Recovery of Functions from Regular Samples via Time-Frequency Methods

Hans G. Feichtinger,
Faculty of Mathematics, University Vienna& ARI (OEAW)
hans.feichtinger@univie.ac.at
www.nuhag.eu

University of Linz, AT, Sept. 1st, 2025

.

OEMG Section: Analysis and Topology

Abstract: Recovery from Regular Samples I

It is the purpose of this note to demonstrate how time-frequency methods can be used in order to get a quantitative understanding of the rate of *recovery for functions (or their Fourier transforms) from sampled versions*.

The approach presented reveals that (once more) the Segal algebra $S_0(R^d)$ can be used as a key player, while on the other hand certain Shubin classes $Q_s(R^d)$ allow a symmetric formulation, due to the fact that they are compactly embedded into $S_0(R^d)$ for $s > d$ and in addition also Fourier-invariant. The are useful because they form a family of Fourier invariant Banach algebras with respect to both convolution and pointwise multiplication.

Abstract: Recovery from Regular Samples II

The key aspect is to relate the recovery from regular samples to the transition from the a-periodic phase space to a flat torus situation and to *periodization in phase space*. This allows to allow well-established method from the theory of Wiener Amalgam spaces over $\mathbb{R}^d$. This manuscript offers the description of first principles, but the approach has the potential of far reaching generalizations. By addressing the problem of transition between discrete and continuous signals it also provides a contribution to the general idea of **Conceptual Harmonic Analysis.**

Some Basic References I

L. Abreu, P. Balazs, N. Holighaus, F. Luef, and M. Speckbacher.
Time-frequency analysis on flat tori and Gabor frames in finite dimensions.
Appl. Comput. Harmon. Anal., page 101622, 2024.

M. Ehler, K. Gröchenig, and A. Klotz.
Quantitative estimates: How well does the discrete Fourier transform approximate the Fourier transform on $\mathbb{R}$.
arXiv preprint arXiv:2403.03810, 2024, to appear SIAM Rev. 2025

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

H. G. Feichtinger and N. Kaiblinger.
Quasi-interpolation in the Fourier algebra.
J. Approx. Theory, 144(1):103–118, 2007.

Some Basic References II

N. Kaiblinger.
Approximation of the Fourier transform and the dual Gabor window.
J. Fourier Anal. Appl., 11(1):25–42, 2005.

E. Cordero, H. G. Feichtinger, and F. Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lect. Notes Math.*, pages 1–33. Springer, Berlin, 2008.

H. G. Feichtinger.
Thoughts on Numerical and Conceptual Harmonic Analysis.
In A. Aldroubi, C. Cabrelli, S. Jaffard, and U. Molter, editors, *New Trends in Applied Harmonic Analysis. Sparse Representations, Compressed Sensing, and Multifractal Analysis*, Applied and Numerical Harmonic Analysis, pages 301–329. Birkhäuser, Cham, 2016.

DFT and the Fourier Transform I

The natural domain for the Fourier transform, given by

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i s x} dx$$

seems to be $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, although the Fourier inversion formula only holds if in addition $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$.
On $\boldsymbol{L}^1 \cap \boldsymbol{L}^2(\mathbb{R}^d)$ this Fourier transform is isometric, and thus can be extended to a unitary automorphism of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. This results is know ans Plancherel's Theorem.

DFT and the Fourier Transform II

One also shows that convolution is going to pointwise multiplication, and that one can recover a band-limited function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}) \subset Q^d = [-1/2, 1/2]^d$ from regular samples $(f(k))_{k \in \mathbb{Z}^d}$ by the Shannon series expansions

$$f(x) = \sum_{k \in \mathbb{Z}} f(k)\, \mathrm{SINC}(x-k),$$

with convergence in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and uniformly, with $\mathrm{SINC} = \mathcal{F}^{-1}(\mathbf{1}_{Q^d})$.

Shannon Sampling Theorem: better localized version I

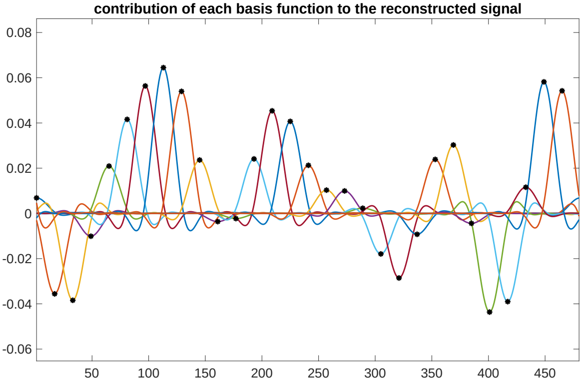
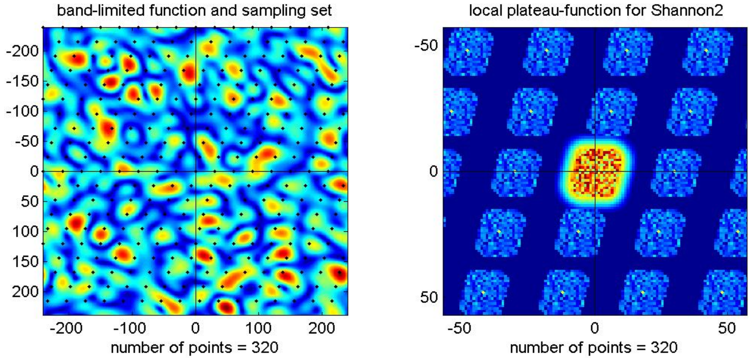


Figure: shannplot009.eps: Shannon's Theorem, better localized

$2D$-version of Shannon's Theorem



What about "Almost Band-limited" Functions? I

It is well known (in engineering and also mathematically) that sampling corresponds to periodization in the frequency domain, and thus it is clear that for the case that the so-called *Nyquist condition* is violated the reconstruction will suffer from the *aliasing effect*.

More concrete: Sampling along some lattice in the signal domain corresponds to periodization with respect to the orthogonal lattice in the frequency domain. Since $\mathbb{Z}^d \lhd \mathbb{R}^d$ has $\mathbb{Z}^d \lhd \widehat{\mathbb{R}^d}$ as the orthogonal lattice one finds easily that $(a\mathbb{Z}^d)^\perp = (1/a)\mathbb{Z}^d$. In other words, taking $a \to 0$, as the sampling lattice gets more and more refined the periodicity on the FT-side expands and tends to be negligible.

What about "Almost Band-limited" Functions? II

Using the localized reconstruction it is easy to show that reduction of the aliasing effect in the uniform norm is controlled by the $\boldsymbol{L}^1$-norms of the tails. Note that this is not a valid estimate if the SINC-function is used.

Unfortunately a similar result in the spirit of Plancherel's Theorem (i.e.for the $\boldsymbol{L}^2$-norms) is not possible, because $\boldsymbol{L}^2$-norms do not control the pointwise sampling sequence which should be controlled in the norm of $\ell^2(\mathbb{Z}^d)$!

However, using suitable *Wiener amalgam spaces* one can estimate the aliasing error in the $\boldsymbol{L}^2(\mathbb{R}^d)$-norm, if the tails are measured in the $\boldsymbol{W}(\boldsymbol{L}^2, \ell^1)(\mathbb{R}^d)$-norm!

BUPUs for (general) Wiener Amalgams (1980) I

Next we discuss BUPUs (Bounded Uniform Partitions of Unity): This are families of the form $T_k\varphi, k \in \mathbb{Z}^d$, arising from a compactly supported function (of some *regularity*), with

$$\sum_{k \in \mathbb{Z}^d} \varphi(x-k) \equiv 1, \quad x \in \mathbb{R}^d.$$

Given a Banach space $\boldsymbol{B} \hookrightarrow \boldsymbol{L}^1_{loc}(\mathbb{R}^d)$ we assume that the action of the members of this family is uniformly bounded: $\exists C > 0$ with:

$$\sup_{k \in \mathbb{Z}^d} \| T_k\varphi \cdot f \|_{\boldsymbol{B}} \leq C \|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{1}$$

In this situation we call the family $(T_k\varphi)_{k \in \mathbb{Z}^d}$ a $\boldsymbol{B}$-BUPU, or a BUPU for $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. For the continuous version we assume $\sup_{k \in \mathbb{R}^d} \| T_k\varphi \cdot f \|_{\boldsymbol{B}}$ in instead of (1).

Definition of $W(B, Y)$ I

With this terminology, and given some **solid BK-space** $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ on the lattice $\mathbb{Z}^d$ we define

Definition

$$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{Y}) := \{f \in \boldsymbol{B}_{loc} \mid \|f \cdot T_k\varphi\|_{\boldsymbol{B}} \in \boldsymbol{Y}\} \tag{2}$$

endowed with the natural norm

$$\|f\|_{\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y})} := \|(\|f \cdot T_k\varphi\|_{\boldsymbol{B}})_{k\in\mathbb{Z}^d}\|_{\boldsymbol{Y}} .$$

We will write φ_k for $T_k\varphi$ in the future.

A slightly smoothed histogram

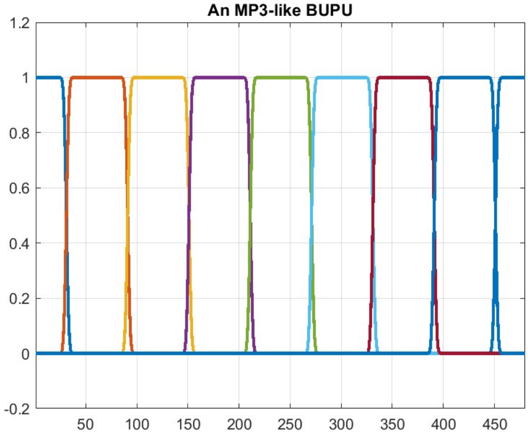


Figure: Indicator Functions with smooth boundaries

Linear and Cubic B-splines

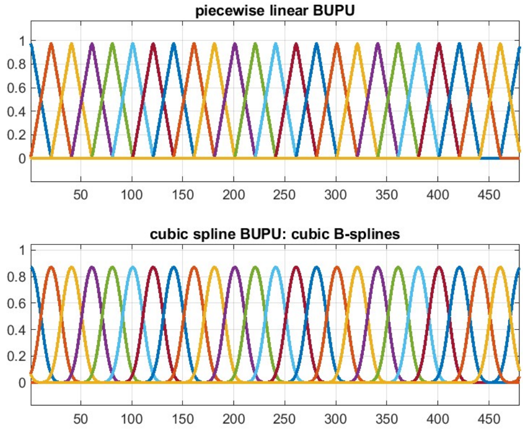


Figure: Two natural types of BUPUs, refinable by dilation

With this terminology, and given some **solid BK-space** $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ on the lattice $\mathbb{Z}^d$ we define

Definition

$$\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y}) := \{f \in \boldsymbol{B}_{loc} \mid (\|f T_k \varphi\|_{\boldsymbol{B}})_{k\in\mathbb{Z}^d} \in \boldsymbol{Y}\} \tag{3}$$

which can be endowed with the natural norm

$$\|f\|_{\boldsymbol{W}(\boldsymbol{B},\boldsymbol{Y})} := \|(\|f \cdot T_k\varphi\|_{\boldsymbol{B}})_{k\in\mathbb{Z}^d}\|_{\boldsymbol{Y}} \,.$$

Of course one can show that different $\boldsymbol{B}-$BUPU define the same space and equivalent norms. It is also not important to choose $\mathbb{Z}^d$ for a BUPU, there is a more general definition, which includes not only different lattices in $\mathbb{R}^d$, but also irregular BUPUs, with $\mathrm{supp}(\psi_i) \subseteq B_\delta(x_i)$ with controlled overlap of these balls.

$\mathcal{F}L^p(\mathbb{R}^d)$ as a local component I

The above setting is well suited in order to define (and study) Wiener amalgam spaces of the form . These spaces correspond to the Triebel-Lizorkin spaces with respect to wavelet, while modulation spaces (below) are the analogue of Besov spaces. They can be characterized using mixed norms in a Gabor setting.

[fe90], H. G. Feichtinger.
Generalized amalgams, with applications to Fourier transform.
Canad. J. Math., 42(3):395–409, 1990.

A useful (!) variant of the Hausdorff-Young Theorem reads then:

Theorem

One has for $1 \leq r \leq p \leq \infty$:

$$\mathcal{F}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^r)\right) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^r, \ell^p).$$

Recovering a function from a periodized version I

We first verify the approximate recover (via localization) of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ by localization. In the case of compact support this is quite easy and hopefully reminds the proof of the Shannon Sampling Theorem (viewed on the Fourier transform side). Formulated as a clear mathematical statement we have, by writing $\sqcup\!\sqcup_\alpha = \sum_{k \in \mathbb{Z}^d} \delta_{\alpha k}$ the following result:

Recovering a function from a periodized version II

Lemma

Given $f \in \mathbf{S}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ there exists a compactly supported function $\varphi \in \mathbf{S}_0(\mathbb{R}^d)$ and $p_0 > 0$ such that one has for any $p \geq p_0$

$$\|f - (f * ⧢_p) \cdot \varphi\|_{\mathbf{S}_0} \leq \varepsilon. \tag{4}$$

We also claim that the family of operators $f \mapsto (f * ⧢_p) \cdot \varphi$ is uniformly bounded on $\mathbf{S}_0(\mathbb{R}^d)$, at least for any fixed $\varphi \in \mathbf{S}_0(\mathbb{R}^d)$.

A simplified version uses just $\varphi \in \mathbf{C}_c(\mathbb{R}^d)$ and makes the claim using $f \in \mathbf{W}(\mathbf{C}_0, \ell^1)(\mathbb{R}^d)$ and the norm convergence in that space. It requires only the boundedness of the periodic version and $\|f - f \cdot \varphi\|_{\mathbf{W}(\mathbf{C}_0,\ell^1)} < \varepsilon$ and using $\|\varphi\|_\infty = 1$.
Corresponding weighted ℓ^1-conditions, such as $\mathbf{W}(\mathbf{C}_0, \ell^1_w)$ allow to claim convergence rates for $p \to \infty$.

Engineering Papers I

For the application of the Fourier transform in an engineering context (often dealing with sampled functions) the use of the DFT (Discrete Fourier transform, implemented as the FFT, the Fast Fourier transform) is typically justified by the argument that "the computer only allows to compute with finite length vectors".

But there are only hand-waving arguments (like the transition from Riemann integrals to Riemannian sums) that really justify this transition, and in fact very few solid results which justify this transition for the case of non-bandlimited (but still smooth) functions.

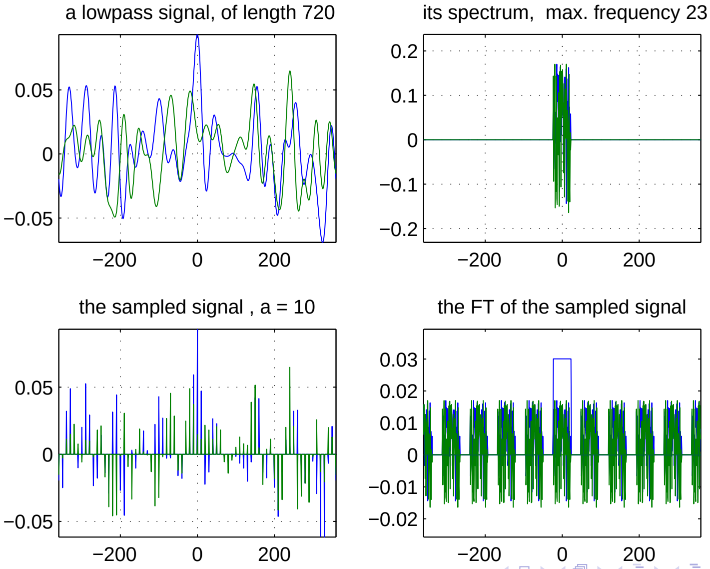
Hence it is a task of approximation theory and Fourier Analysis to address this transition, making use of suitable function spaces.

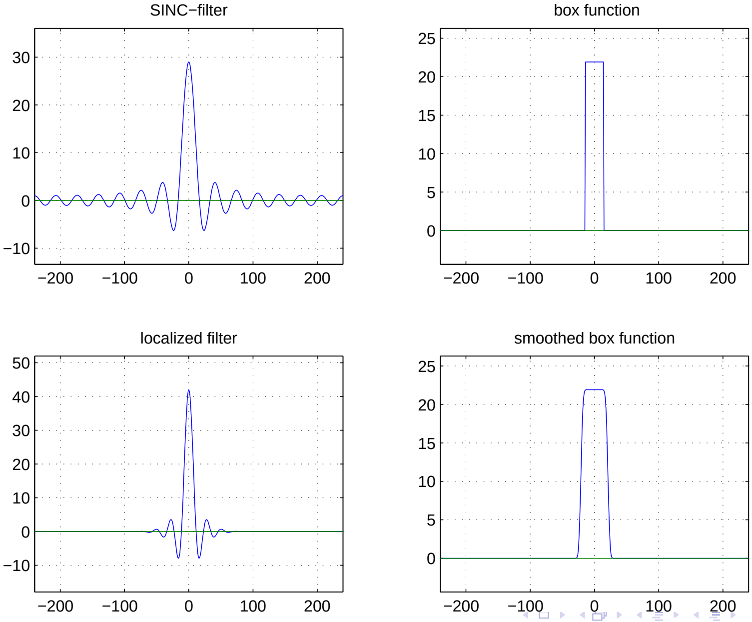
Sampling and Periodization

H. G. Feichtinger. Sampling via the Banach Gelfand Triple.
In Stephen D. Casey, Maurice Dodson, Paulo J.S.G. Ferreira, and Ahmed Zayed, editors, *Sampling, Approximation, and Signal Analysis Harmonic Analysis in the Spirit of J. Rowland Higgins*, Appl. Num. Harm. Anal., pages 211–242. Cham: Springer International Publishing, 2024.

The key-player for the description of the process of regular sampling (in its normalized from, i.e. sampling over $\mathbb{Z}^d$) is the Dirac comb: $⊔⊔ := \sum_{k \in \mathbb{Z}^d} \delta_k$.
For any bounded function $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ the product is a weighted Dirac comb: $h \cdot ⊔⊔ = \sum_{k \in \mathbb{Z}^d} h(k)\delta_k$.
Using the fact that (by Poisson's formula!) the Dirac comb is Fourier invariant we have by the *convolution theorem*:

$$\mathcal{F}(⊔⊔ \cdot h) = \mathcal{F}(⊔⊔) * \mathcal{F}(h) = ⊔⊔ * \widehat{h} = \sum_{k \in \mathbb{Z}^d} T_k \widehat{h}.$$


a lowpass signal, of length 720
its spectrum, max. frequency 23
the sampled signal , a = 10
the FT of the sampled signal
0.05
0
−0.05
−200
200
0.2
0.1
−0.1
−0.2
0.03
0.02
0.01
−0.01
−0.02


SINC−filter
box function
localized filter
smoothed box function

Symmetric Viewpoint

Of course this relationship is symmetric: In other words, since convolution goes into pointwise multiplication under the (generalized) Fourier transform we can describe periodization as sampling on the Fourier transform side.
And of course it makes sense to combine the two operators: We talk about Sampling-Periodization or Periodization-Sampling operators. In the normalized situation we will assume that the period is a natural number (!) $p \in \mathbb{N}$. In general we assume that the period is an integer multiple of the sampling distance.

If we want to obtain a maximally symmetric version providing N samples we should choose $p = \sqrt{N}$ and the sampling distance $h = 1/p = 1/\sqrt{N}$. We write PS_N for the operator which outputs the sampling sequence $F(0), F(h), \ldots F((N-1)h)$, with F being the p-periodic version of f: $F(x) = \sum_{k \in \mathbb{Z}^d} f(x - kp)$.

Function space estimates I

Obviously the order of sampling and periodization does not matter if the period p is an integer multiple of h!!!

Next we want to verify the boundedness of this PS-operator on Wiener's algebra $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$.

Given any function g with support in some ball of radius $R > 0$ such that $B_R(x) \cap kp + B_R(x) = \emptyset$ for $k \neq 0$ it is clear that the periodization satisfies $\|\sum_{k\in\mathbb{Z}^d} T_{kp}g\|_\infty = \|g\|_\infty$. By construction of $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ there exists $C = C(R) > 0$ such that any $f \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)$ can be decomposed into an absolutely convergent sum of functions g_n as above, with $\sum_{n=1}^{\infty} \|g_n\|_\infty \leq C\|f\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^1)}$, and consequently $\|F\|_\infty \leq C\|f\|_{\boldsymbol{W}(\boldsymbol{C}_0,\ell^1)}$.

Thus the weighted Dirac comb $\sum_{k\in\mathbb{Z}^d} F(hk)\delta_{hk}$ belongs to the space of translation bounded measures $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d)$.

Function space estimates II

Next we introduce the Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ (Feichtinger's algebra), which has been introduced as Wiener amalgam space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$. We have a dense embedding

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}),$$

and consequently, the opposite inclusion of the dual spaces

$$(\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d).$$

Any band-limited function in $\boldsymbol{L}^1(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$, but also any compactly supported function in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. Moreover, one has

$$\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d) \quad \text{and} \quad \mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d).$$

The elements of $\boldsymbol{S}_0'(\mathbb{R}^d)$ are called mild distributions.

Kernel Theorem and Spreading Function

An important additional feature of these spaces is the fact that (as an extension of the kernel theorem for Hilbert-Schmidt operators, or as a much more simple version of the Schwartz Kernel Theorem) with $Tf(g) = \sigma(f \otimes g)$, for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, or:

Theorem

Any operator $T \in \mathcal{L}\left(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\right)$ *has a (distributional) kernel* $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$. *In other words, there exist bounded functions* $K_\rho(x,y)$ *on* $\mathbb{R}^{2d}$ *such that for any* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the output* $T(f)$ *is approximated (in the* w^*-*sense) by the integral operator*

$$T_\rho f(x) = \int_{\mathbb{R}^d} K_\rho(x,y)\, f(y) dy.$$

The Spreading Representation I

Given all the invariance properties of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual space it is also possible to associate to each $T \in \mathcal{L}\left(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\right)$ a unique spreading function, which kind of represents T as a (limit of) superposition(s) of time-frequency shifts $\pi(\lambda) = M_s T_t,\ t, s \in \mathbb{R}^d$:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(T)(\lambda)\pi(\lambda) d\lambda.$$

For $\mathcal{HS}$-operators one has $\|T\|_{\mathcal{HS}} = \|etaT\|_{\boldsymbol{L}^2(\mathbb{R}^{2d})}$.
It is well known that a convolution operator C_μ (commuting with all translations) can be constituted from convolutions only, so $\eta(C_\mu)$ is supported on $\mathbb{R}^d \times \{0\}$, while a multiplication operator M_h is constituted from pure modulations, so $\text{supp}(\eta(M_h)) \subset \{0\} \times \mathbb{R}^d$.

The Spreading Representation II

Lemma

Given positive constants $a, b > 0$ we obtain the spreading representation of the $PS - SP$-operators from the following facts:

1. *The periodization operator* $\mathrm{Per}_b : f \mapsto f * \sqcup\!\sqcup_b = \sum_{k\in\mathbb{Z}^d} T_{bk} f$ *has the spreading representation*

$$\eta(\mathrm{Per}_b) = \sum_{k\in\mathbb{Z}^d} \delta_{bk,0} \in \boldsymbol{S}_0'(\mathbb{R}^{2d}),$$

which is supported on the discrete subgroup $H_1 = b\mathbb{Z}^d \times \{0\} \lhd \mathbb{R}^{2d}$.

2. *The sampling operator* $\mathrm{Smp}_a : f \mapsto \sqcup\!\sqcup_a \cdot f = \sum_{l\in\mathbb{Z}^d} f(al)\delta_{al}$ *can be described as the Fourier version of the periodization operator with* $a = 1/b$*, hence it has the spreading representation*

Lemma

The sampling and the periodization operator commute if and only if b is a natural multiple of $1/a$. In the positive case the spreading symbol for the (equal) operators $\mathrm{PS}_{p,\alpha}$ or $\mathrm{SP}_{\alpha,p}$ is (up to normalization) just $⊔⊔_{b\mathbb{Z}^d \times (1/a)\mathbb{Z}^d}$, which is a commutative group of TF-operators.

$$\eta(\mathrm{PS}_{p,\alpha}) = \eta(\mathrm{SP}_{\alpha,p}) = ⊔⊔_{b\mathbb{Z}^d \times (1/a)\mathbb{Z}^d} = \delta_{kb,n/a} \in \boldsymbol{S}_0'(\mathbb{R}^{2d}).$$

In other words, the spreading function of this operator is the indicator function of a commutative lattice of TF-shifts!!
Note: This situation is quite similar to the case of the spreading function for a Gabor frame operator $S(g,a,b)$ associated to a triple (g,a,b), which supported on the adjoint lattice $\Lambda = (1/b)\mathbb{Z}^d \times (1/a)\mathbb{Z}^d$. If this is a commutative lattice (integer oversampling) we can make use of the *Zak transform.*

Spreading and twisted convolution I

If we want to visualize the effect of periodization of a function $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ by looking at the corresponding spectrograms we find that it corresponds to periodization of the spectrogram.
In a similar way, sampling, or periodization in the frequency domain, corresponds to

From Heuristics to Mathematical Analysis I

We have to express some WARNINGS in order to avoid a naive overinterpretation of the visual impression obtained by MATLAB experiments. We take care of the concerns by the following reasoning:

- First we recall how periodization in the $2d$-dimensional setting can be realized;
- Secondly we relate the situation to the $2d$-sampling problem
- Third: We get rid of the appearance of phase factors for valid approximate recovery (by localization in phase space).

more next slide!

Doubling the period and the sampling rate I

In the approximation of functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ from (finite) samples the combination of periodization and sampling plays an important role. For us the interchange with the DFT is also important for a description.

Obviously there are many possible ways of choosing the periods and the sampling rates, but it is clear that some of them are in a nice/natural relationsship (of algebraic nature), namely if one period p_2 is the (integer) multiple of another period p_1, and similar with sampling rates s_2 could be an integer fraction of s_1.

Specifically one can ask, what happens when one starts with a given operator and then doubles the period and the sampling rate. In other words, how to the corresponding vectors of length N and $4N$ relate to each other naturally.

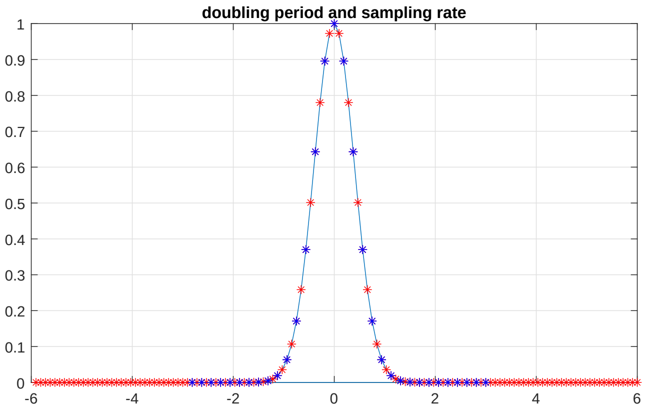


Figure: Adding the red values: period is twice as long and the sampling rate is twice as big, i.e. two time the new step-width is the original (blue) one.

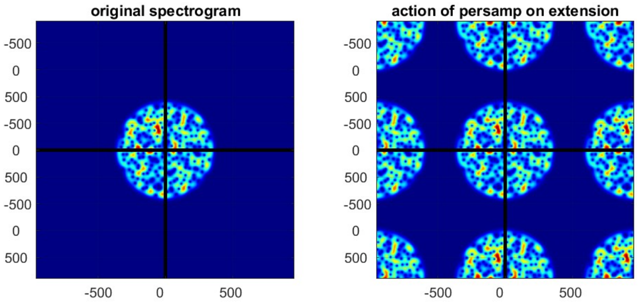


Figure: The picture in the STFT domain. Actually, it is the (twisted) convolution with a 2D Dirac comb with 4 points, so it is a 2D-periodization of the spectrogram.

Periodization and Sampling at the spectrogram level I

Theorem

We consider a tight Gabor family $(g_\lambda)_{\lambda\in\Lambda}$ *with* $\Lambda = 1/2\mathbb{Z}^d$ *and* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. *Thus any* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *has a representation of the from* $f = \sum_{\lambda\in\Lambda} V_g f(\lambda)\pi(\lambda)g$, *and the coefficient norm of* $(V_g f(\lambda))_{\lambda\in\Lambda}$ *in* $(\ell^1(\Lambda), \|\cdot\|_1)$ *defines an equivalent norm on* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. *In addition the* $\boldsymbol{S}_0'$*-norm of* $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ *is equivalent to the* $\ell^\infty(\Lambda)$*-norm of its Gabor coefficients* $V_g(\sigma)|_\Lambda$.
In this case any periodization operator PS_p *realizing a periodization with period* p *and sampling at the rate* $1/p$, *for some* $p \in 2\mathbb{N}$ *(any even natural number) defines a bounded operator from* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *into* $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ *which commutes with the Fourier transform. At the level of Gabor coefficients it corresponds to periodization with respect to the lattice* $p\mathbb{Z}^{2d}$.

Periodization and Sampling at the spectrogram level II

The decisive argument can be compressed in the following chain of equations: Let us write H_p for the group $p\mathbb{Z}^d \times p\mathbb{Z}^d \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ for some even $p \in \mathbb{N}$. Consequently we have for any $\lambda \in 0.5 \cdot \mathbb{Z}^{2d}$:

$$V_g(\mathrm{PS}_p f)(\lambda) = \langle \sum h \in H_p \pi(h) f, \pi(\lambda) g \rangle =$$

$$= \langle \sum_{h \in H_p} f, \pi(\lambda - h) g \rangle = \sum_{h \in H_p} V_g(\lambda - h).$$

Periodization and Sampling at the spectrogram level III

Given any lattice $\Lambda = a\mathbb{Z}^d$ with $a < 1$ there exists tight Gabor frames (perfect reconstruction) with very good TF-localization, using the $S^{-1/2}$-method. Partial sums approximate functions $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ in the norm of this Segal algebra, hence in any $\boldsymbol{L}^p(\mathbb{R}^d)$.

$$\mathrm{GM}_p(f) := \sum_{\lambda \in Q_p} V_g f(\lambda)\pi(\lambda)g. \tag{5}$$

Since for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ the Gabor coefficients are in $\ell^1(\Lambda)$ it is clear that we have

$$\lim_{p\to\infty} \|\mathrm{GM}_p(f) - f\|_{\boldsymbol{S}_0} = 0, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{6}$$

Periodization and Sampling at the spectrogram level IV

The MAIN RESULT obtained by these methods reads as follows:

Theorem

Under the assumptions of Theorem 8 we have: the sequence of operators $f \mapsto \mathrm{GM}_p(\mathrm{PS}_p f)$, with $p \in 2\mathbb{N}$ is uniformly bounded and satisfies

$$\lim_{p \to \infty} \|\mathrm{GM}_p(\mathrm{PS}_p f) - f\|_{\mathbf{S}_0} = 0, \quad f \in \mathbf{S}_0(\mathbb{R}^d). \tag{7}$$

Featuring Properties of $S_0(\mathbb{R}^d)$ I

By definition $\boldsymbol{S}_0(\mathbb{R}^d)$ consists of those functions in $\boldsymbol{L}^2(\mathbb{R}^d)$ (or $\boldsymbol{L}^1(\mathbb{R}^d)$, this is equivalent) whose STFT (Short-Time Fourier Transform) $V_g f$ with respect to - say - the Gaussian window g belongs to $\boldsymbol{L}^1(\mathbb{R}^{2d})$, with the norm $\|f\|_{\boldsymbol{S}_0} := \|V_g f\|_{\boldsymbol{L}^1(\mathbb{R}^{2d})}$. However, the internal structure of the STFT (which is highly redundant!) can be used to derive that this STFT has automatically some amount of "smoothness" and thus this condition only dictates a global behaviour. More precisely we have: Given any $0 \neq g \in \boldsymbol{S}_0(\mathbb{R}^d)$ (e.g. a Gaussian or any Schwartz function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ we have: $V_g f \in \boldsymbol{S}_0(\mathbb{R}^{2d}) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)()$ and for some universal constant $C > 0$ (depending only on the choice of norms), one has

$$\|V_g f\|_{\leq} C\|f\|_{\boldsymbol{S}_0}\|g\|_{\boldsymbol{S}_0}, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

Featuring Properties of $S_0(\mathbb{R}^d)$ II

This characterization makes use of a window g (or Gabor atom), there is also a description which does not depend on a window g, using the Wigner distributions associated with f.
As a corollary, one has:

Corollary

For any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ one finds a constant $C_\Lambda > 0$ such that one has

$$\|(V_g f(\lambda))_{\lambda \in \Lambda}\|_{\ell^1(\Lambda)} \leq C_\Lambda \|f\|_{S_0} \|g\|_{S_0}, \quad f \in S_0(\mathbb{R}^d).$$

Gabor families

For the recovery of the periodized version (in the TF-sense!) over the lattice $0.5\mathbb{Z} \times 0.5\mathbb{Z}$ we will make use of the tight Gabor atom derived from the Gauss function.

This lattice is a Gabor lattice of redundancy 4 (chosen for simplicity), and suitably normalized the Gabor frame operator S is in fact very close to the identity operator. Hence easily invertible, and also $S^{-1/2}g$ can by also computed and is also very close to the Gaussian.

Thus we have full reconstruction of f from the sampled STFT over the given lattice $1/2(\mathbb{Z} \times \mathbb{Z})$ or even $\sqrt{1/2}(\mathbb{Z} \times \mathbb{Z})$.

Due to the estimate of the ℓ^1-norm over the lattice it is easy to find a compact domain which gives good reconstruction.

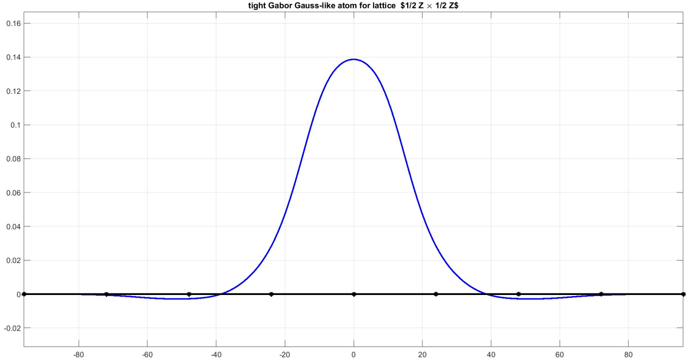


Figure: gausstgt4.jpg

Summary

The qualitative key ingredients of the proposed approach:

1. The periodization-sampling operator as a typical example of an operator in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, whose spreading symbol is the Haar measure of a commutative TF-lattice, e.g. $p\mathbb{Z}^{2d}$,$p \in \mathbb{N}$.
2. When restricted to TF-lattice $0.5 \cdot \mathbb{Z}^{2d}$ the STFT $V_g f$ *commutes with periodization*, *without phase factors p* even;
3. There are compatible Gabor systems of controlled redundancy ≤ 4 (e.g. $\sqrt(2)$) allowing localized phase space reconstruction;
4. The discrete $\ell^1(0.5 \cdot \mathbb{Z}^{2d})$-norm for $V_g f$ is equivalent to $\|f\|_{\boldsymbol{S}_0}$!
5. Wiener amalgam estimates allow to control the approximation rate as $p \to \infty$ within $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

A Compact Notational Summary I

PERSAMP-operator $\quad ⧢(f) := \sqcup\sqcup \cdot (\sqcup\sqcup * f) = \sqcup\sqcup * (\sqcup\sqcup \cdot f), \quad (8)$

$$⧢_\rho(f) := \sqcup\sqcup_{1/\rho} \cdot (\sqcup\sqcup_\rho * f) = \sqcup\sqcup_{1/\rho} * (\sqcup\sqcup_\rho \cdot f). \tag{9}$$

For $\rho \in \mathbb{N}$ one has, by rescaling equation (9):

$$D_\rho \sqcup\sqcup \cdot (St_\rho \sqcup\sqcup * f) = St_\rho \sqcup\sqcup * (D_\rho \sqcup\sqcup \cdot f) \tag{10}$$

Taking limits in the w^*-sense one actually gets:

$$\mathbf{1} \cdot (\delta_0 * f) = f = \delta_0 * (\mathbf{1} \cdot f).$$

A Compact Notational Summary II

In other words, we can expect to have the following relation:

$$\lim_{\rho\to\infty} Ш_\rho(f) = f, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d), \tag{11}$$

$$\text{or } \lim_{\rho\to\infty} Ш_\rho = \mathrm{Id}_{\boldsymbol{S}_0} \quad \text{in the strong operator topology.} \tag{12}$$

Note that $Ш_\rho$ is an operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$ commutes with the Fourier transform, or in symbols

$$Ш_\rho \circ \mathcal{F} = \mathcal{F} \circ Ш_\rho. \tag{13}$$

THANKS to the audience

THANKS you for your attention

All my papers are found at
https://nuhagphp.univie.ac.at/home/feipub$_d$*b.php*
and my talks are at
https://nuhagphp.univie.ac.at/home/feitalks.php
NOTE: Talks require access code:
"visitor" and "nuhagtalks"