The Mathematics of Signal Processing via Mild Distributions: : From Lin.Alg. to TFA

Hans G. Feichtinger, Univ. Vienna and ARI
hans.feichtinger@univie.ac.at
www.nuhag.eu TALKS at www.nuhag.eu/talks

LMCRC Workshop on Mathematics in Signal Processing
May 15th, 2025, PARIS

NuHAG Numerical Harmonic Analysis Group

The Mathematics of Signal Processing via Mild Distributions: : From Lin.Alg. to TFA

Hans G. Feichtinger, Univ. Vienna and ARI
hans.feichtinger@univie.ac.at
www.nuhag.eu TALKS at www.nuhag.eu/talks

LMCRC Workshop on Mathematics in Signal Processing
May 15th, 2025, PARIS

Overall Motivation

Most of the time mathematical talks outline a specific set of mathematical problems and describe ways how to come up with solutions, or describe new methods to attack those problems, mention the obstacles on the way and the improvements achieved by the authors.

Survey talks summarize the field, provide a summary of the state of the art and outline possible new methods for the solution of a well-defined set of problems, e.g. characterizing functions with a particular form of smoothness and so on.

THIS is a(onther) perspective talk putting established problem settings into question and suggesting new ways to look at well-known and less well-known mathematical problems.

Connections to Paris

As most of you know Paris has been very influential to the development of wavelets, due to the work of Yves Meyer.

`Yves Meyer: De la recherche petrolier et la geometrie des espaces de Banach en passant par les paraproduite. In: Seminaire sur les equations aux derivees partielles, 1985-1986 Exp. No. I, Ecole Polytechnique (1986), p.11.`

Visiting Yver Meyer in Feb. 1986 he told me: We just have two different constructions of wavelets (with P.G. Lemarie), AND:

Function spaces are (only) good for the description of operators!

For him the classical function spaces were good for the description of Calderon-Zygmund operators.

A Couple of Provocative Statements

Let me start with a couple of provicative statements:

1. Is it relevant to know the $\boldsymbol{L}^p$-behaviour of linear operators (e.g. Hausorff-Young)?
2. Is it really necessary to properly study topological vector spaces in order to understand the Dirac Deltas δ_x?
3. Is the Kernel Theorem (based on the *nuclearity of the Frechet-space* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$) relevant for engineering applications?
4. What is the role of (rigged) Hilbert spaces $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')$?
5. Is the space $\boldsymbol{L}(\mathcal{H})$ the correct *universe* for the world of all relevant operators (with say trace class operators or Hilbert Schmidt operatorsas a crucial subfamilies).

The usual argument is that integrability is crucical for the definition of the Fourier integral or the pointwise definition of convolution. And the $\boldsymbol{L}^2(\mathbb{R}^d)$-setting makes the sesqui-linear pairing of functions symmetric and generates a *Hilbert space*.

The Concepts to be Popularized

Work in the context of Time-Frequency Analysis and extensive studies of the subject using various function spaces, but also performing quite a few numerical explorations have lead to the following perspective and terminology:

- CONCEPTUAL HARMONIC ANALYSIS (integrating abstract and computational harmonic analysis)
- Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, for signals and operators and their different representations;
- The term mild distributions for members of the space $\boldsymbol{S}_0'(\mathbb{R}^d)$ of all *signals*, and Feichtinger's algebra as set of measurements.
- The kernel theorem characterizing bounded linear operators with mild distributions on $\mathbb{R}^{2d}$.

Original Abstract by HGFei I

For most purposes one may view SIGNALS as (not clearly defined) objects which depend on a parameter, such as time, or location, or even coordinates for a three dimensional space. Just think of audio signals, pictures, or the temperature in a closed room. We can record the audio signal (maybe compress it via MP3), and we can compute the spectrogram (STFT) in real time. We can take pixel images of a scene or measure the room temperature along an array of thermometers in the room. But we could also think of the task of synthesising a three-dimensional sound field using an array of load-speakers. For all this the theory of L^p-spaces does not really help, nor a simplistic derivation of translation invariant linear operators as convolution operators, using the impulse response of the system, or the description as a Fourier multiplier, using the transfer function model for L^1-functions. Here and in other places

Original Abstract by HGFei II

the mysterious Dirac Delta is appearing, usually with a hint to the theory of tempered distributions developed by Laurent Schwartz. The thesis of this talk is the claim that the best model to-date for general signals is that of mild distributions, i.e. the linear space of objects which have a bounded STFT. They also have a naturally defined Fourier transform. One just has to establish clear rules how to handle these objects, and how to define convergence. In the background is the concept of Banach Gelfand Triples. We will explain how this leads to the identification of the signal space with the dual of Feichtinger's algebra $\boldsymbol{S}_0(\mathbb{R}^d)$. The correct view on this is to consider the elements of $\boldsymbol{S}_0(\mathbb{R}^d)$ as the collection of all reasonable linear measurements! In a similar way the kernel theorem arising in this context provides the insight that general linear operators, namely operators which assign to each test function $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ some signal $T(f)$ can be described by a

Original Abstract by HGFei III

kernel, which is simply a mild distribution of $2d$ variables, in the form of an abstract integral operator. Such operators also have a spreading function and a Kohn-Nirenberg symbol. Furthermore one can show that the Wigner distribution of a signal is a well-defined mild distribution. Various examples of this situation will be listed, emphasizing that all this can be developed without use of either $\boldsymbol{L}^1(\mathbb{R}^d)$ or even $\boldsymbol{L}^2(\mathbb{R}^d)$.
Finally the approximation if signals by finite vectors (typically fine regular samples of a continuous function over a sufficiently long interval) will be indicated, and how discretizations (e.g. of the Wigner distribution) can be obtained. Such questions are hard to understand from the point of view of abstract harmonic analysis, but also from a purely computational approach. But isn't a high-resolution TV-set providing a good approximation of images arising in the real world?

Overview

Depending on the approach to Fourier Analysis or the education received during the studies people take predominantly the following approach to Fourier Analysis

- The historical viewpoint (Fourier series to Schwartz theory);
- The technical viewpoint (integration theory,..);
- The analysis viewpoint (distribution theory);
- The applied viewpoint (e.g. optics);
- The engineering viewpoint (e.g. communication theory);
- The physics background (continuous Dirac basis);
- The computational approach (FFT);
- The quantum world (simulation, quantum computer).

Alternative Approach: APPLICATION ORIENTED FA

Over the years I have learned that the developments in Fourier Analysis in application areas and in pure mathematics drift away from each other, more and more! Depending on the application area (from engineering, to physics, chemistry, optics or astrophysics) there is a *huge variety* of tasks. What is common to most of them?

1. It is about *analyzing received signals;*
2. *filtering components from a given signal;*
3. *Detect a signal hidden in noise (LIGO!);*
4. *Understanding the behaviour of e.g. solutions to the Schrödinger equation;*
5. *Simulate optical systems using coheren light.*

Which Mathematical Tools

UNFORTUNATELY the separation of communities has the effect that we rarely have a discussion of the following very natural issues:

Which mathematical tools serve which purpose?

1. Integration theory secures the "existence" of a FT;
2. The FFT is used to compute (?) the Fourier transform;
3. The theory of *tempered distributions* (L. Schwartz) allows a generlized FT, including Dirac measures;
4. L. Hörmander established the connection to PDE;
5. The theory of *pseudo-differential operators* is closely related to *slowly varying systems, Kohn-Nirenberg calculus*;
6. the *metaplectic group (including the Fractional FT (FrFt) appear e.g. in optics, e.g. Fresnel transforms, fiber optics;*
7. *There is a lot of interest in generalized Wigner transforms (Weyl calculus, quantum theory) recently.*

Common Ground: Linear Spaces

When it comes to the discussion of SIGNALs we often encounter *signal spaces*, i.e. (complex) vector spaces of signals, e.g. trig. polynomials etc.. We learn from linear algebra that such spaces have a basis and thus uniquely determined coordinates, once a basis is fixed. Linear mappings between vector spaces can thus be described by their action on a given basis, stored in the form of a *matrix* $\boldsymbol{A}$.

Composition (and hence inversion) of linear mappings is thus described by matrix multiplication or inversion of matrices.

Variants include the use of *generating systems* (frames) or linear independent families (Riesz sequences) and thus the use of the *pseudo-inverse* based on the SVD of a general matrix, givig a simple approach to dual frames or biorthogonal systems.

The Infinite-Dimensional Situation

Coming to the situation of continuous variable, be it functions on the torus $\mathbb{T}$ or on the integers $\mathbb{Z}$ one encounters already the problem of *infinite dimensional vector spaces, which should be better called not anymore finite dimensional (because they are to "big" to contain a finite basis!).*

It is then necessary to consider not only finite sums, but admit at least some infinite sums, or series. All this opens the discussion of convergence (in some norm, or in some topology, conditional or unconditional, etc.) and leads to the need of (linear) functional analysis.

It is no surprise (at least in retrospect) to realize the influence of Fourier Analysis on the development of Functional Analysis and thus on Mathematical Analysis in general.

A Couple of Examples

1. Classical Fourier series found an "easy" explanation once it was understood that it is just an *orthogonal expansion* in some Hilbert space. Plancherel's Theorem (using the complex exponantials) simply describe a *unitary isomorphism* between $(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2)$ and $\ell^2(\mathbf{Z}_N)$.
2. On the space $\mathcal{S}(\mathbb{R}^d)$ introduced by L. Schwartz the Fourier transform defines a topological automorphism which extends to the dual space $\mathcal{S}'(\mathbb{R}^d)$ (tempered distributions);
3. Any PDE can be viewed as a multiplication operator on the FT side (Hörmander).
4. *Gabor Analysis* requires the discussion of (Banach) *frames in* $\boldsymbol{L}^2(\mathbb{R}^d)$ *and other modulation spaces* $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$, *such* $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d)$ *and* $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^{\infty,\infty}(\mathbb{R}^d)$.

Related Facts at the Operator Level

1. The characterization of Hilbert-Schmidt operators (so-to-say integral operators with kernels of finite energy) appears as the analogue for the matrix representation of linear mappings, but it only allows to describe certain *compact* operators on Hilbert spaces. ... In contrast we have for $\mathcal{S}(\mathbb{R}^d)$:
2. The **Kernel Theorem** is a well-established tool in order to describe operators from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. It also allows to derive a Wey calculus or to define the Wigner distribution (in $\mathcal{S}'(\mathbb{R}^{2d})$) of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$.
3. Also the *Kohn-Nirenberg* calculus or the representation of an operator using the *spreading representation* (in the ultra-weak sense) can be developed on this basis.

What is done at the Finite/Discrete Side

While many of the expressions described so far have an analogue in the setting of LCA groups (some of those are relevant for quantum groups etc.) relatively little has been done (so far) in order to use computational methods based on (numerical) linear algebra in order to approximate (e.g. simulate) the continuous, infinite dimensional situation).
Often the naive approach is taken: Following a "continuous heuristic" in an engineering paper one finds simulations, making use of samples of a given continuous function and applying the FFT whenever a Fourier transform is required.
This raises the question: **When is the output of the discrete computation indicative for the discussed output of continuous operations**. AND: If so, how would we describe (or measure) the approximation quality, etc..

Obtaining a Result at given Resolution I

The typical performance of a given algorithm involves some reduction to a finite dimensional situation, such as taking samples, then a linear algebra operation (matrix multiplication, FFT, composition of spreading matrix, etc.) and finally the return to the continuous domain, e.g. by some kind of (quasi-)interpolation. Ideally one can fix a domain space $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and a target space $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ and would like to have the constructively realizable approximation of $T(f)$ up to given error $\varepsilon > 0$.
Often the proofs provide a uniformly bounded family of operators T_α (of finite rank) which approximate the operator T in the strong operator sense, i.e. for a finite set of input vectors $f_1, ..., f_n$ and up to a given precision $\varepsilon > 0$.
In such a case one can extend the approximation to compact subsets of $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$, such as the unit ball of compactly embedded normed spaces inside $\boldsymbol{X}$.

Well established examples relate the operator $T =$ Fourier Transform (via Riemann integrals on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to the corresponding FFT algorithm, using regular sampling (over a sufficiently large domain) and at the end corresponding interpolation (or quasi-interpolation) methods, such as piecewise linear interpolation.
Correspondingly operations like convolution of taking the STFT (with window and signal in $\boldsymbol{S}_0(\mathbb{R}^d)$) can be realized as well.
A little bit more work (in progress) is required for the approximation of so-called *Anti-Wick operators* (STFT-multipliers) so that it is plausible, that the corresponding eigenvalues and eigenvectors behave well if one takes the limit (more and more samples at a higher rate).
Experimentally (using MATLAB) things look fine!

The key-players for time-frequency analysis I

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \; \lambda = (t, \omega);$$

Some important properties of the STFT I

Starting from the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ with scalar product

$$\langle f, g\rangle_{\boldsymbol{L}^2} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$$

one can find that for fixed g the linear mapping $f \mapsto V_g f$ is isometric from $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$. But also

$$\|V_g f\|_\infty = \max_{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \leq \|f\|_2 \|g\|_2$$

by the Cauchy-Schwarz inequality, since $\|\pi(\lambda)g\|_2 = \|g\|_2$, for any $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
There is also orthogonality for the windows, since we have

$$V_g f(\lambda) = \langle f, \pi(\lambda)g\rangle = \langle \pi(\lambda)^* f, g\rangle,$$

Some important properties of the STFT II

which gives essentially symmetry between f and g:

$$V_g f(t,s) = e^{2\pi i t s} V_{\widehat{g}} \widehat{f}(s,-t).$$

The next property is sometimes called the *covariance property* of the STFT.

Lemma (3.1.3)

Whenever $V_g f$ is defined, we have

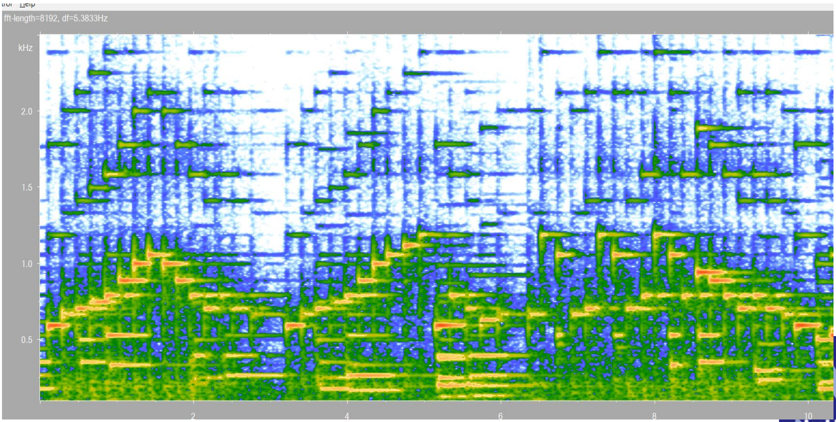
$$V_g(T_u M_\eta f)(x,\omega) = e^{-2\pi i u\cdot\omega} V_g f(x-u,\omega-\eta) \quad (3.14)$$

for $x,u,\omega,\eta \in \mathbb{R}^d$. In particular,

$$\left|V_g(T_u M_\eta f)(x,\omega)\right| = \left|V_g f(x-u,\omega-\eta)\right|.$$

A Typical Musical STFT

A typical piano spectrogram (Mozart), from recording



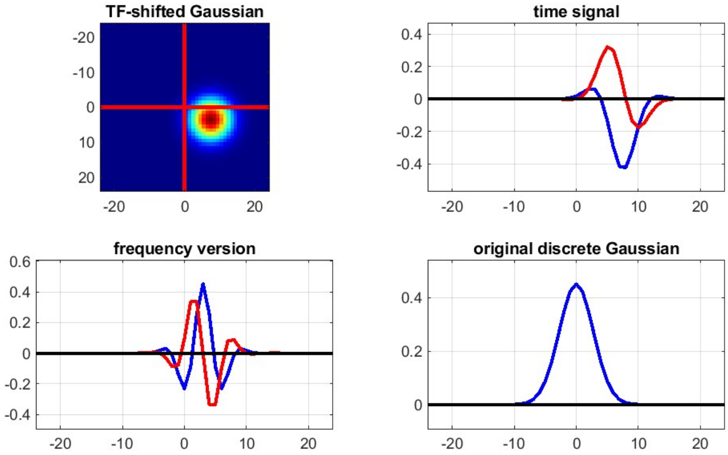
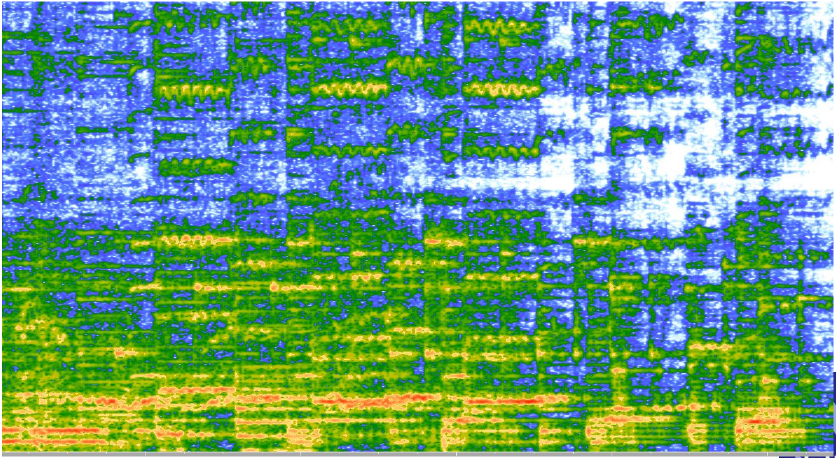
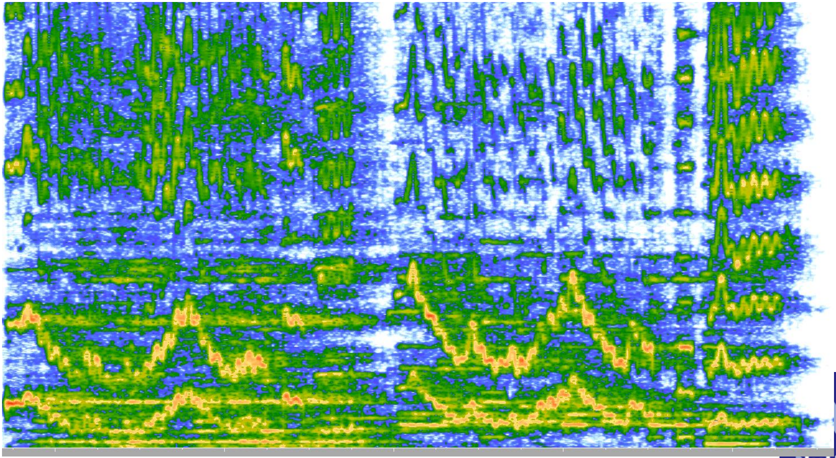


Abbildung: g48TFshifts.jpg

A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979) I

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

A Banach Space of Test Functions (Fei 1979) II

Since one has for any pair $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2,$$

as a simple consequence of the Cauchy-Schwarz inequality, this is stronger then the corresponding norm in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. In fact one has

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

This implies that the range of V_g is a closed, invariant subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the projection operator is (twisted convolution operator), mapping $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$ onto $V_g(\boldsymbol{L}^2(\mathbb{R}^d))$. If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the convolution kernel is in $\boldsymbol{L}^1(\mathbb{R}^{2d})$.
Assuming $\|g\|_2 = 1$ we have the *reconstruction formula*:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda)g,$$

which can be approximated in $\boldsymbol{L}^2$ by Riemannian sums.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ *(dense embedding);*
2. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images), for $1 \leq p \leq \infty$.

Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

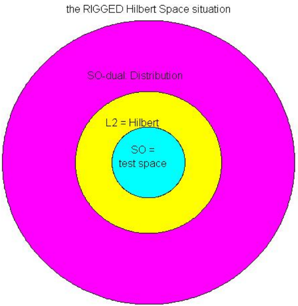


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$,

known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (**[lo83-1]**).
For $\mathbb{R}^d$ the most elegant way (which is describe in **[gr01]** or **[ja18]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}\,dy$$

and the corresponding norm

$$\|f\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $S_0(\mathbb{R}^d)$.

Theorem

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \big(\boldsymbol{W}(\boldsymbol{C}_0,\ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$;
2. $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d)$ *(isometrically);*
3. *Isometrically invariant under TF-shifts*

$$\|\pi(\lambda)(f)\|_{\boldsymbol{S}_0} = \|M_s T_t f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

4. $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ *is an essential double module (convolution and multiplication)*

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d),$$

   in fact a Banach ideal and hence a double Banach algebra.
5. *Tensor product property* $\boldsymbol{S}_0(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{S}_0(\mathbb{R}^d) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(\boldsymbol{S}_0(G)) = \boldsymbol{S}_0(H)$.

1. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ has various equivalent descriptions, e.g.
   - as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$;
   - via *atomic decompositions* of the form

$$f = \sum_{i \in I} c_i \pi(\lambda_i) g \text{ with } (c_i)_{i \in I} \in \ell^1(I).$$

2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under group automorphism;
3. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is invariant under the *metaplectic group*, and thus under the *Fractional Fourier transform* as well as the multiplication with *chirp signals*: $t \mapsto exp(-i\alpha t^2)$, for $\alpha \geq 0$.

In addition $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is quite universally useful in Classical Fourier Analysis and of course for *Time-Frequency Analysis* and *Gabor Analysis*, and as I am going to show also for QHA: Quantum Harmonic Analysis. In short, it is easier to handle than the Schwartz-Bruhat space or even the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, and since $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ it is (much) bigger.

Theorem

1. $\mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$ *via* $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0'$.
2. *Identification of TLIS:* $\boldsymbol{H}_G(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$, $(\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \to \boldsymbol{S}_0(H) \hookrightarrow \boldsymbol{S}_0(G)$ *via* $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \boldsymbol{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \boldsymbol{S}_0(G) \,|\, \mathrm{supp}(\tau) \subset H\}$.

Theorem

1. $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}) = (\mathbf{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^\infty})$, *with* $V_g(\sigma)$ *and* $\|\sigma\|_{\mathbf{S}_0'} = \|V_g(\sigma)\|_\infty$, *hence norm convergence corresponds to uniform convergence on pahse space. Also* w^*-*convergence is uniform convergence over compact subsets of phase space.*
2. $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}) \hookrightarrow (\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$, *with density for* $1 \leq p < \infty$, *and* w^*-*density in* $\mathbf{S}_0'$. *Hence, facts valid for* $\mathbf{S}_0$ *can be extended to* $\mathbf{S}_0'$ *via* w^*-*limits.*
3. *Periodic elements* $(T_h\sigma = \sigma, h \in H)$ *correspond exactly to those with* $\tau = \mathcal{F}(\sigma)$ *having* $\mathrm{supp}(\tau) \subseteq H^\perp$.
4. *The (unique) spreading representation* $T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda) d\lambda$, $F \in \mathbf{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *for* $T \in \mathcal{B}$ *extends to the isomorphism* $T \leftrightarrow \eta(T)$ $\eta : \mathcal{B} \approx \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$, *uniquely determined by the correspondence with*
$$\eta(\pi(\lambda)) = \delta_\lambda, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Some conventions

Scalar product in $\mathcal{HS}$:

$$\langle T, S\rangle_{\mathcal{HS}} = \mathrm{trace}(T * S^*)$$

In **[feko98]** the notation

$$\alpha(\lambda)(T) = [\pi \otimes \pi^*(\lambda)](T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

and the covariance of the KNS-symbol is decisive:

$$\sigma(\pi \otimes \pi^*(\lambda)(T)) = T_\lambda(\sigma(T)), \quad T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Signals ARE mild distributions

Thinking of actual signals and their measurements we can think of a **bilinear pairing**.
The vague definition of a signal is something that can be measured.
The THESIS which works extremely well according to my experience from the last 20 years is: Mild distributions, i.e. continuous linear functionals provide a very good mathematical model for the kind of signals occuring in the real world!! They can be applied (by definition) to test functions $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ (the Feichtinger algebra) in exactly the way we expect the pairing between BRAs and KETs.

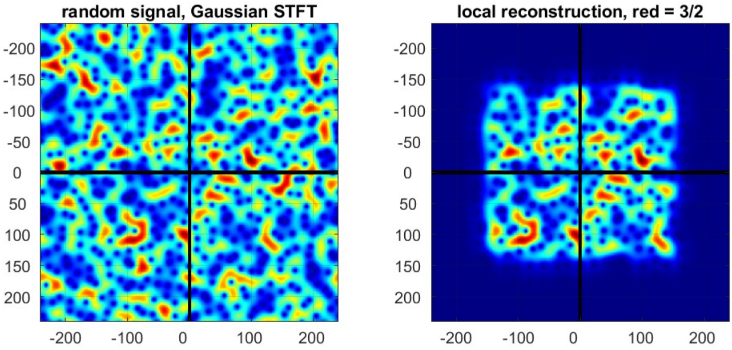


Abbildung: GabLocRec01.jpg

A Couple of Natural Questions

We are facing the following natural questions:

1. What is the influence of the choice of the window? Maybe one has for each different window another family of "signals"?
2. Is it necessary to know the boundedness for each offset in both time and frequency? Or is it enough to know it for a certain lattice (sampling in the time domain and the frequency domain). Could there be a blow-up "in between" those lattice points?
3. What about convergence of signals? When should we view signals close to each other? What does it mean that the CD allows to realize a perfect reconstruction of the concert in your home?

Mild convergence and w^*-convergence

It is easy to recognize that the described convergence is just the w^*-convergence in a dual Banach space, which is correctly described by nets (generalized sequences). However, since $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a *separable* Banach space, it is enough to work with sequences. Consequently the pointwise convergence of functionals (meaning application to any given $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ implies norm boundedness of the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$. Recall that the dual norm on $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is just

$$\|\sigma\|_{\boldsymbol{S}_0'(\mathbb{R}^d)} := \sup_{\|f\|_{\boldsymbol{S}_0}\leq 1} |\sigma(f)|.$$

As a matter of fact we have

$$\|\sigma\|_{\boldsymbol{S}_0'(\mathbb{R}^d)} = \sup_{(t,s)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |V_g\sigma(t,s)|.$$

Creating Discrete Hermite Functions

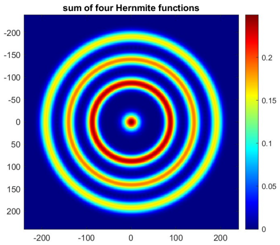


Abbildung: The spectrogram of a sum of four discrete Hermite functions

Discrete Hermite Functions

It is well known that the Hermite functions are (among others) joint eigenvectors of Anti-Wick operators arising from rotationally symmetric *upper symbols* (multipliers). These operators commute with the Fourier transform, and in fact with Fractional Fourier transform. Hence it is plausible that the corresponding discrete version (which can be realized in MATLAB) using the covariance of the Kohn-Nirenberg symbol of matrices. They commute with the DFT and are thus also eigenvectors of the (normalized) FFT. Such discrete Hermite functions (DHFs) can also be used to realize (a discrete variant of) the Fractional Fourier Transform (FrFT) by building the corresponding *Hermite Multipliers*.

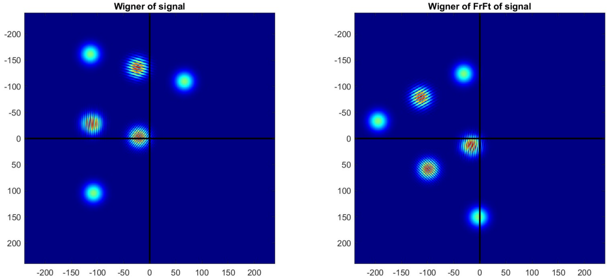


Abbildung: DemWigHerm20A.jpg: A discrete Wigner function of a signal and the signal after Fractional Fourier transform.

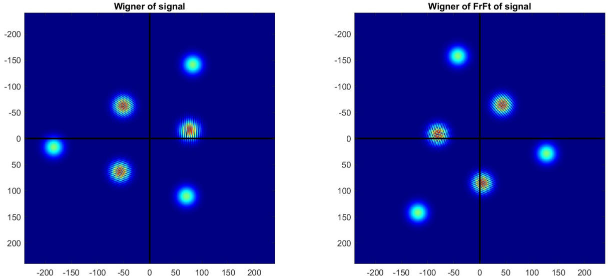


Abbildung: DemWigHerm20B.jpg: A discrete Wigner function of a signal and the signal after Fractional Fourier transform.

Conclusion

In the current situation there is too little exchange between the different communities. Specialization is going ahead without too much reconnection to the **"need" of say the applied side**.
On the other hand one has the impression that in some of the modern and fashionable areas of research things are reinvented with new names or without verifying the existing literature.
Conceptual Harmonic Analysis may be a way to mitigate these discrepancies a bit and allow simulations which lead to **quantitative results** based on the combination of actual computations and accompanying analytic considerations (estimates).
Reliable transfer between the discrete and continuous setting helps to obtain quantitative results.

THANKS to the audience

THANKS you for your attention

maybe you visit www.nuhag.eu

and checkout other talks and related material. (password protected!) hgfei

`www.nuhag.eu/bibtex`: all papers

`www.nuhag.eu/talks`: many talks on related subjects

NOTE: Talks require access code:

"visitor" and "nuhagtalks"