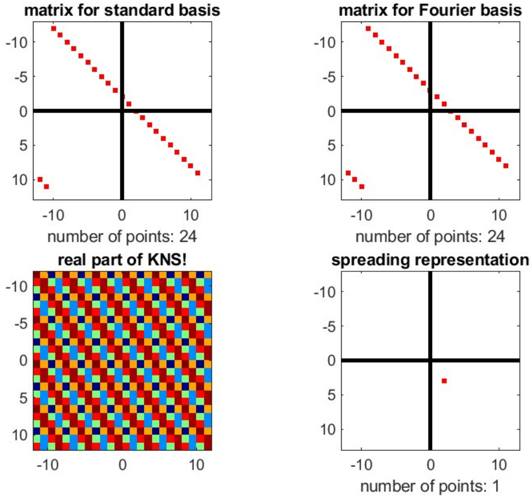
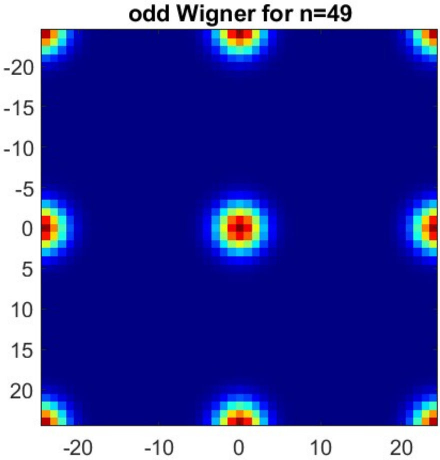
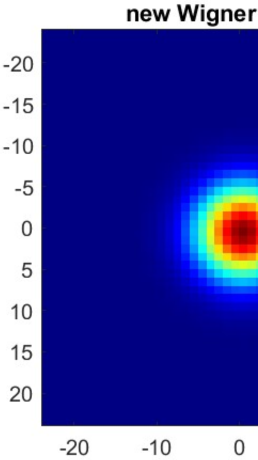


Abbildung: TF24m23A.jpg: Concentration of TF-shift: n=24, t=2, s=3, KNS symbol showing only real part (plane wave).


odd Wigner for n=49
-20
-15
-10
-5
0
5
10
15
20
-20 -10 0 10 20


new Wigner
-20
-15
-10
-5
0
5
10
15
20
-20 -10 0

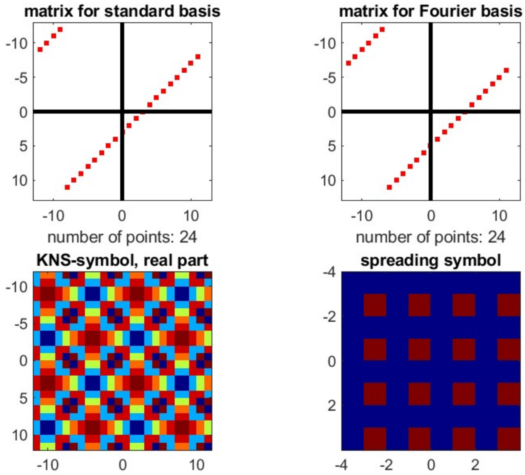


Abbildung: W35spy4A.jpg

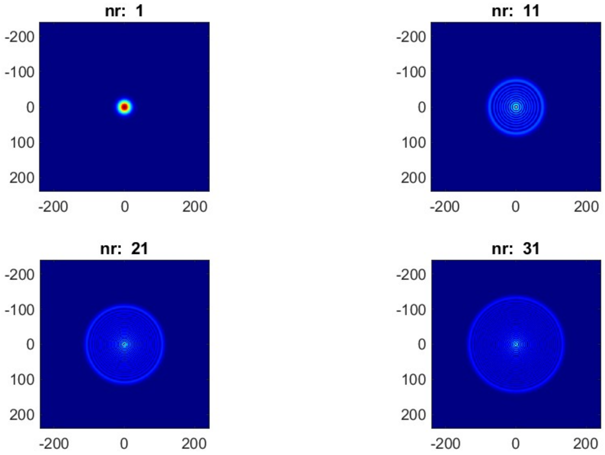


Abbildung: hermwig1to31.jpg: Wigner functions associated with low order Hermite functions.

The Concepts to be Popularized

Work in the context of Time-Frequency Analysis and extensive studies of the subject using various function spaces, but also performing quite a few numerical explorations have lead to the following perspective and terminology:

- CONCEPTUAL HARMONIC ANALYSIS (integrating abstract and computational harmonic analysis)
- Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, for signals and operators and their different representations;
- The term mild distributions for members of the space $\boldsymbol{S}_0'(\mathbb{R}^d)$ of all *signals*, and Feichtinger's algebra as set of measurements.
- The kernel theorem characterizing bounded linear operators with mild distributions on $\mathbb{R}^{2d}$.

Which Mathematical Tools

Which mathematical tools serve which purpose?

1. Integration theory secures the "existence" of a FT;
2. The FFT is used to compute (?) the Fourier transform;
3. The theory of *tempered distributions* (L. Schwartz) allows a generalized FT, including Dirac measures;
4. L. Hörmander established the connection to PDE;
5. The theory of *pseudo-differential operators* is closely related to *slowly varying systems, Kohn-Nirenberg calculus*;
6. the *metaplectic group* (including the *Fractional FT* (*FrFt*) appear e.g. in *optics*, e.g. *Fresnel transforms*, fiber optics;
7. There is a lot of interest in generalized *Wigner transforms* (Weyl calculus, quantum theory) recently.

Common Ground: Linear Spaces

When it comes to the discussion of SIGNALs we often encounter *signal spaces*, i.e. (complex) vector spaces of signals, e.g. trig. polynomials etc.. We learn from linear algebra that such spaces have a basis and thus uniquely determined coordinates, once a basis is fixed. Linear mappings between vector spaces can thus be described by their action on a given basis, stored in the form of a *matrix* $\boldsymbol{A}$.

Composition (and hence inversion) of linear mappings is thus described by matrix multiplication or inversion of matrices. Variants include the use of *generating systems* (frames) or linear independent families (Riesz sequences) and thus the use of the *pseudo-inverse* based on the SVD of a general matrix, givig a simple approach to *dual frames* or *biorthogonal systems*.

The Infinite-Dimensional Situation

Coming to the situation of continuous variable, be it functions on the torus $\mathbb{T}$ or on the integers $\mathbb{Z}$ one encounters already the problem of *infinite dimensional* vector spaces, which should be better called *not anymore finite dimensional* (because they are to "big" to contain a finite basis!).
It is then necessary to consider not only finite sums, but admit at least some infinite sums, or series. All this opens the discussion of convergence (in some norm, or in some topology, conditional or unconditional, etc.) and leads to the need of (linear) functional analysis.
It is no surprise (at least in retrospect) to realize the influence of Fourier Analysis on the development of Functional Analysis and thus on Mathematical Analysis in general.

Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

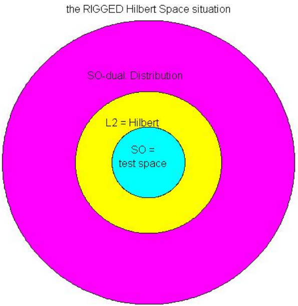


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra*

$\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$,
known as space of *mild distributions. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable functorial properties, see the early work of V. Losert (***[lo83-1]***).*
*For $\mathbb{R}^d$ the most elegant way (which is describe in **[gr01]** or **[ja18]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT*

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi isy}dy$$

and the corresponding norm

$$\|f\|_{\boldsymbol{S}_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|dxdy < \infty.$$

From a practical point of view one can argue that one has the *following list of good properties of $\boldsymbol{S}_0(\mathbb{R}^d)$.*

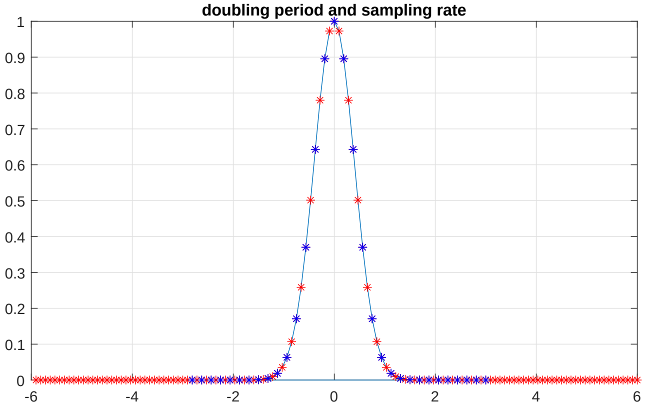


Abbildung: *Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.*

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ I

It is a harmless but important observation that the squares of the elements of $\mathbb{Z}_n$ (rotation by multiples of $2\pi/n$) are just the elements of $\mathbb{Z}_{n/2}$ (only for n is even!), repeated twice.
Thus for us the operator which replaces a given function (or matrix) by its 2-periodic and 2-sampled version will be of big relevance. Also, since all the information comes twice (for matrices in both the world of column AND the world of rows) we have to understand how to extract properly the subsequence of indices "most representative" for such a reduction (turning vectors of length n into vectors of length $n/2$) or just of length $2n$ into vectors of length n and matrices of size $2n \times 2n$ into matrices of size n, in a compatible way.
We will illustrate this by some plots and also verify that this procedure is well compatible with many of the representations of functions of operators.

Functions on $\mathbb{Z}_n$ versus $\mathbb{Z}_{n/2}$ II

As a basic example let us take a function with small support, then produce its p-periodic version, and then sample at the rate of $1/p$, $p \in \mathbb{N}$. Then you will find that the "representing sequence" of the Fourier version of such a function, treated in the same way, will be just (suitable normalized) the FFT of the finite vector (of length p^2, of course) of the vector in $\mathbb{C}^{p^2}$ representing the discrete and periodic signal on $\mathbb{R}$.

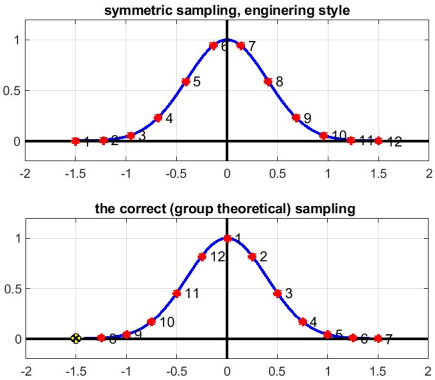


Abbildung: *Naive versus correct (group theoretical) sampling. There is natural behavior with respect to refinement of the sampling and taking multiples of the period*

THANKS!

www.nuhag.eu/ETH20 *Course Notes ETH 2020!!*
www.nuhag.eu/talks: *many talks on related subjects*
NOTE: **Talks require access code:**
"visitor" and "nuhagtalks"

At the page
https://www.univie.ac.at/nuhag-php/home/db.php
you also find recorded talks from some of my courses, e.g.
FROM LINEAR ALGEBRA TO GELFAND TRIPLES
(towards the end).
Various Lecture Notes can be downloaded from
https://www.univie.ac.at/nuhag-php/home/skripten.php