Treating Dirac-like continuous bases in the context of mild distributions

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Submitted ABSTRACT

It is a common method used in many engineering disciplines or in physics to the consider the collection of Dirac measures $\delta_x, x \in R^d$ as a continuous (orthonormal?) basis, e.g. by starting the discussion of time-invariant systems from the sifting property of the Dirac delta, which could be written as

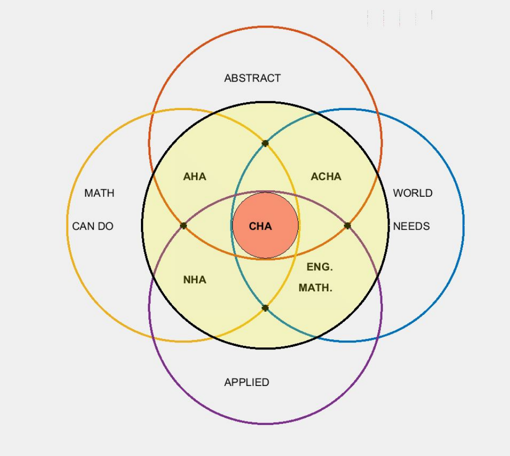
$$f = \delta_0 * f \quad [= T_0 f = Id(f)].$$

Although sometimes quite intuitive, at least in the context of (continuous) functions which are well described by their pointwise values, the existence of values is almost lost for the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and completely in the realm of (mild) distributions.

Abstract Part II

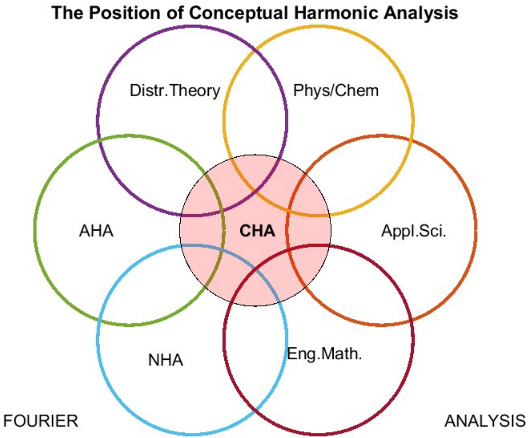
Nevertheless there is a kernel theorem for operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to the dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$ of mild distributions, which can be viewed as the representations of the operator, using on both sides (the domain and the target space) the Dirac basis.
The basic properties of the Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ allow to discuss other bases (e.g. by making use of the concept of Fractional Fourier transform) and thus initiate the discussion about the similarity (in the sense of linear algebra) of different representations of this type, which we call mild bases (by abuse of language), or Dirac-like bases. In this way we can shed some light on what we call the Kirkwood-Dirac Principle (KDP), which seems to get large attention in the quantum community recently.

The IKIGAI Diagram for Conceptual Harmonic Analysis



The IKIGAI Principle

According to WIKIPEDIA we have:
Ikigai can describe having a sense of purpose in life, as well as being motivated According to a study by Michiko Kumano, feeling ikigai as described in Japanese usually means the *feeling of accomplishment and fulfillment that follows when people pursue their passions*.
Activities that generate the feeling of ikigai are not forced on an individual; they are perceived as being spontaneous and undertaken willingly, and thus are personal and depend on a person's inner self.
According to psychologist Katsuya Inoue, ikigai is a concept consisting of two aspects: "sources or objects that bring value or meaning to life" and "a feeling that one's life has value or meaning because of the existence of its source or object".


The Position of Conceptual Harmonic Analysis
Distr.Theory
Phys/Chem
AHA
CHA
Appl.Sci.
NHA
Eng.Math.
FOURIER
ANALYSIS

Overview and Motivation I

Wiener Amalgams and Modulation Spaces have shown their usefulness in the context of time-frequency analysis and in particular for Gabor Analysis, the study of Gabor multipliers or Anti-Wick operators and so on. The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, based on the Feichtinger algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, with the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ in the middle, has been useful from the very beginning. The standard principle which can be applied based on this triple is to verify identities which hold true in the case of finite Abelian groups for the space $\boldsymbol{S}_0(\mathbb{R}^d)$ of test functions, by replacing sums by integrals. For example, the Fourier inversion theorem applies pointwise for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Overview and Motivation II

Going on one extends the FT to all of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (where it defines a unitary operator, known as Plancherel's Theorem) and finally duality applies to extend it in a unique way to all of $\boldsymbol{S}_0'(\mathbb{R}^d)$ by setting $\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in \boldsymbol{S}_0(\mathbb{R}^d)$. This is a w^*-w^*-continous extension, and as such it is uniquele determined. Altogether one can thus describe the Fourier Transform in this setting as a unitary Banach Gelfand Triple Automorphism (**UBGTA**).
Taking the finite dimensional version it should be seen just as a change of bases, from the basis of unit vectors in $\mathbb{C}^n$ to the Fourier basis, the system of column vectors of the DFT-matrix.

In this talk we want to discuss the properties of the DIRAC basis and related mild bases, and provide several concrete examples.

What are continuous orthonormal bases

We will proceed as follows:

1. First recall some concepts that are found in the literature
2. Inspect concrete special cases, like the Dirac or Fourier basis
3. Learn from this setting how to describe it in the context of the Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$, using the concept of *mild convergence* for the space $\mathbf{S}_0'(\mathbb{R}^d)$ of mild distributions;
4. Use the family of (unitary) automorphism in the category of BGTs in order to define mild or *Dirac-like* bases, which then inherit their (good) properties from the Dirac basis.

Overall Motivation

Most of the time mathematical talks outline a specific set of mathematical problems and describe ways how to come up with solutions, or describe new methods to attack those problems, mention the obstacles on the way and the improvements achieved by the authors.

Survey talks summarize the field, provide a summary of the state of the art and outline possible new methods for the solution of a well-defined set of problems, e.g. characterizing functions with a particular form of smoothness and so on.

THIS is a(onther) perspective talk putting established problem settings into question and suggesting new ways to look at well-known and less well-known mathematical problems.

Some Linear Algebra Memories I

We recall that linear mappings T from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be compactly described as real $n \times n$-matrices $\boldsymbol{A}$. The rows or columns describe the image of the corresponding unit vectors under right/left matrix-vector multiplication, described again in the standard basis of $\mathbb{R}^n$.

Of particular interest are the so-called *orthogonal matrices* which in fact should be called *orthogonality preserving matrices*. They can be characterized by the fact that the rows (or columns) form an orthonormal basis for $\mathbb{R}^n$, and since the inversion can be realized by transposition the two choices do not matter. Obviously these matrices form a group under composition.

The corresponding terminology for the complex situation and also for Hilbert spaces is the notion of *unitary matrices*.

Some Linear Algebra Memories II

Going from finite dimension or the discrete, countable situation to the continuous setting is not so obvious. While the most interesting examples of (discrete) frames or Banach frames appear in connection with the sampling of abstract voice transforms (wavelet frames, Gabor frames, etc.), where typically one has first a continuous collection of *coherent elements*, the orbit of an element in some Hilbert space under a group action, and has to work if one wants to demonstrate that discrete subfamilies form a frame in the now classical setting (this is done in *coorbit theory*) the current trend seems to go into the opposite direction, and continuous frames are introduced, and not so much continuous bases. Of course, one can could view the collection of Dirac measures $\delta_\tau, \tau \in \mathbb{T}$ as a kind of basis for $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$, but obviously Dirac measure do not belong to $\boldsymbol{L}^2(\mathbb{T})$.

Some Linear Algebra Memories III

Asking AI or consulting physics books one often finds the following description of the collection $(\delta_x)_{x \in \mathbb{R}}$ as a continuous orthonormal basis:

Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

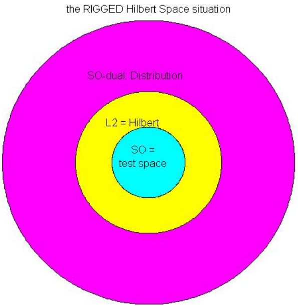


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!

What we are going to suggest/present is the Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of *Feichtinger's algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$,

known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (**[lo83-1]**).
For $\mathbb{R}^d$ the most elegant way (which is describe in **[gr01]** or **[ja18]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}\, dy$$

and the corresponding norm

$$\|f\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)| dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$;
2. $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d)$ *(isometrically)*;
3. *Isometrically invariant under TF-shifts*
$$\|\pi(\lambda)(f)\|_{\boldsymbol{S}_0} = \|M_s T_t f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$
4. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *is an essential double module (convolution and multiplication)*
$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d),$$
in fact a Banach ideal and hence a double Banach algebra.
5. *Tensor product property* $\boldsymbol{S}_0(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{S}_0(\mathbb{R}^d) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(\boldsymbol{S}_0(G)) = \boldsymbol{S}_0(H)$.

1. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ has various equivalent descriptions, e.g.
   - as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d)$;
   - via *atomic decompositions* of the form

$$f = \sum_{i \in I} c_i \pi(\lambda_i) g \text{ with } (c_i)_{i \in I} \in \boldsymbol{\ell}^1(I).$$

2. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is invariant under group automorphism;
3. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is invariant under the *metaplectic group*, and thus under the *Fractional Fourier transform* as well as the multiplication with *chirp signals*: $t \mapsto exp(-i\alpha t^2)$, for $\alpha \geq 0$.

In addition $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is quite universally useful in Classical Fourier Analysis and of course for *Time-Frequency Analysis* and *Gabor Analysis*, and as I am going to show also for QHA: Quantum Harmonic Analysis. In short, it is easier to handle than the Schwartz-Bruhat space or even the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, and since $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ it is (much) bigger.

Theorem

1. $\mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$ *via* $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0'$.
2. *Identification of TLIS:* $\boldsymbol{H}_G(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0, \ (\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \to \boldsymbol{S}_0(H) \hookrightarrow \boldsymbol{S}_0(G)$ *via* $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \boldsymbol{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \boldsymbol{S}_0(G) \mid \mathrm{supp}(\tau) \subset H\}$.

Theorem

1. $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}) = (\boldsymbol{M}^\infty(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^\infty})$, *with* $V_g(\sigma)$ *and* $\|\sigma\|_{\boldsymbol{S}_0'} = \|V_g(\sigma)\|_\infty$, *hence norm convergence corresponds to uniform convergence on pahse space. Also* w^*-*convergence is uniform convergence over compact subsets of phase space.*
2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, *with density for* $1 \leq p < \infty$, *and* w^*-*density in* $\boldsymbol{S}_0'$. *Hence, facts valid for* $\boldsymbol{S}_0$ *can be extended to* $\boldsymbol{S}_0'$ *via* w^*-*limits.*
3. *Periodic elements* $(T_h\sigma = \sigma, h \in H)$ *correspond exactly to those with* $\tau = \mathcal{F}(\sigma)$ *having* $\mathrm{supp}(\tau) \subseteq H^\perp$.
4. *The (unique) spreading representation* $T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)d\lambda$, $F \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *for* $T \in \mathcal{B}$ *extends to the isomorphism* $T \leftrightarrow \eta(T)$ $\eta : \mathcal{B} \approx \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, *uniquely determined by the correspondence with*
$$\eta(\pi(\lambda)) = \delta_\lambda, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Some conventions

Scalar product in $\mathcal{H}$:

$$\langle T, S\rangle_{\mathcal{H}} = \mathrm{trace}(T * S^*)$$

In **[feko98]** the notation

$$\alpha(\lambda)(T) = [\pi \otimes \pi^*(\lambda)](T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

and the covariance of the KNS-symbol is decisive:

$$\sigma(\pi \otimes \pi^*(\lambda)(T)) = T_\lambda(\sigma(T)), \quad T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

The attempt of the definition mild ONBs I

Taking the family $(\delta_x)_{x\in\mathbb{R}^d}$ as a prototype one might extend the concept of orthonormal bases to a continuous BGT-setting as follows.

Definition

A family $(\tau_x)_{x\in\mathbb{R}^d}$ in $\boldsymbol{S}_0'(\mathbb{R}^d)$ is called a mild ONB if it forms a tight, (normalized) continuous frame in $\boldsymbol{S}_0(\mathbb{R}^d)$, meaning that

$$\|f\|_2 = \left(\int_{\mathbb{R}^d} |\tau_x(f)|^2 dx\right)^{1/2}, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{1}$$

with the extra property that the range of the coordinate mapping $f \mapsto \tau(f)$: with $\tau(f)(x) := \tau_x(f), f \in \boldsymbol{S}_0(\mathbb{R}^d)$ is dense in $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$.

The attempt of the definition mild ONBs II

Let us recall the situation of frames in the discrete setting, even in the finite-dimensional. Frame in $\mathbb{C}^n$ are simply generating families of vectors in $\mathbb{C}^n$, which means we have to have at least $m \geq n$ such vectors. If the frame is not a basis (or $m > n$) then we will have some linear dependencies, but than the range space of the coefficient mapping will be n-dimensional, so it is not dense in $\mathbb{C}^m$. For the case $m = n$ however we recognize that the analogue of (1) in fact implies that our frame is an ONB (and otherwise not!). Similar consideration apply for the case of countable tight frames in separable Hilbert spaces.

The attempt of the definition mild ONBs III

So let us consider the situation above: We are more or less in the situation of the proof of Plancherel's Theorem. First we extend $\tau : f \mapsto (\tau_x(f))_{x\in\mathbb{R}^d}$ to all of $\boldsymbol{L}^2(\mathbb{R}^d)$ (since $\boldsymbol{S}_0(\mathbb{R}^d)$ is dense in $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$) by a density argument. Surjectivity result from the density in the range. For Plancherel's Theorem this follows by the Weierstrass Theorem.

We thus have an isometric embedding into $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$.

Let us now assume **in addition** that τ maps $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$ (?do we need to assume density also here?). Then by a Theorem established in 1998 (with W.Kozek) we can extend this *unitary automorphism* to a unitary BGT automorphism! Clearly in this case the value of $\tau(f)$ at $x \in \mathbb{R}^d$, or $\delta_x(\tau(f)) = \tau_x(f)$, and thus the inverse of the mentioned BGT-automorphism determines the family $(\tau_x)_{x\in\mathbb{R}^d}$.

Orthogonal Mild Basis

Definition

A family $(\tau_x)_{x\in\mathbb{R}^d}$ in $\boldsymbol{S}_0'(\mathbb{R}^d)$ is called a OMB (orthogonal mild basis) on $\mathbb{R}^d$ if there exists some unitary BGT-automorphism α such that $\tau_x = \alpha(\delta_x), x \in \mathbb{R}^d$.

Obviously this is a "cheap definition" (once the concept of BGTs is available) which implies immediately a variety of basic results, using the analogy between the unitary group over $\mathbb{C}^n$ and the group of unitary BGT-automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. However, one has to show that there are interesting examples.
As the first example let us mention the Fourier basis. Using the above concept and the fact that the (inverse) Fourier transform defines a unitary BGT-automorphism, which obviously maps δ_x to χ_s (pure frequencies) implies immediately that these pure frequencies form an OMB.

Metaplectic Dirac-like bases I

An important group of unitary BGT-automorphism of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ arises from the action of the *metaplectic group*. This group contains a variety of important integral transformation, such as the fractional Fourier transform (FrFT). Although it is easier to understand the fact that the form a *compact group of unitary operators* on $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ it is more intuitive to view them as rotation operators in the the phase-space domain (e.g. via the Fock space version), leaving thus the spaces $\left(\mathbf{M}^p(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^p}\right)$, $1 \leq p \leq \infty$ invariant. Recall that $\mathbf{M}^1(\mathbb{R}^d) = \mathbf{S}_0(\mathbb{R}^d)$ and $\mathbf{M}^\infty(\mathbb{R}^d) = \mathbf{S}_0'(\mathbb{R}^d)$.

The Kernel Theorem I

Going one level higher we move from signals $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ to operators (or perhaps *observables* in Quantum Theory) $T \in \mathcal{L}\left(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\right)$.
Obviously the tensor product of two Dirac bases in $\mathbb{R}^k$ an $\mathbb{R}^l$ provide the Dirac basis for $\mathbb{R}^n$, with $n = k + l$. This can be used to generate new bases, which are connected with partial Fourier transforms in the different possible variables, which also form unitary BGTs.
One could define the metaplectic group (usually described as double covering of the symplectic group) as the group of unitary operators generated by dilations (induced by non-singular matrices), multiplication operators by chirp signals (*character's of second degree*, according to A. Weil) and the (partial) Fourier transforms.

The Kernel Theorem II

MAYBE hints to the theory of Gelfand-Dirac bases: **[boga89]** (really 1989??) tells us in the context of Rigged Hilbert Spaces that there is a *complete set of generalized eigenvectors*.
They mention an obvious *short-coming* of Neumann's approach to Quantum Theory: He suggests to identify *physical observables* with self-adjoint operators $\boldsymbol{A}$ on some Hilbert space $\mathcal{H}$, and *(pure) states* with unit vectors ψ resp. the corresponding one-dimensiona projection operators $P_\psi(f) = \langle f, \psi \rangle_{\mathcal{H}} \psi$.
The (probabilistic term) *expectation value of the observable $\boldsymbol{A}$ in the state ψ* is the expressed by the term (using Dirac's Bra-Ket notation)

$$\langle \psi | \boldsymbol{A} | \psi \rangle = \langle \psi, \boldsymbol{A}(\psi) \rangle_{\mathcal{H}} = \mathrm{trace}(\boldsymbol{A} \circ P_).$$

Spreading function and Kohn-Nirenberg Symbol I

The analogy to the decomposition of any $n \times n$-matrix into a linear combination of the n^2 TF-shift operators over $\mathbb{Z}_n$ gives the spreading representation of operators, or better the unitary BGT-isomorphism mapping kernels of operators to the corresponding *spreading operators* quite plausible.
Here the TF-shift operators (which act as unitary BGT-automorphism each, by themselves) $\pi(\lambda) = M_s T_t$ correspond to the Dirac measures at $\lambda = (t, s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
The KNS-symbol of the operators T with kernel σ_T then can be obtained by the symplectic Fourier transform of the spreading symbol. The corresponding building-blocks are the rank one operators which arise as a product of a Dirac measure with a pure frequency (cf. partial Fourier transforms).

The Weyl-Wigner connection I

The *Wigner transform* (usually defined for so-called "states", i.e. functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|f\|_2 = 1$ can be extended to the cross Wigner transform by polarization and gives at the end a unitary BGT morphism, say again from kernels of operators to their Wigner symbol. This is arranged in such a way that it is real-valued for Hermitean operators and also behaves in a covariant way under metaplectic operators (corresponding to certain symplectic matrices acting on phase space).

The corresponding family of self-adjoint operators are the essentially the TF-shifts combined with the *parity operator*: $f \mapsto f^{\checkmark} : f^{\checkmark}(x) = f(x), x \in \mathbb{R}^d$.

The corresponding OMB consists than (described as operators) of operators derived from the parity operator by conjugation with TF-shifts. For the discrete setting (n even) one has to choose a slightly different approach to Weyl-Heisenberg operators.

Theorem

- *Any mild basis $(\tau_x)_{x\in\mathbb{R}^d}$ constitutes a bounded subset of $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$.*
- *For very mild basis the mapping $x \mapsto \tau_x$ is continuous from $\mathbb{R}^d$ into $\mathbf{S}_0'(\mathbb{R}^d)$, endowed with the w^*-topology. It can be characterized by the convergence of sequences, i.e. by the fact, that one has for any sequence $x_n \to x_0$ in $\mathbb{R}^d$: $\lim_{n\to\infty} \tau_{x_n}(f) = \tau_{x_0}(f)$, forf $\in \mathbf{S}_0(\mathbb{R}^d)$.*
- *Given $f^1, f^2 \in \mathbf{S}_0(\mathbb{R}^d)$ we have $f^1 = f^2$ (as functions, or in $\mathbf{S}_0(\mathbb{R}^d)$) if and only if $\tau_x(f^1) = \tau_x(f^2)$, $x \in \mathbb{R}^d$ for one (and hence all) mild bases $(\tau_x)_{x\in\mathbb{R}^d}$.*
- *Given to mild basis $(\tau_x^1)_{x\in\mathbb{R}^d}$ and $(\tau_x^2)_{x\in\mathbb{R}^d}$ for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ there exists a unique BGT-automorphism γ with the property that $\gamma(\tau_x^1) = \tau_x^2$, for $x \in \mathbb{R}^d$.*

Tensor product of mild bases: I

By analogy to the possibility of obtaining a basis representation of matrices as a sum of rank-one operators using elementary tensors we can expect a similar situation for the continuous case.
As in the matrix case one can describe the Kirkwood-Dirac representation of a matrix simply as the representation of a matrix in a *tensor product basis*. Consequently it is natural to first define the tensor product of two mild bases as a mild basis on $\mathbb{R}^{2d}$ (for simplicity, the general situation is easily derived using a slightly more complicated notation).

Definition

Given two mild bases $(\tau^1_x)_{x\in\mathbb{R}^d}$ and $(\tau^2_y)_{y\in\mathbb{R}^d}$ we can define their tensor product $(\tau^1_x \otimes \tau^2_y)_{(x,y)\in\mathbb{R}^{2d}}$.

Tensor product of mild bases: II

Due to the simple fact that

$$\mathbf{S}_0'(\mathbb{R}^d) \otimes \mathbf{S}_0'(\mathbb{R}^d) \subset \mathbf{S}_0'(\mathbb{R}^{2d}),$$

in fact even a proper inclusion

$$\mathbf{S}_0'(\mathbb{R}^d) \widehat{\otimes} \mathbf{S}_0'(\mathbb{R}^d) \subset \mathbf{S}_0'(\mathbb{R}^{2d}),$$

this is a well-defined, bounded family in $\mathbf{S}_0'(\mathbb{R}^{2d})$.

Lemma

Given two unitary mild bases $(\tau_x^1)_{x \in \mathbb{R}^d}$ and $(\tau_x^2)_{x \in \mathbb{R}^d}$ for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ their tensor product $\tau_{\mathbf{x}}^1 \otimes \tau_{\mathbf{y}}^2$, with $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}$, defines a mild basis for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. We call it a TUMB.

Tensor product of mild bases: III

Proof.

First we unify the notation and replace the generic Dirac basis $(\delta_{\mathbf{z}})_{\mathbf{z}\in\mathbb{R}^{2d}}$ for $\mathbb{R}^{2d}$ by the usual double index $\delta_{\mathbf{x},\mathbf{y}}$ with $(\mathbf{x},\mathbf{y})=\mathbf{z}\in\mathbb{R}^{2d}$. This simply relies an the given (maybe to be adapted to be a particular situation) splitting, which identifies $\mathbb{R}^{2d}$ with $\mathbb{R}^d\bigoplus\mathbb{R}^d$.

Given the two automorphisms of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, let us call them α^1 and α^2 which are involved in the creation of the individual mild bases we can apply them to the components.

Based on the tensor product property for $\boldsymbol{S}_0$, which allows to naturally identify $\boldsymbol{S}_0(\mathbb{R}^{2d})$ with $\boldsymbol{S}_0(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{S}_0(\mathbb{R}^d)$ (with equivalence of natural norms), we can easily establish the action of $\alpha^1\otimes\alpha^2$ (respectively their inverse automorphisms, β^1 and β^2) on $\boldsymbol{S}_0(\mathbb{R}^{2d})$.

□

Tensor product of mild bases: IV

Justified by this consideration we use the following terminology:

Definition

Mild bases for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ arising as a tensor products of some decomposition of $\mathbb{R}^{2d}$ as a direct sum of two copies of $\mathbb{R}^d$ are called separable mild bases. We call it a separable unitary mild basis (or separable MOB? so SMOB??) if both mild bases of mild orthonormal bases.

Lemma

A Kirkwood-Dirac representation of an operator is just the description of an operator from $\mathcal{L}\left(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d)\right)$ *using a suitable separable unitary mild basis.*

Tensor product of mild bases: V

We are going to demonstrate that a Kirkwood-Dirac distribution is - for the most relevant examples, where only mild distributions are involved - are JUST particular basis expansion using tensor mild product bases

Tensor product of mild bases: VI

Theorem

Given any mild coordinate system $(\tau_x)_{x\in\mathbb{R}^d}$ the coordinate functions $(\tau_x(f))_{x\in\mathbb{R}^d}$ belong to $\boldsymbol{S}_0(\mathbb{R}^d)$ for any $f\in\boldsymbol{S}_0(\mathbb{R}^d)$. Hence, any metaplectic transformation $\mu\in\mathrm{Mp}(2d,\mathbb{R}^d)$ can be applied and defines a bounded operator into $\left(\boldsymbol{S}_0(\mathbb{R}^d),\|\cdot\|_{\boldsymbol{S}_0}\right)$. Taking it back to $\boldsymbol{S}_0(\mathbb{R}^d)$ via the synthesis function

$$\mathrm{SYN}(F)(f) := \int_{\mathbb{R}^d} F(x)\tau_x(f)dx, \quad f\in\boldsymbol{S}_0(\mathbb{R}^d), \tag{2}$$

μ induces an invertible mapping T_μ on $\left(\boldsymbol{S}_0(\mathbb{R}^d),\|\cdot\|_{\boldsymbol{S}_0}\right)$. In fact, T_μ extends in a unique fashion to an automorphism of $(\boldsymbol{S}_0,\boldsymbol{L}^2,\boldsymbol{S}_0')(\mathbb{R}^d)$, which can be described as

$$f\mapsto T_\mu(f) = \int_{\mathbb{R}^d}\mu(\tau(f))(x)\tau_x\,dx. \tag{3}$$

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Links and Refs

Let us provide a few links:

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This talk provides some thought related to the
CONCEPTUAL HARMONIC ANALYSIS idea.