Signals and Operators: the World of Mild Distributions

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September 23rd, 2025

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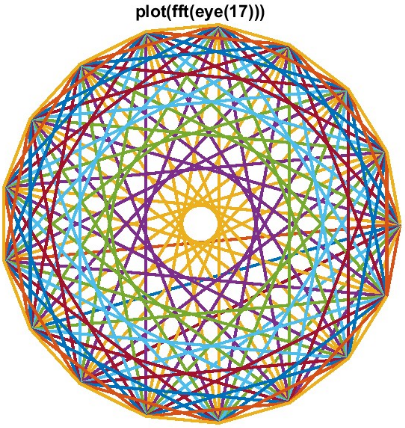
Workshop on Mathematical Signal Processing



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Links to my Homepage

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https://nuhagphp.univie.ac.at/home/feipub_db.php
and my talks are at www.nuhag.eu/feitalks (OR)
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Maybe my alternative approach to Fourier analysis as provided by

www.nuhag.eu/ETH20 is found interesting (look for ETH EXTRA Lecture Notes 2024 in the list of publications).

Other lecture notes should be found at

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Abstract Submitted I

Taking an axiomatic viewpoint mathematicians avoid to define what certain "*objects are*". Instead one starts from certain relations and structures, typically algebraic or analytic, which allow to perform certain computations or carry out some reasoning. We all know the classical examples: The field of real numbers is made up in such a way that the classical manipulations known for integers or rational numbers, can be performed, with the additional observation that it is also complete with respect to the Euclidean metric. Once we have formulated such axioms we need not (but of course we can) case about the existence of such objects, e.g. by introducing infinite decimal expressions and their computational rules, but this is just one *model*, a completely different one is obtained by Dedekind sections. Of course two such models are isomorphic in a natural sense.

Abstract Submitted II

Another well known example is the theory of *finite dimensional vector spaces*, as is developed in a typical linear algebra course. Aside from additional properties valid only for concrete examples one does not care, whether the objects are column or row vectors, or polynomials or function inside some infinite dimensional vector space. The common structure can be described using finite bases, and matrices characterizing general *linear mappings* using such basis on both the domain and the target space.
In this sense we want to propose a heuristic approach to the world of signals and operators. At the end we will suggest that so-called *mild distributions*, or the elements of the dual of Feichtinger's algebra are a suitable setting for signals.
The basic idea is that signals are objects (not to be specified here!) which can be "tested" by suitable measurements. Although a simplified approach models a signal as a continuous functions,

Abstract Submitted III

which allows point evaluations as simple measurements, more complex situations are more realistic. Think of the temperature at a given location as a function of time, obtained by placing a thermometer for a little while at a concrete spot and wait for the display to stabilize. This is a typical measurement. Another very natural object is a sound signal, which can be recorded with the help of a microphone (using your smart-phone or notebook). The digital data obtained in this way are not the exact value of what (?), the air pressure at the microphone, but still one can insert the resulting signal (ideally stored on a CD at the rate of 44100 Hz) into a mathematical software program and analyze the signal in this way.

The underlying reasoning (explained in fe24-6 in more detail) is the assumptions, that both signals and measurements form linear spaces. In fact, the sum are a scalar multiple of a given signal (by

Abstract Submitted IV

amplification) should be signal as well. But also the average of certain (we assume now *linear*!) measurements or other linear combinations describe linear mappings defined on the vector space of all signals, thus providing a bilinear mapping defined on the product space, generated from signals and (admissible) measurements. In addition, this setting provides a very natural notion of similarity of signals: Given a finite set F of measurements and some $\varepsilon > 0$ we say that two signals are close in the (F, ε)-sense if the difference for each of the measurements of the two signals is at most ε, for all measurements from F.
It is plausible that such bilinear (and continuous) pairings can be realized using spaces of test functions (as measurements) and generalized functions (the dual space) as space of signals. In the concrete scenari we will use the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (the *Feichtinger algebra) as the Fourier invariant Banach algebra of test*

Abstract Submitted V

functions, and the dual space, the space of mild distributions, as the space of signals. $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ *can also be characterized as the subspace of the Schwartz space* $\mathcal{S}'(\mathbb{R}^d)$ *of tempered distributions with a bounded STFT (short-time Fourier transform). In fact, for a general* $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ *and any (real-valued) Schwartz function* $g \in \mathcal{S}(\mathbb{R}^d)$ *(say a Gaussian), we have*

$$V_g(\sigma)(t,s) = \sigma(exp(-2\pi is\cdot) \cdot g(\cdot - t)), \quad (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

It is also natural to view the "most general" linear operators as those mapping $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ *into* $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ *(in a continuous way). For such operators we have a so-called kernel theorem: They can be described as limits of a bounded sequence of integral kernels* $K_n(x,y)$ *in the sense of* $\boldsymbol{S}_0'(\mathbb{R}^{2d})$*, where each* K_n *is represented by a bounded and continuous function. This is a*

Abstract Submitted VI

very useful analogue of the Schwartz kernel theorem characterizing continuous linear mappings from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, by "kernels" $K \in \mathcal{S}'(\mathbb{R}^{2d})$.

[fe19] H. G. Feichtinger. *Classical Fourier Analysis via mild distributions.* MESA, Non-linear Studies, *26(4):783–804, 2019.*

[fe20-1] H. G. Feichtinger. *A sequential approach to mild distributions.* Axioms, *9(1):1–25, 2020.*

[fe24-5] H. G. Feichtinger. *Fourier analysis via mild distributions: group theoretical aspects. Trends in Mathematics by Birkhäuser., pages 1–6, 2024.*

Abstract Submitted VII

[fe24] H. G. Feichtinger. Sampling via the Banach Gelfand Triple. In Stephen D. Casey, Maurice Dodson, et al. editors, Sampling, Approximation, and Signal Analysis Harmonic Analysis in the Spirit of J. Rowland Higgins, *211–242. Cham: Springer, 2024.*

[fe24-6] H. G. Feichtinger. Modelling Signals as Mild Distributions (Abdul Siddiqi Memorial Volume), *pages 1–26. Birkhäuser/Springer, Cham, 2024.*

General Views

To get started a few personal comments:
I consider myself an *application oriented harmonic analyst*. From my education I was starting mathematically with abstract harmonic analysis (AHA) under Hans Reiter, while I was also a teacher student in *physics*.
Over the years I have supervised quite a few PhD students and realized many joint projects with applied people from different areas, especially in the context of engineering applications.
While I consider myself and expert on function spaces the goal of most of my research is to make methods from Fourier Analysis useful for the applied community, to reveal mathematical structures to make dubious objects like the Dirac delta accessible, and so on.
Sometimes it was necessary to develop new methods, sometime it is interesting to explore the usefulness of new tools to classical questions.

The Key Topics relevant for this talk

Short Historical Overview:

1. Fourier Series (1822)
2. Fourier Transforms via Lebesgue (ca. 1922)
3. Fourier Transform for tempered distributions (around 1950)
4. the Fast Fourier Transform (1965)
5. Gabor Analysis (1946, mathematics starting 1980)
6. Coherent states, phase space analysis (Folland 1989)
7. Wiener Amalgam spaces (HGFei, 1980)
8. $\boldsymbol{S}_0(G)$ (Feichtinger algebra), *modulation spaces* (1983)
9. Wavelet Analysis (1986), painless expansions
10. Coorbit Theory (1989) (with K.Gröchenig)

A quick approach to Mild Distributions I

Throughout I will use the normalization of the Fourier Transform

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s t} dt = \int_{\mathbb{R}^d} f(t)\overline{\chi_s(t)} dt.$$

Using the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ of *rapidly decreasing functions* the extended *Fourier transform* for the dual space, the space of *tempered distributions* is defined via $\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.
It is convenient to use the Gauss function $g_0(t) = e^{-\pi|t|^2}$ because it is Fourier invariant: $\mathcal{F}(g_0) = \widehat{g}_0 = g_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.
Then we can define the STFT (Short Time Fourier Transform)

$$V_g\sigma(t,s) = \sigma(M_{-s} T_t g_0),$$

with $T_t f(x) = f(x-t)$, $M_s f(t) = \chi_s(t) f(t)$. It is knwon that $V_g\sigma$ of at most polynomial growth for any $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$.

Setting the stage I

In this situation we can define the modulation space $\left(\boldsymbol{M}^p(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^p}\right)$, for $1 \leq p \leq \infty$, by setting

$$\boldsymbol{M}^p(\mathbb{R}^d) := \{\sigma \in \boldsymbol{S}'(\mathbb{R}^d) | V_g\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d})\}$$

and endow each such space with the norm

$$\|\sigma\|_{\boldsymbol{M}^p(\mathbb{R}^d)} := \|V_g\sigma\|_{\boldsymbol{L}^p(\mathbb{R}^{2d})}$$

It is not difficult to show that each of these spaces is a Banach space, and in fact also Fourier invariant.
Furthermore one has the duality results

$$\left(\boldsymbol{M}^p(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^p}\right)' = \left(\boldsymbol{M}^q(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^q}\right), \text{ for } 1 \leq p < \infty.$$

Feichtinger Algebra and Mild Distributions

We then have the following chain of dense embeddings:

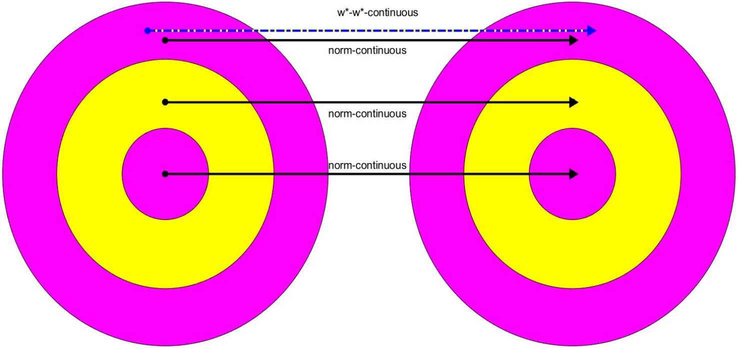
$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^1(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^p(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^\infty(\mathbb{R}^d) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$$

and using my more "classical notation"
$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) = (\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$ (Feichtinger algebra)
and $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}) = (\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$ (mild distributions) we have the following "rigged Hilbert space" situation:

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow \boldsymbol{M}^2(\mathbb{R}^d) = \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}),$$

the last embedding being w^*-dense only.
We thus call $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ THE Banach Gelfand Triple.
Via *Wilson bases* it can be identified with $(\ell^1, \ell^2, \ell^\infty)$
(the *canonical Banach Gelfand Triple*).

Banach-Gelfand-Tripel-Homomorphisms



Wiener's Algebra

In a way $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ takes the role of Wiener's algebra $\left(\boldsymbol{A}(\mathrm{T}), \|\cdot\|_{\boldsymbol{A}}\right)$ of *absolutely convergent Fourier series* for the non-compact and non-discrete case. It has much better properties than the Fourier algebra $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ or the space $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

Recall that the classical Fourier transform $f \mapsto (\widehat{f}(k))_{k\in\mathbb{Z}}$ providing an orthogonal expansion in the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ (Plancherel's Theorem) can be viewed as a unitary BGT isomorphism between $(\boldsymbol{A}^2, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$. Here $\boldsymbol{PM}(\mathbb{T})$ is the space of pseudo-measures over the torus, which can be defined as the dual space of $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$. Mild convergence corresponds in this case to coordinatewise convergence in the Fourier domain.

Official Abstract Provided I

Taking the current viewpoint classical Fourier Analysis started from the study the orthogonal expansion of periodic functions in the Hilbert space $\boldsymbol{L}^2(\mathbb{T})$, using the orthonormal system of pure frequencies (described by complex exponential functions $(\chi_k)_{k\in\mathbb{Z}}$). Moving on to the real line it was natural to view the Fourier transform, given by the usual integral formula, as a linear mapping defined on the Lebesgue space $\boldsymbol{L}^1(\mathbb{R}^d)$, mapping into the space $\boldsymbol{C}_0(\mathbb{R}^d)$ of continuous functions, vanishing at infinity (by the Riemann-Lebesgue Lemma). The convolution theorem then goes on and describes this mapping as an injective Banach algebra homomorphism, converting convolution into pointwise multiplication. This is also the basis for the study of summability methods which are used in order to verify the validity of Fourier inversion formulas.

Official Abstract Provided II

As it turned out, $\boldsymbol{L}^p$-spaces are not so well suited for the study of the Fourier transform (only $\boldsymbol{L}^2(\mathbb{R}^d)$ is invariant) and even less for the modern theory of Time-Frequency Analysis (TFA) or Gabor Analysis (a branch of TFA dealing with discrete representations). Instead, modulation spaces are better suited. The prototypes in this family are the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, the so-called Feichtinger algebra, and its dual, meanwhile known as Banach space $\boldsymbol{S}_0'(\mathbb{R}^d)$ of *mild distributions*. Together with the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ they form **THE BGT: Banach Gelfand Triple** $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which behaves in many ways like the triple $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ of $\mathbb{Q}$ (rational), $\mathbb{R}$ (real) and ($\mathbb{C}$: complex) numbers. Although these spaces can be defined and are useful in the context of general locally compact Abelian (LCA) groups we will discuss a (long) list of features making them extremely useful, for the study of both theoretical and applied (say in signal processing) problems.

Official Abstract Provided III

THE BGT allows for an elegant derivation of the basic principles of Fourier Analysis, but it is also a central tool for TFA and Gabor Analysis. It also provides a solid foundation of a mind-set called Conceptual Harmonic Analysis (CHA). Although it is pursued by the author now for several years, it still lacks popularity, competing with traditional approaches.
Overall CHA suggests to promote the integration of concepts from basic functional and harmonic analysis with numerical computations via efficient algorithms. It also supports the modelling real world problems (such as recovery of signals from sampling), thus avoiding purely heuristic "hand-waving" arguments.

Executive Summary I

We use the term *Mild Distributions* for the elements of the dual space of the Feichtinger algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, also known as *modulation space* $(\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$.
It is well known that many problems in Fourier and Time-Frequency Analysis can be well described using *THE Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. The many good properties of $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ allow to use it, typically as a replacement to the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, especially in the context of LCA groups. It is used in the context of *Gabor Analysis* and appears implicitly in Classical Fourier Analysis, since summability kernels belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.

It is the purpose of this talk to feature the usefulness of mild distributions, e.g., by emphasizing the problems of the traditional approach using the *Lebesgue spaces* $\boldsymbol{L}^p(\mathbb{R}^d)$.

Using the Fourier Transform I

One can look at *Fourier Analysis* or more generally *Time-Frequency Analysis* from many different angles.
The technical/mathematical viewpoint takes the Fourier transform as an integral transform and tries to extend it to the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ (a unitary automorphism, by Plancherel's Theorem), or to (tempered) distributions (via duality).

While engineers typically do similar things in a *discrete context* it is in the world of physics where "*continuous bases*" of Dirac measures $(\delta_x)_{x\in\mathbb{R}^d}$ are used for heuristic arguments.
Abstract Harmonic Analysis (AHA) classifies the different types of Fourier transforms as realizations of a general abstract principle (dual group of "pure frequencies"), with the DFT/FFT corresponding to the use of the cyclic group $\mathbb{Z}_N$ of order N.

Using the Fourier Transform II

Looking around in the real world and without formally defining what a "signal" IS we can say:
Very few signals in the real world are either periodic (in a strict sense), or well defined almost everywhere and square integrable (meaning belonging to $\boldsymbol{L}^2(\mathbb{R}^d)$). Nevertheless we have nowadays very good signal processing algorithms which help us to record, transmit, modify signals in many ways, even in real time. But is the established mathematical theory helping us (except for heuristic considerations)?

Recall, that TF-shifts are defined for $\lambda = (t, s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$:

$$\pi(\lambda)g(x) = [M_s T_t]g(x) = e^{2\pi isx} g(x-t).$$

Basic Considerations concerning Mild Distributions I

Just recall the definition of $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$. First we need the definition of the STFT of a tempered distribution with respect to the Gaussian window $g_0(t) = \exp(-\pi|t|^2)$: For $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ set:

$$V_{g_o}\sigma(t,s) = \sigma(M_{-s}T_t g_0), \;\; \text{with } \lambda = (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{1}$$

We can define $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^\infty(\mathbb{R}^d)$ by setting

$$\boldsymbol{S}_0'(\mathbb{R}^d) := \{\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \,|\, V_{g_o}\sigma \in \boldsymbol{C}_b(\mathbb{R}^{2d})\}, \;\; \|\sigma\|_{\boldsymbol{S}_0'} := \|V_{g_o}\sigma\|_\infty < \infty. \tag{2}$$

We have many good properties, such as invariance under the Fourier transform (defined in many equivalent ways), e.g. via

$$\mathcal{F}(\sigma)(f) = \widehat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

Basic Operations on Mild Distributions

1. First: $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is a Banach space with the dual norm;
2. TF-shifts $\pi(\lambda) = M_s T_t$ act isometrically, automorphisms (e.g. dilations) act boundedly on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$;
3. Every $\sigma \in \boldsymbol{S}_0'$ has a Fourier transform; in fact, $\mathcal{F}$ is an isometric automorphism of $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, preserving mild convergence, characterized by $\mathcal{F}(\chi_s) = \delta_s$;
4. Every $\sigma \in \boldsymbol{S}_0'$ is the mild limit of test functions, e.g. from $\boldsymbol{S}_0(\mathbb{R}^d)$, which may be obtained by *regularization*, or of (finite discrete) measures, hence the unique extension of the usual integral version of the FT;
5. Fractional FTs, even *metaplectic operators* [or LCTs] act as *unitary BGT automorphisms* on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

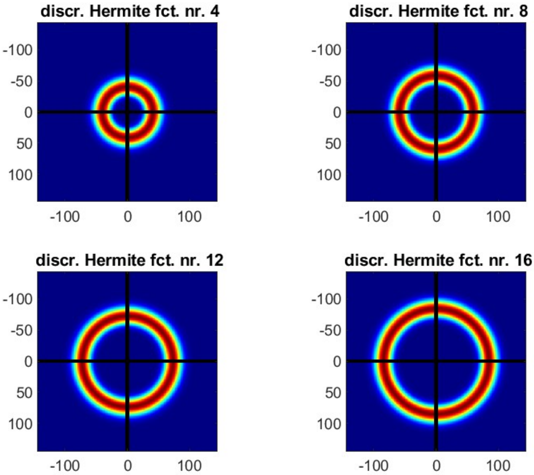


Figure: herm1040A.jpg: Zoom into TF-plane

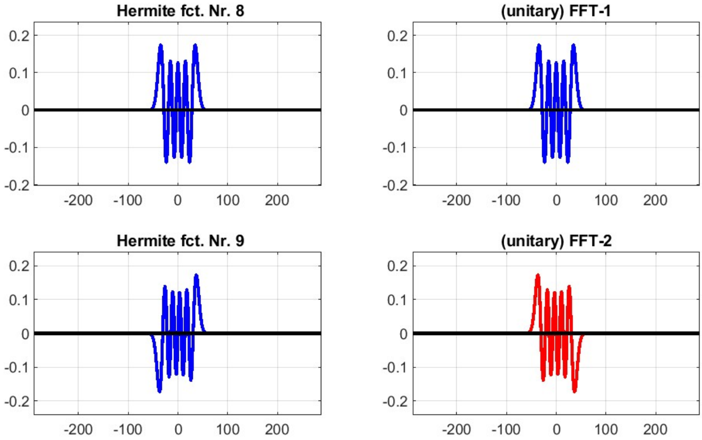


Figure: hermiteN89.jpg

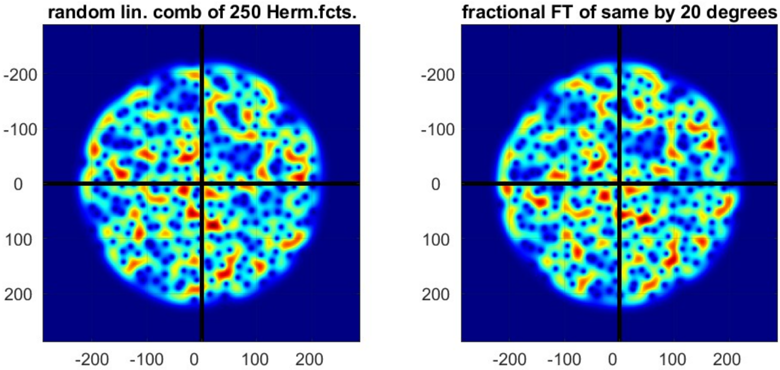


Figure: hermrot250A.jpg: Fractional Fourier transform by 20 degrees,

Regularizations I

There are many ways to regularize a given mild distributions, such as a *Dirac comb* $⊔⊔ = \sum_{k\in\mathbb{Z}^d} \delta_k$, which shows neither decay nor smoothness (decay in the frequency direction).
The typical methods to approximate (in the "mild sense") a given $\sigma \in \boldsymbol{S}_0'$ is to convolve it with a Dirac sequence, normalized in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ but sitting inside of $\boldsymbol{S}_0(\mathbb{R}^d)$, hence its FT (a summability kernel) is a bounded approximate unit in the pointwise sense. Thus it is good to know that $\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) \subset \boldsymbol{C}_b(\mathbb{R}^d)$ with corresponding norm estimates, but more importantly

$$(\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$$

$$(\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d).$$

Regularizations II

There is also a discrete alternative, which makes use of *Gabor frames* with windows $g \in \mathcal{S}(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$. Think, for example of $g = g_0$, the standard Gaussian, with lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, for $ab < 1$.

Then the dual window $\widetilde{g}$ also belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ (by a result of Gröchenig and Leinert) and consequently on has:

$$\sigma \mapsto V_g\sigma|_\Lambda \in \ell^\infty(\Lambda)$$

and

$$\sigma = \sum_{\lambda\in\Lambda} V_g\sigma(\lambda)\pi(\lambda)\widetilde{g},$$

is unconditionally mildly convergent.

Regularizations III

Obviously the finite partial sums belong to $\boldsymbol{S}_0(\mathbb{R}^d)$ (or even to $\mathcal{S}(\mathbb{R}^d)$) and so it is not surprising that the family of partial sums

$$\sigma_F = \sigma_{f,g,\Lambda,F} := \sum_{\lambda \in F} V_g\sigma(\lambda)\pi(\lambda)\widetilde{g}$$

are mildly convergent to σ. In other words:
Given a compact domain $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and $\varepsilon > 0$ there exists some finite set F_0 (typically $\Lambda \cap (B_R(0) + Q)$, where R depends only on the concentration properties of g and $\widetilde{g}$): such that one has for any finite set $F \supseteq F_0$:

$$|V_g\sigma(q) - V_g\sigma_F(q)| \leq \varepsilon, \quad \forall q \in Q.$$

Context of Gabor Analysis I

It is pretty well established these days that there are many good reasons for assuming the window used for the STFT to belong to $\mathbf{S}_0(\mathbb{R}^d)$, even if the first, natural appearing definition is given in the setting of the Hilbert space , by setting

$$V_g f(\lambda) = \langle f, \pi(\lambda) g \rangle_{\mathbf{L}^2}, \quad \lambda = (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

For $f, g \in \mathbf{L}^2(\mathbb{R}^d)$ we know that $V_g f \in \mathbf{L}^2 \cap \mathbf{C}_0(\mathbb{R}^{2d})$, with

$$\|V_g f\|_{\mathbf{L}^2(\mathbb{R}^{2d})} = \|f\|_{\mathbf{L}^2(\mathbb{R}^d)} \cdot \|g\|_{\mathbf{L}^2(\mathbb{R}^d)}, \quad \text{Moyal's Identity.}$$

This implies that $f \mapsto V_g f$ is an isometry for $\|g\|_2 = 1$, and the *spectrogram* $|V_g f|^2$ is a probability distribution if also f is normalized with $\|f\|_2 = 1$.
However, once we start to sample $V_g f$ the map $f \mapsto V_g f|_\Lambda$ may not be unbounded from $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ into $\ell^2(\Lambda)$.

Context of Gabor Analysis II

In contrast, one can guarantee that one has uniform boundedness of this mapping for all lattices of the form $= a\mathbb{Z}^d \times b\mathbb{Z}^d$, if normalized properly (in the spirit of Riemann sums), whenever $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. It is also one of the cornerstones of standard Gabor Analysis that the following is true:

Theorem

Assume that (g, Λ) generates a Gabor frame for $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, then also the dual Gabor atom $\widetilde{g} = S_{g,\Lambda}^{-1}(g)$ and the tight Gabor window $S_{g,\Lambda}^{-1}(g)$ also belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.
In particular, the mapping $f \mapsto V_g f|_\Lambda$ defines a Banach frame for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ with respect to $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Context of Gabor Analysis III

[fezi98] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, 1998.

[feko98] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

Riemann sum representation I

The classical Fourier inversion theoremis often described as

$$f = \int_{\widehat{\mathbb{R}^d}} \langle f, \chi_s \rangle \chi_s ds, \quad f \in \boldsymbol{L}^1(\mathbb{R}^d).$$

Here the scalar product is understood as the extended pairing of $\boldsymbol{L}^1(\mathbb{R}^d) \times \boldsymbol{L}^\infty(\mathbb{R}^d)$, but *very unfortunately* the finite Riemannien sums of the form

$$\sum_{s \in F} \widehat{f}(s) \chi_s$$

are trigonometric polynomials which do not belong to either $\boldsymbol{L}^1(\mathbb{R}^d)$ or $\boldsymbol{L}^2(\mathbb{R}^d)$, so the meaning of the integral, or more or less equivalently, the *resolution of identity* property of characters (pure frequencies) appears as a kind of mystification:

$$\mathrm{Id} = \int_{\widehat{\mathbb{R}^d}} |\chi_s\rangle \langle \chi_s ds.$$

Riemann sum representation II

In contrast, the isometric property of $f \mapsto V_g f$ (for g normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$) implies that the adjoint mapping $F \mapsto V_g^*(F)$ defines the inverse of V_g on the range, hence (in the weak sense):

$$f = V_g^*(V_g f) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda) \widetilde{g}_\lambda d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

However for windows $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one can turn this weak representation of $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ into a strong one, using finite Riemannian sums: $f = \lim_{n\to\infty} \sum_{F_n} V_g f(\lambda) \widetilde{g}_\lambda$.
where F_n a sufficiently large finite' set (defined geometrically, depending on f), inside some sufficiently fine lattice Λ_n.

[fewe06-3] H. G. Feichtinger and F. Weisz. Inversion formulas for the short-time Fourier transform.
J. Geom. Anal., 16(3):507–521, 2006.

Representation of Linear Systems I

The motivation to study the Fourier transform and convolution in the context of *linear, time-invariant systems* stems from the fact that such operators can be characterized as convolution operators (using the so-called impulse response) or by the transfer function. For such operators T on $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ the *impulse response* description is more suitable. They can be described as convolution operators with bounded measure μ, obtained as a w^*-limit of $T(g_\alpha)$, for a Dirac sequence (g_α).

In contrast, we have an easy description via *transfer functions* in $\boldsymbol{L}^\infty(\mathbb{R}^d)$ on $\boldsymbol{L}^2(\mathbb{R}^d)$. The impulse response is then called a *pseudo-measure*, which is simply $\mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$.

For general linear operators from $\boldsymbol{L}^p(\mathbb{R}^d)$ to $\boldsymbol{L}^q(\mathbb{R}^d)$ (with $q \geq p$) the theory of *quasi-measures* has been introduced in the 1960s, by G. Gaudry, which is more complicated than the theory of mild distributions which can be described as follows:

Representation of Linear Systems II

Theorem

Every operator $T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ which commutes with translation (or all convolutions from a sufficiently comprehensive system of functions) is a convolution operator, i.e.:

$$Tf = \sigma * f, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

with impulse response σ, realized via

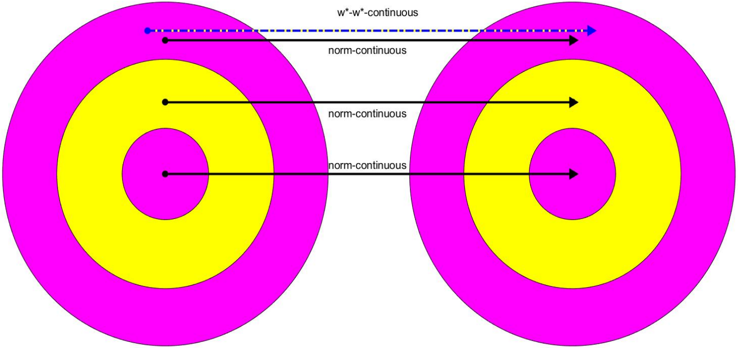
$$Tf(x) = \sigma(T_x f^{\checkmark}), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

using $f^{\checkmark}(x) = f(-x)$, or (on the Fourier transform side)

$$Tf = \mathcal{F}^{-1}(\widehat{\sigma} \cdot \widehat{f}).$$

for some unique $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ given by $\sigma(f) = T(f^{\checkmark})(0)$.

Banach-Gelfand-Tripel-Homomorphisms



DIAGRAMS I

Think of $\boldsymbol{X}$ as $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, and $\boldsymbol{Y} = (\ell^1, \ell^2, \ell^\infty)$:

BANACH (Gelfand Triple) FRAME case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range Y_0 of $\mathcal{C}$ in $\boldsymbol{Y}$:

$$
\begin{tikzcd}
 & \boldsymbol{Y} \arrow[dl, "\mathcal{R}"'] \arrow[d, "P"] \\
\boldsymbol{X} \arrow[r, shift right, "\mathcal{C}"'] & Y_0 \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Modulation Spaces and Gabor Expansions I

The problem of Gabor expansions of "signals" is *equivalent* to the question of reconstructing a STFT $V_g\sigma$ from samples.
If the samples are taken from a lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ in phase space we call this a *regular Gabor expansions*. Often one takes $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$. Otherwise we talk of *irregular Gabor families*.
Anyway, the usual consideration is that of a problem of frame in Hilbert space context, where the frame condition can be expressed by a simple pair of inequalities. In words:
`The sampling energy is supposed to be equivalent to to energy (`$\boldsymbol{L}^2$`-norm) of the signal`.
But it turns out that at least for regular Gabor families generated by some $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ that then one has in fact such a retract situation according to the above diagram.

Gabor Expansions of Mild signals I

Being defined as an adjoint mapping, which in fact extends the classical integral transform given by

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s t} dt,$$

which leaves $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ isometrically invariant, we know that the (extended) FT is w^*-w^*--continuous, i.e. it preserves *mild convergence*.
But there is a more natural equivalent expression for this: A sequence (or bounded net) σ_α is mildly convergent to σ_0 if the corresponding STFTs are *uniformly convergent over compact subsets of* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (phase space).

Gabor Expansions of Mild signals II

In the context of Gabor Analysis, let us choose on of the most simple cases, namely a Gaussian family of redundancy 2, with $a = 1/\sqrt{2} = b$. Then the elements of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R})$ can be characterized by the possibility of representing them as double series over the lattice , with coefficients in $(\ell^1, \ell^2, \ell^\infty)$.
Now there is a 'tight Gabor family' $(g_{n,k}) = (M_{bn} T_{ak} g_t)_{(n,k)\in\mathbb{Z}^2}$ (red shape below) which allows to write any $f \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ as

$$f = \sum_{(k,n)\in\mathbb{Z}^2} \langle f, g_{n,k}\rangle g_{n,k}. \tag{3}$$

The Fourier invariance of g_0 and the symmetry $a = b$ implies that also the tight version, i.e. this g is Fourier invariant, and thus

$$\mathcal{F}(g_{n,k}) = \mathcal{F}(M_{nb} T_{ka} g) = T_{nb} M_{-ka} \mathcal{F}(g) = e^{2\pi i knab} g_{-k,n}.$$

Gabor Expansions of Mild signals III

This implies that the Fourier transform can be simply applied (up to a well explained phase factor) as a simple permutation of Gabor atoms.

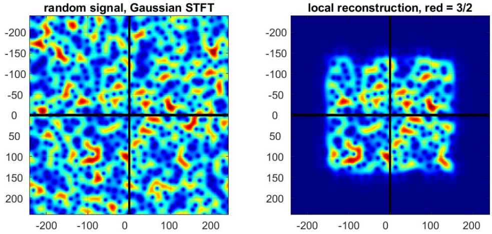


Figure: GabLocRec01.jpg

What are the USPs for the space of mild distributions?

A couple of bold claims!

1. It is a perfect *model for general SIGNALS*
2. It is large enough to contain (almost) “everything”
3. It covers discrete and continuous, periodic and non-periodic
4. It supports the engineering or physicists viewpoint
5. It is small and simple enough (a Banach space)
6. It allows to describe *general OPERATORS*
7. It suggests *structure preserving approximation schemes*
8. It can be defined on any LCA group, from scratch (ETH20)
9. It is easy/intuitively to use (no Lebesgue, little topology)

P1: Simplicity I

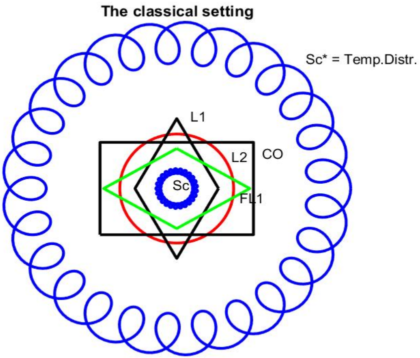


Figure: Classical spaces and tempered distributions

Classical spaces and Fourier

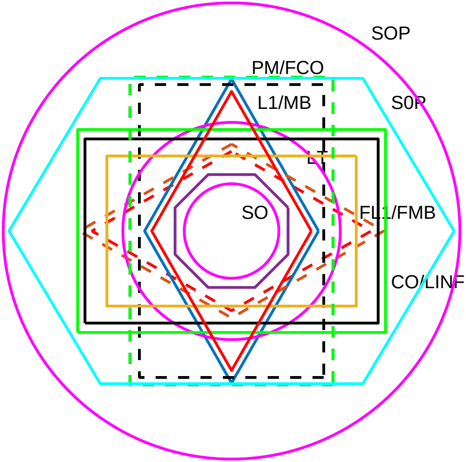


Figure: Classical spaces and Fourier

TOPIC P1: SIMPLICITY

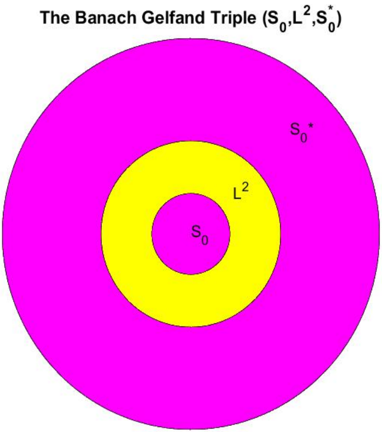


Figure: THE (simple) Banach Gelfand Triple

P2: Historical Remarks, Schwartz-Bruhat I

The space $\left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right)$ of *translation bounded quasi-measures* has been introduced shortly after the paper on the "New Segal Algebra" $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (which goes now by the name of Feichtinger's algebra) in [4]. It appears in the Compt. Rend. Note [3]. This paper from 1980 has as many/few citations as the recent survey article [9] by Mads Jakobsen, which is a sign that recognition came rather late.

In many earlier talks I have tried to emphasize the possible role of the so-called Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, especially for the Euclidean case, as a universal tool for many questions arising naturally in Fourier Analysis and Time-Frequency Analysis, in particular in the context of Gabor Analysis.

This construction (also called a "rigged Hilbert space" setting) is an important corner stone for the development of the idea of "Conceptual Harmonic Analysis", because very often

P2: Historical Remarks, Schwartz-Bruhat II

Banach Gelfand Triple Morphisms (BGT-morphisms) allow to widen the view on *unitary mappings*. The prototypical example is Plancherel's Theorem. It can be obtained by observing that $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a space of continuous, Riemann integrable functions which is invariant under the Fourier transform. Thus the Fourier Inversion Theorem applies pointwise (even with Riemann integrals), and by observing the $\boldsymbol{L}^2$-isometric property one naturally extends the integral transform to a unitary automorphism of $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$. But finally the (unique, w^*-w^*-continuous) extension to all of $\boldsymbol{S}_0'(\mathbb{R}^d)$ allows to observe that it just maps pure frequencies into Dirac Deltas, similar to the fact that the DFT maps pure frequencies on $\mathbb{Z}_N$ into corresponding unit vectors in the frequency domain.

P2: Historical Remarks, Schwartz-Bruhat III

So this space is around for 45 years, and I have been using it a lot for various applications. But still, its usefulness outside of time-frequency analysis (where it is usually described as the modulation space) is not so clear within a wider community. Therefore it is the purpose of this talk to provide a list of facts and properties of $\boldsymbol{S}_0'(G)$ which make it so useful. We will also explain the effect of fundamental properties of $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ for the various descriptions of $\boldsymbol{S}_0'(G)$, including the characterization of w^*-convergent sequences (or better bounded nets).
As a final introductory remark let me note that (for a similar reason) I decided to call the elements of $\boldsymbol{S}_0'(\mathbb{R}^d)$ "mild distributions" (because they behave better than general tempered distributions), see [6] or [7].

P2: Mild versus Tempered Distributions I

Although it is clear that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and its (topological) dual, the space $\mathcal{S}'(\mathbb{R}^d)$ of *tempered distributions* not only Fourier invariant, but also closed under differentiation, and thus are the ideal tool for finding solutions of PDEs, it is quite cumbersome to work with countable families of (semi-)norms instead of just one simple norm as in the case of $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or $\left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right)$. However, for general LCA groups the so-called *Schwartz-Bruhat space* is really cumbersome and may not have much to do with differentiation (see [12]), which has been used in the famous Acta paper by Andre Weil ([16]) about the metaplectic group (including Fractional Fourier transforms as a compact subgroup of unitary operators).
It was then H. Reiter who picked up this property in his revision of Weil's work in the Lecture notes [15].

P3: Extending the FT to all $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ I

We have the following sandwich situation for $1 \leq p \leq \infty$:

$$(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^p(G), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}), \quad ,$$

and thus it is no problem to define the FT for any such function. This is in contrast to the limitations arising with the classical $\boldsymbol{L}^p$-spaces, where only the *Hausdorff-Young* estimate is valid (with the usual convention $1/q + 1/p = 1$):

$$\mathcal{F}(\boldsymbol{L}^p(G)) \hookrightarrow \boldsymbol{L}^q(\widehat{G}), \quad 1 \leq p \leq 2.$$

But, for example, with the observation that $\mathcal{F}(\boldsymbol{L}^p(\mathbb{R}^d)) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$ we know much more than just the membership of $\widehat{f} \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$. Of course, for $p > 2$ $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$ will contain true distributions which are not represented by locally integrable functions!

P3: Extending the FT to all $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ II

Among the spaces discussed above we have $p = \infty$ as an important special case. Obviously we have a continuous and dense embedding of $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ (it is a true Segal algebra in Reiter's sense), and thus by duality we have

$$\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right) \hookrightarrow \left(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}\right).$$

Hence any $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$ has a Fourier transform in $\boldsymbol{S}_0(\mathbb{R}^d)$, and any such mild distribution has a support, defined in the usual way, as one is used from the classical theory of distributions.
The idea to call $\mathrm{supp}(\widehat{h})$ the "spectrum of" h (we write $\mathrm{spec}(h)$) can be justified by demonstrating that this natural viewpoint is compatible with Reiter's definition of the spectrum, using closed ideals of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (cf. Reiter's Ideal Theorem for Segal algebras, see [13] and [14]).

P4: Translation-bounded measures

Of course there as mild distributions which are not regular, i.e. not represented by (locally) integrable functions, but true, potentially unbounded measures.
Obvious examples are Dirac measures $\delta_x : f \mapsto f(x)$, but also Dirac combs (more or less Haar measures on lattices $\Lambda \lhd \mathbb{R}^d$): $⧢_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$.
Poisson's formula (valid for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$) implies

$$\mathcal{F}(⧢) = ⧢, \quad \text{and} \quad \mathcal{F}(⧢_\Lambda) = C_\Lambda ⧢_{\Lambda^\perp},$$

where $\Lambda^\perp$ is the *orthogonal group* to Λ in $\widehat{\mathbb{R}^d} = \mathbb{R}^d$.
I am currently engaged in a project with colleagues active in *quasi-crystals* who find this form of FT quite convenient.

P5: Sampling Theory, Shannon's Theorem I

One of the basic principles of digital signal processing is the fact that a band-limited signal can be recovered well from discrete (regular) samples if only the *Nyquist condition* is satisfied. At the basis (also for the finite discrete setting) is Poisson's formula, giving us

$$\mathcal{F}(⧢) = ⧢.$$

By applying a dilation on obtains a similar statements for general lattices of the form $⧢_a := ⧢_{a\mathbb{Z}^d}$, namely (up to normalization)

$$\mathcal{F}⧢_a = C_a ⧢_b, \quad b = 1/a.$$

and obviously one has (by the extended convolution theorem):

$$\mathcal{F}(⧢_a \cdot f) = C_a(\widehat{f} * ⧢_b), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

P5: Sampling Theory, Shannon's Theorem II

Since a bandlimited function $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ we can state that the information in the sampled signal $\sum_{k\in\mathbb{Z}^d} f(ak)\delta_{ak}$ is the same as in the b-periodic version of $\widehat{f}$. So if the periodized spectrum is well separated it allows to recover $\widehat{f}$ by a simple convolution on the time side (resulting in Shannon's reconstruction formula).

Next we formulate the most simple version. It does not make use of the classical SINC-function, but requires instead a little bit of oversampling, which in turn guarantees much better locality of the reconstruction (better control on the truncation errors).

Shannon's Theorem for band-limited signals

Theorem

Given a compact set $\Omega \subset \widehat{\mathbb{R}^d}$ *and* b *large enough such that* $bk + \Omega \cap \Omega = \emptyset$ *for* $k \neq 0$, *one can recover any band-limited signal* $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$ *with* $\mathrm{spec}(f) = \mathrm{supp}(\widehat{f}) \subseteq \Omega$ *from the samples taken over any of the lattices* $a\mathbb{Z}^d$ *with* $a = 1/b$
This can be achieved using of any band-limited function $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ *with* $\widehat{g}(s) \equiv 1$ *on* Ω *and* $\widehat{g}(u) \equiv 0$ *for* $u \in bk + \Omega, k \neq 0$ *by forming the series*

$$f(t) = C\,(\sqcup\!\sqcup_a \cdot f) * g = C \cdot \sum_{k \in \mathbb{Z}^d} f(ak) T_{ak} g(t).$$

For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the sum is absolutely convergent in* $\boldsymbol{S}_0(\mathbb{R}^d)$.

P6: Translation invariant Systems and Convolution

In the work on multipliers between $\boldsymbol{L}^p$-spaces Gaudry has introduced the space of *quasi-measures*, which (according to Cowling, see [1]) can be identified with $\mathcal{F}\boldsymbol{L}^\infty_{loc}$).
As one can find in the book of Larsen about Multipliers (see [11]) any operator between (reflexive) $\boldsymbol{L}^p$-spaces can be represented as a convolution operator with a (uniquely determined) quasi-measure, but since there are no global restrictions in this space one cannot extend the Fourier transform to all the quasi-measures. Thus it is not clear, whether such a "multiplier", understood as Fourier multiplier, can be described by pointwise multiplication with some "transfer function" on the Fourier side. WE HAVE:
Multipliers T from $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ to $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$ are exactly convolution operators by unique elements $\sigma \in \boldsymbol{S}_0'(G)$, (*impulse response*) or as a pointwise multiplier on the Fourier side by $\widehat{\sigma} \in \boldsymbol{S}_0'(\mathbb{R}^d)$ (which takes the role of a *transfer function*).

P7: All the classical cases are just Special Cases I

It is clear that periodization provides a *periodic* function, and pointwise multiplication of $h \in \mathbf{C}_b(\mathbb{R}^d)$ with a Dirac comb $⊔⊔_\Lambda$ defines a weighted Dirac comb (with bounded coefficients), which belong to $\mathbf{S}_0'(\mathbb{R}^d)$ and satisfy $\operatorname{supp}(h \cdot ⊔⊔_\Lambda) \subseteq \Lambda$.
For the case of tempered distributions this does not mean that we have a distribution which is already defined on the subgroup!
Similar for continuous subgroups like the x-axis in $\mathbb{R}^2$. But - this is meant by claiming that $\mathbf{S}_0'(\mathbb{R}^d)$ is not too big - we have

- Any $\sigma \in \mathbf{S}_0'$ with $\operatorname{supp}(\sigma) \subseteq H \lhd \mathbb{R}^d$ arises from a unique $\tau \in \mathbf{S}_0'(H)$ via $\sigma(f) = \tau(f|_H), f \in \mathbf{S}_0(\mathbb{R}^d)$.
- The above situation is equivalent to the claim that $\widehat{\sigma}$ is a $H^\perp$ periodic distribution in $\mathbf{S}_0'(\mathbb{R}^d)$.

These facts ARE IN CONTRAST with tempered distributions!

There is just one Fourier Transform (for each dimension) I

The reservoir $\boldsymbol{S}_0'(\mathbb{R}^d)$ contains all the typical cases, where usually for each type a specialized Fourier transform seems to be in place:

1. First the "continuous non-periodic" case: $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ for $1 \leq p \leq 2$, hence the extended Fourier transform contains the Plancherel transform and the Hausdorff-Young setting.
2. Periodic functions (with whatever period) in $\boldsymbol{L}^p$ (locally) belong to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ and the corresponding FT is concentrated on the dual lattice (classical Fourier Series).
3. Periodic and discrete signals belong to $\boldsymbol{S}_0'(\mathbb{R}^d)$ and their FT can be obtained by the usual DFT/FFT algorithm.
4. Similar claims apply for *discrete signals* (DTFT) or *almost periodic functions*.

There is just one Fourier Transform (for each dimension) II

More importantly, we can (easily) show that the heuristic transitions performed usually correspond to "mild convergence" (i.e. w^*-convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$), such as:

1. lattice constants which are convergent, including to infinity, i.e. to the non-periodic case (FT for $\boldsymbol{L}^1(\mathbb{R}^d)$);
2. approximation of discrete FTs (via Dirac sequences) from the continuous case;
3. approximating the discrete and periodic case from non-periodic discrete version by periodization
4. approximation of continuous densities of bounded measures by discrete measures (using a simple discretization procedure), whose inverse is smoothing using BUPUs (such as piecewise linear interpolation), and so on.

Q1: The Kernel Theorem I

Clearly a linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^m$ have a matrix representation: $T(\mathbf{x}) = \boldsymbol{A} * \mathbf{x}$, where the entries are of the form

$$a_{j,k} = \langle T(\mathbf{e}_k), \mathbf{e}_j \rangle, 1 \leq k \leq m, 1 \leq j \leq n.$$

Hence one can expect that the continuous version allows to write at least (certain integral) operators as

$$T(f) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

It turns out, that for $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ these operators map $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$ in a w^*-to-norm continuous fashion and *vice versa*. Moreover in analogy to the discrete case one has

$$K(x,y) = \delta_x(T(\delta_y)), \quad x, y \in \mathbb{R}^d.$$

Q1: The Kernel Theorem II

Extending to the Hilbert space setting one finds that kernels in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ give rise to the well-known Hilbert Schmidt operators. In fact this is a unitary mapping, using the fact

$$\|K\|_{\boldsymbol{L}^2} = \|T\|_{\mathcal{HS}} := \mathrm{trace}(T \circ T*).$$

The outer layer describes the most general operator. The correspondence identifies $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ with the space of all bounded linear operators from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$. In this setting one can even describe multiplication or convolution operators, in particular the identity operator, which corresponds to the distribution $F \mapsto \int_{\mathbb{R}^d} F(x,x)dx, \quad F \in \boldsymbol{S}_0(\mathbb{R}^{2d})$, which is well defined since the restriction of $F \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ to the diagonal is in $\boldsymbol{S}_0(\mathbb{R}^d)$ and hence integrable.

Q1: The Kernel Theorem III

If one tries to rewrite the functional (representing the identity operator) in the usual way (or observing that of course the identity operator is an operator which commutes with translation, and thus has to be a convolution operator, with the usual Dirac measure $\delta_0 : f \mapsto f(0)$) we have

$$f(t) = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

which is only possible if one has in each row

$$K(x,.) = \delta_x, \quad x \in \mathbb{R}^d.$$

(this is more or less the transition from the *Kronecker delta* describing the unit-matrix to the Dirac delta, and is another way of expressing the "sifting property" of δ_0.)

Q1: The Kernel Theorem IV

The composition law for matrices is the unique way of combining information about two linear mappings which can be composed (Domino rule) into a new matrix scheme, via standard matrix multiplication rules: $\boldsymbol{C} = \boldsymbol{A} * \boldsymbol{B}$. Thus one expects for the composition of operators a similar composition law for their kernels, something like

$$K(x,y) = \int_{\mathbb{R}^d} K_1(x,z)K_2(z,y)dz, \quad x,y \in \mathbb{R}^d.$$

If one make use of the kernel for the Fourier transform, i.e. $K_2(z,y) = exp(-2\pi i\langle y,z\rangle)$ and $K_1(x,z) = exp(2\pi i\langle x,z\rangle)$, then, even if the integrals do not make sense anymore in the Lebesgue sense, it still suggest to claim that the resulting product operator is the identity operator, which gives a meaning to formulas appearing in engineering books on the Fourier transform.

Smaller can be better I

While sometimes (e.g. for PDE) it is of interest to know that the FT can be extended to e.g. partial derivatives of δ (turning them into polynomials) there are other cases where it has advantages to have the kernel theorem for mild distributions at hand, because membership of a kernel (or some other representation of an operator) it tells us more about the operator than just the knowledge that it belongs to .

Example 1: Multiplication operators by bounded elements $\tau \in \boldsymbol{S}_0'(\mathbb{R}^d)$ obviously correspond to kernels which are concentrated on the diagonal $\Delta \lhd \mathbb{R}^d \times \mathbb{R}^d$. For the case of mild distributions the converse is true as well. Operators satisfying $\mathrm{supp}(K) \subseteq \Delta$ are just multiplication operators.

Smaller can be better II

Example 2: Traditionally the discussion for Gabor frames starts with the treatment of the Gabor frame operator $S_{g,\Lambda}(f) = \sum_{\lambda\in\Lambda} V_g f(\lambda) g_\lambda$. The usual procedure is to assume that the Gabor family $(g,\Lambda) = (g_\lambda)_{\lambda\in\Lambda}$ is a *Bessel family*, i.e. that the coefficient mapping $f \mapsto (V_g f(\lambda))_{\lambda\in\Lambda}$ as bounded from $\boldsymbol{L}^2(\mathbb{R}^d)$ to $\ell^2(\Lambda)$. Then one goes on to show that it commutes with the TF-shifts from $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, which implies that the support of its *spreading symbol* $\eta(S_{g,\Lambda})$ is contained in $\Lambda^\perp$.

Even for general $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ one can show the boundedness of in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ and derive (without the Bessel condition) that

$$\eta(S_{g,\Lambda}) = \sum_{\lambda^\circ\in\Lambda^\circ} \lambda^\circ \delta_{\lambda^\circ}$$

which eventually leads to the Janssen representation and the Wexler-Raz biorthogonality relationship for Gabor frames.

Some relevant references

Example 2 above was motivating the use of the BGT in [feko98] (see above). The Fundamental Relationship of Gabor Analysis is

discussed in detail in [felu06] H. G. Feichtinger and F. Luef. Wiener amalgam spaces for the Fundamental Identity of Gabor Analysis. *Collect. Math.*, 57(Extra Volume (2006)):233–253, 2006.

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Piecewise linear interpolation and approximate recovery in the norm of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is discussed in [feka07] H. G. Feichtinger

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Q2: Spreading, Weyl, KNS-Calculus I

Within the three layer model the inner layer offers itself for approximation from finite sampling sequences (e.g. by piecewise linear interpolation).
It is easy to explain the concept of the *spreading function* corresponding to a given $N \times N$ matrix $\boldsymbol{A}$, by decomposing it into (N cyclic) side-diagonals and then applying a Fourier transform on them. The KNS symbol $\kappa(\boldsymbol{A})$ (Kohn-Nirenberg) then arises as the (!*symplectic*) FT of the spreading function. It satisfies the following important *covariance property*:

$$\kappa(\pi(\lambda) \circ \boldsymbol{A} \circ \pi(\lambda)^*) = T_\lambda \kappa(\boldsymbol{A}).$$

This makes these descriptions suitable tools for the description of e.g. *Anti-Wick operators* (STFT-multipliers).

Q3: Mild distributions as upper symbols I

There are many good reasons to restrict the windows for the STFT to the space $\boldsymbol{S}_0(\mathbb{R}^d)$. Then one can assure that any $\sigma \in \boldsymbol{S}_0'$ has a bounded and continuous STFT, denoted by $V_g\sigma$.

Anti-Wick operators are based on the consideration that (for a normalized window $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\|g\|_2 = 1$) the inverse for the *isometric* embedding $f \mapsto V_g f$ (from $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$) is just the adjoint mapping V_g^*, of the form

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda,$$

for $F \in \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Thus we have the reconstruction formula

$$f = V_g^*(V_g f) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda)g \, d\lambda.$$

Q3: Mild distributions as upper symbols II

For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ this integral can even be approximated by corresponding Riemannian sums (according to F. Weisz). Thus it is natural to define for bounded functions $m \in \boldsymbol{C}_b(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ an (Anti-Wick) operator of the form

$$\boldsymbol{A}_m(f) = V_g^*(V_g f) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} m(\lambda) V_g f(\lambda) \pi(\lambda) g \, d\lambda.$$

This obviously defines a bounded linear operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, with

$$|||\boldsymbol{A}_m|||_{\boldsymbol{L}^2(\mathbb{R}^d)} \leq C\|m\|_\infty,$$

but it defines in fact a BGT-homomorphism, thus acting boundedly also on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or $\left(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}\right)$.

Q3: The "natural choice of upper symbols I

As it turns out, the correlation between neighboring elements from the family $\pi(\lambda)g, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ introduces some smearing of the (upper) symbol. One can show that $\boldsymbol{L}^2$-symbols give Hilbert Schmidt operators, but the converse is not true, it is enough that a smeared version of m belongs to $\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

With the idea that a "smeared version" of the upper symbol is supposed to be in $\boldsymbol{C}_b(\mathbb{R}^{2d})$ we may (correctly) expect that ANY SIGNAL, i.e. any $m \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ defines a BGTr homorphism, with control of the norms through $\|m\|_{\boldsymbol{S}_0'}$ (at all three levels!).

The key for a verification is the following observation:

The restriction of : $f \mapsto V_g f$, which acts isometrically on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, is also bounded from $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, and not just into $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$. In fact, one has $V_g f \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if f and g belong to $\boldsymbol{S}_0(\mathbb{R}^d)$.

Q3: The "natural choice of upper symbols II

While it is convenient to assume that an Anti-Wick operator (we call it sometimes an STFT-multiplier) arises from a bounded and continuous function, thus giving a bounded operator on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, it is only the global behaviour of the upper symbol which counts.
In fact, Gabor multipliers can be viewed as Anti-Wick operators whose upper symbol is a weighted Dirac comb, say of the form

$$\sum_{(n,k)\in\mathbb{Z}^d\times\mathbb{Z}^d} m_{n,k}\delta_{ak,bn}$$

The action of such an operator can be described as

$$A_m(f) = \sum_{(n,k)\in\mathbb{Z}^d\times\mathbb{Z}^d} m_{n,k}\langle f, g_{n,k}\rangle g_{n,k}.$$

Q3: The “natural choice of upper symbols III

But much more is true: Given any $\mu \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ the Anti-Wick operator

$$A_\mu(f) = V_g^*(\mu \cdot V_g f)$$

defines a bounded BGT-homomorphism on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. Moreover, w^*-convergence of symbols μ_α implies strong convergence of the corresponding operators.
For $\mu \in \boldsymbol{L}^2(\mathbb{R}^{2d})$ one obtains Hilbert-Schmidt operators (and corresponding convergence), and for $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ one obtains trace class operators.

Operators which are not well defined on $L^2(\mathbb{R}^d)$

The "mild kernel theorem" obviously allows to deal with operators which are not well-defined or unbounded on $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Let us discuss it for the case of the real line. Given $N \in \mathbb{N}$ the natural mapping from $S_0(\mathbb{R})$ to $\mathbb{C}^N = \ell^2(\mathbb{Z}_N)$ can be obtained by periodization with period $p = \sqrt(N)$ and samping at the distance $s = 1/p$. This operator can be described as

$$PS_N(f) = ⧢_p * (⧢_s \cdot f) = ⧢_s \cdot (⧢_p * f), \quad f \in S_0(\mathbb{R}).$$

The resulting periodic, discrete signal belongs to $S_0'(\mathbb{R})$ and thus has a Fourier transform, which by the rules for the extended FT on $S_0'(\mathbb{R})$ can be realized using the FFT.
The question of approximately recovering $\widehat{f}$ from the FFT/DFT applied to this sequence of sampling values can be well described and is subject of current investigations. (see upcoming SIAM review paper by Ehler, Gröchenig and Klotz).

Q4: Generalized Stochastic Processes I

As I learned from many discussions with engineers one should sometimes take a probabilistic viewpoint. A signal is not "a true signal" but just a *stochastic realization* of a process. So to say, even "measuring the same thing several times" does not mean that one gets exactly the same values.
Thus it is natural to ask what a "stochastic signal" could be.
Recall that a (continuous) function assigns to each point t a value $f(t)$. A *generalized function* σ assigns to each test function k (average of point masses) some value $\sigma(k)$. A *stochastic process* $\rho(t)$ assigns to each value t a probability measure, ideally a member of some abstract Hilbert space $\mathcal{H}$ (defined in a probabilistic manner). So a *generalized stochastic signal* (**GSP**) is a bounded linear operator $\rho : \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \to (\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ (see work with W.Hörmann, 1989). [8]).

Q4: Generalized Stochastic Processes II

For each GSP there is a Fourier transform, the associated *spectral process* and a *spectral representation* (inverse Fourier transform), by the usual rule

$$\widehat{\rho}(f) = \rho(\widehat{f}), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

For every GSP we have a *autocorrelation* $\sigma_\rho \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ with

$$\sigma_\rho(f \otimes g) = \langle f, g \rangle_{\mathcal{H}}, \quad f, g \in \boldsymbol{S}_0(\mathbb{R}^d).$$

As expected the autocorrelation of $\widehat{\rho}$ is just $\mathcal{F}_{\mathbb{R}^{2d}}(\sigma_\rho)$.
Wide sense stationarity means:

$$\langle \rho(x), \rho(y) \rangle_{\mathcal{H}} = \langle \rho(x+h), \rho(y+h) \rangle_{\mathcal{H}}, \quad x, y, h \in \mathbb{R}^d.$$

Or in the case of generalized stochastic processes

$$\langle \rho(f), \rho(g) \rangle_{\mathcal{H}} = \langle \rho(T_h f), \rho(T_h(g)) \rangle_{\mathcal{H}}, \quad h \in \mathbb{R}^d, f, g \in \boldsymbol{S}_0.$$

Where do MILD DISTRIBUTIONS appear naturally

We list topics according to relevance for engineering students!

1. Signals ARE mild distributions (talk at LMU, 7.6.2024)
2. Mild distributions describe *impulse response* and transfer function. for TILS (*translation invariant linear systems*).
3. This approach allows a natural approach to convolution and the Fourier transform (convolution theorem), see [5].
4. The interpretation of the FT as *unitary BGTr automorphism* unifies different FFT variants and reveals connections.
5. This gives easy approach to *Shannon Theorem*.
6. The kernel theorem (also KNS symbols, spreading function) opens a natural approach to “continuous matrix theory”.
7. Such considerations help in *Quantum Harmonic Analysis*.

The Kirkwood-Dirac Principle

Recently it is observed that ideas named after J. Kirkwood ([10]) and P. Dirac ([2]) can play an important role in quantum theory. What is called in such papers the Kirkwood-Dirac distribution (in analogy to the description of states by the Wigner distribution) is rather a general principle, we call it the *Kirkwood-Dirac Principle* (KDP) comparable to the choice of two bases in linear algebra in order to describe a linear operator by a matrix.

Obviously the Dirac basis $(\delta_x)_{x\in\mathbb{R}^d}$ (it is the prototype of a *mild basis*) and the Fourier basis $(\chi_s)_{s\in\mathbb{R}^d}$ are two natural bases, used in this context. But as a matter of fact such a description is known in the engineering literature as *Rihaczek distribution*:

$$\mathrm{Rih}(f,g)(t,s) = e^{2\pi i st} f(t)\widehat{g}(s).$$

An appropriate theory of *mild bases* is on the way!

Kernels for Fractional Fourier Transforms

Obviously the discrete Fourier transform (FFT) can be described as the transition from the Dirac basis to the Fourier basis consisting of pure frequencies, as it maps unit vectors to the characters of $\mathbb{Z}_N$. Unit vectors are a joint basis diagonalizing multiplication operators, while the Fourier basis is the joint basis for convolution operators.

The fractional Fourier transform (FrFT) of order $\alpha \in \mathbb{R}$ is

$$\mathcal{F}_\alpha[f](u) = e^{-i\pi \operatorname{sgn}(\sin\alpha)/4} \int_{-\infty}^{\infty} f(t) K_\alpha(t,u)\, dt,$$

where (for $\alpha \neq k\pi, k \in \mathbb{Z}$):) the kernel is given by:

$$K_\alpha(t,u) = \sqrt{1 - i\cot\alpha}\, e^{i\pi(t^2\cot\alpha - 2tu\csc\alpha + u^2\cot\alpha)}.$$

The involved chirp functions are eigenvectors to TF-shifts!

Thanks and LINKS

Thank you for your attention!

All relevant papers are downloadable from my homepage

https://nuhagphp.univie.ac.at/home/feipub_db.php

Course notes (including YouTube Links) are found at

www.nuhag.eu/ETH20

Many other talks on related subjects can be downloaded from

https://nuhagphp.univie.ac.at/home/feitalks.php

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