NuHAG Numerical Harmonic Analysis Group

A Gauss-Gabor approach to Mild Distributions

New Perspectives on the Extended Fourier Transforms

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CSASC 2026: PILSEN Talk held Feb. 3rd, 2026

SUBMITTED CONTRIBUTION for PILSEN 2026 I

In the context of Time-Frequency Analysis or Quantum Harmonic Analysis the Banach Gelfand Triple (BGT, or rigged Hilbert space) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ plays an important and natural role, with the Feichtinger algebra $\boldsymbol{S}_0$ and its dual as key ingredients. The elements of $\boldsymbol{S}_0'$ are called mild distributions. During the last decade this BGT has emerged as a versatile tool contributing to a better understanding of classical questions in Fourier Analysis, in the context of digital signal processing (engineering) or physics (using the Dirac continuous "orthonormal basis" or Wigner distributions). Thus the question arose *how to provide a physics motivated description of this BGT*.

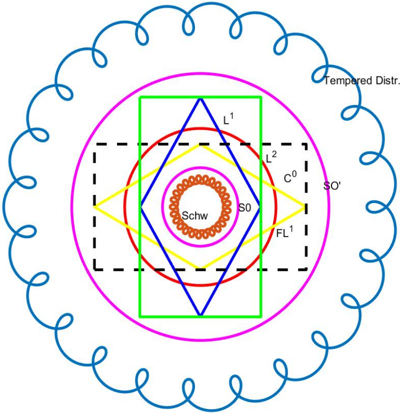
The meanwhile well established theory of Gabor Analysis allows to provide such a description. Starting from a lattice $\Lambda = aZ \times bZ$ in the complex plan (phase space), ideally with $a = b = 1/\sqrt{2}$,

SUBMITTED CONTRIBUTION for PILSEN 2026 II

and a Gaussian window one can characterize the space of mild distributions as the space of tempered distributions which allow a (non-unique) description as infinite double-series of Gaussian coherent states, with bounded coefficients. The atoms are thus of the form $exp(2\pi ibnt)g(t-bk)$, with $g(t)=exp(-\pi|t|^2)$, the Fourier invariant Gaussian.

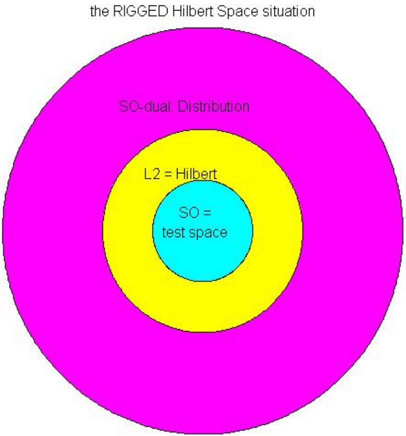
The talk will highlight an intuitive description of the BGT, backed up by established mathematical functional analytic facts. In particular, the very natural description of the concept of mild convergence (a particular form of distributional convergence) and its relevance will be explained, by comparing the situation with the handling of real numbers as infinite decimals. Among others, the sequential approach to mild distributions can be re-introduced with this perspective.

A schematic description: all the spaces



A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



Goals of this lecture

- Recall the view on Dirac bases.
- Discuss how problematic they are.
- Remind the concept of Banach Gelfand Triples.
- Specifically $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
- The Fourier transform as a basic
  unitary Banach Gelfand Triple automorphism.
- Alternative Gaussian perspective
- Definition of *mild* (Dirac-like) bases.
- Examples and illustrations, e.g.
  Fractional Fourier Transforms as a change of bases!
- Signals $\sigma \in \boldsymbol{S}_0'$ are measured by evaluation $\mapsto \sigma(f)$!

MATHEMATICAL SIGNALS I

Asking a mathematician or even a mathematically well-trained engineer what a "signal is" most likely the answer be: a function $f \in \boldsymbol{L}^2(\mathbb{R})$. As a justification there is the comment that "every real-life signal has finite energy" and $\int_{-\infty}^{\infty} |f(t)|^2 dt$ is considered the "energy in the signal f". Plancherel's Theorem (stating that the Fourier transform is a unitary automorphism of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$) is interpreted as "energy preservation".
Of course, when it comes to do actual computations the engineer will say: Since the computer only allows to deal with finite length vectors one has to replace the integral transform by the FFT, and it is natural to obtain such a finite sequence by sampling the function (regularly and) sufficiently "fine" over a sufficiently "long interval".
Discussions about the connections between the discrete and the continuous setting are still rare (would be important)!

Different Settings of Signals I

Engineering Books

Engineers are use to have signals of different types:

- Signals are either continuous of discrete;
- Signals are either periodic or non-periodic;
- or combinations: finite = discrete + periodic;

For each setting we have a corresponding Fourier transform.

20th Century Mathematical Analysis

In Abstract Harmonic (Fourier) Analysis we view such signals as functions (or distributions) over suitable LCA groups: e.g., $\mathbb{R}^d$ versus $\mathbb{Z}^d$, or $\mathbb{R}$ versus $\mathbb{T}$ (torus: Fourier series!).
In this general setting one finds that compact groups have discrete dual groups ("pure frequencies") and vice versa. $\mathbb{R}^d$ is non-compact and non-discrete. Periodic and discrete signals live on finite Abelian groups (such as the unit roots of order $N \in \mathbb{N}$).

Different Settings of Signals II

In the classical setting $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$, the space of absolutely convergent Fourier series, which is dense in $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$. Together with its dual space, known as the space $\boldsymbol{PM}(\mathbb{T})$ of so-called pseudo-measures we have the "prototypical Banach Gelfand triple" $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})$, and the mapping from functions (on the torus) to their Fourier coefficients can be considered the first BGT-isomorphism. It identifies (via distribution theory) the space of pseudo-measures with the set of (periodic, tempered) distributions which have bounded Fourier. Thus the target is $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\mathbb{Z})$.
We observe, that the unitary mapping at the Hilbert space level restricts to an identification of $\boldsymbol{A}(\mathbb{T})$ with $\boldsymbol{\ell}^1(\mathbb{Z})$, and extends (in a "unique way") by duality to the "outer spaces".

Integration Theory: Why Lebesgues Spaces? I

It is clear that integration theory plays a significant role in the development of Fourier Analysis, first through the Riemann integral and later via the Lebesgue integral, which is also a cornerstone of functional analysis.

After all, it APPEARS as obvious that one needs the (perfect) integration methods if one wants to study the Fourier transforms (and its inversion), as it IS (?) an integral transform, given by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t}\, dt \quad \text{and} \quad f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega}\, d\omega, \tag{1}$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, but it is not natural, because $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ does NOT imply $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, hence summability methods had to be used!

Integration Theory: Why Lebesgues Spaces? II

Later on the Fourier transform (over $\mathbb{R}^d$, for this discussion) was extended to $\boldsymbol{L}^p(\mathbb{R}^d)$, but with the restriction to $1 \leq p \leq 2$, known as the *Hausdorff-Young* Theorem. Nowadays it is derived using the "complex interpolation method" (going back to Riesz Thorin). Boundedness of quite general operators (Fourier multipliers, pseudo-differential operators, etc.) are still considered over $\boldsymbol{L}^p$-spaces (with the Hilbert space case $p = 2$ as the cornerstones). Many of the modern function spaces, such as Besov-Triebel-Lizorkin spaces are obtained using Fourier methods, but basically using norms defined naturally on $\boldsymbol{L}^p(\mathbb{R}^d)$, and using *dyadic partitions of unity* (Paley-Littlewood Theory).

But $\boldsymbol{L}^p$-spaces and Fourier transforms do not go along too well!
Only $\boldsymbol{L}^2(\mathbb{R}^d)$ is invariant under the Fourier transform!

REAL LIFE signals I

Let us look at different *real-life signals*. Is it really true that we have (almost everywhere) well-defined functions if we consider a signal? In a few cases we could consider a signal as a bounded and continuous functions, say the temperature as measured by a classical thermometer. BUT in reality this is already an average of the *fluctuating temperature* that we might sense! (due to an open window, or hot air from the hair dryer).

A more typical case is an audio signal, which can be recorded, stored, compressed nowadays without problems (MP3). The HiFi standard used for the reproduction of music stored in a digital format on a CD (or transmitted for streaming) makes use of a sampling rate of 44100 samples per second, which (based on the Shannon Sampling Theorem) allows a (*quasi real-time*) reconstruction of the underlying "function" up to the audible range (resolution) up to $20kHz$ (using spectrograms).

REAL LIFE signals II

In summary: an acoustic signal (of time) is “something” that can be recorded (using a microphone) which (with the help of an A/D-converter) allows to compute a spectrogram (or Short-Time Fourier Transform, or Sliding-Window Fourier Transform).

Two-dimensional signal analysis is seen as the basis for digital image processing. Here we have a similar situation. While the standard mathematical approach talks about the Hilbert spaces $\boldsymbol{L}^2(\mathbb{R}^2)$ it is not realistic to define the (gray or color) value at a physical point but only at local average values as obtained with the help of photonic sensors. Depending on the quality of such a digital camera one has different resolution levels.

Thus in a way an IMAGE can be seen as a collection of possible pixel images of given scene, again up to some resolution.

REAL LIFE signals III

As a final example, emphasizing the idea that a "signal" is something which can be recorded or measured, let us think of the gravitational force acting on mass (i.e. materials of different types). Here is is clear that we cannot measure the infinitesimal action of a force, say on an individual atom. But we can observe the acceleration of a piece a falling stone on earth or the moon, and deduce, using physical laws and some mathematics (going back to Newton) the strength of the gravitational force at this location. On the other hand it is common in physics to talk about point-charge or in our case of point mass, i.e. the whole mass of body concentrated in one point. For long distances this simplifies a lot computations and is good enough to explain the action of the moon on the ocean (tides), or the path of a stone thrown into the air, following a parabolic curve, or "dithering" (print).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ (hence w^*-dense there) is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is a unitary isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

Banach Gelfand Triples

In principle every CONB ($=$ *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

Sufficient conditions for BGT-automorphisms

A couple of basic facts resulting direct from the definition, combined with the general fact that the space of w^*-continuous linear functionals on a dual space $\boldsymbol{B}'$ coincides naturally with the original Banach space, gives the following facts.
We apply them to the concrete Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.

- For every BGTr-mapping there is an adjoint BGTr-mapping T^*, and $T^{**} = T$ for each of them;
- If a bounded linear mapping on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ leaves $\boldsymbol{S}_0(\mathbb{R}^d)$ invariant and so does T^* (the Hilbert adjoint), then T (and also T^*) define BGTr-homomorphism.

Among others this principle can be used to show that the even *fractional Fourier transforms* define (unitary) BGTr-automorphisms on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

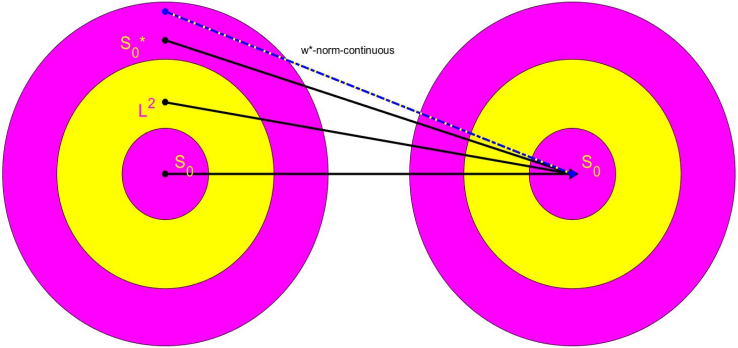
Some general scenarios arising in Gabor Analysis

There are situations, where one is forced to view even more general situations than BGTr-operators (BGOs), even if one is a priori interested only in the Hilbert space behaviour:
Here are some examples:

1. Let $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ be given, and any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the coefficient operator
$$C : f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$$
only maps $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\mathrm{cosp}(\Lambda)$, but $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\ell^2(\Lambda)$.
2. similar situation for the synthesis operator, which is its adjoint.
3. Finally the frame-operator is a priori an operator which maps $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.

But this is good enough to show that it has a Janssen representation.

Regularizing Operators



Regularizers help to approximate distributions

An important family of regularizing operators are those bounded families of BGOs, where each of them maps $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ into $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, and which form an approximating the identity. Abstractly speaking they form bounded nets of BGOs which converge strongly (e.g. in their action on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) to the identity. Since we have

$$(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subset \boldsymbol{S}_0 \quad \text{and} \quad (\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subset \boldsymbol{S}_0$$

this can be product-convolution (or convolution-product) operators.
A similar property is shared by Gabor multipliers with finitely many symbols (using lattice points in a bounded domain).

Gabor Regularization Approach

Comparable to the situation with $\mathbb{R}$, the set of all infinite decimals, where one just uses a finite, maybe even small number digits in practice, one can view the (double) infinite representations of $\sigma \in \boldsymbol{S}_0'$ as the limit of the finite sums, describing the "signal" $\sigma \in \boldsymbol{S}_0'$ up to some resolution, starting from

$$\sigma = \sum_{\lambda \in \Lambda} \sigma(\widetilde{g}_\lambda) g_\lambda$$

where $\widetilde{g}_\lambda = \overline{\pi(\lambda)\widetilde{g}}$ are the TF-shifts of some dual window $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$, and the coefficient mapping $\sigma \to \sigma(\widetilde{g}_\lambda)$ is a bounded mapping from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\ell^\infty(\Lambda)$. For us: $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$.
Mapping properties of linear operators can be described by corresponding properties of their Gabor matrix representations.

Mild Bases

Definition

A family $(\tau_x)_{x\in\mathbb{R}^d}$ in $\boldsymbol{S}_0'(\mathbb{R}^d)$ is called a *mild basis*[a] for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if there exists some BGT-automorphism α such that $\tau_x = \alpha(\delta_x)$ for all $x \in \mathbb{R}^d$. We call it a *unitary mild basis* (for short **UMB**) for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if α is a unitary BGT-automorphism.

[a] This terminology relates well to the use of the Dirac basis in physics and other application areas. Strictly speaking it would be more correct to call it a *tight, mild frame*, because a basis is supposed to be minimal, while a frame is a generating system which allows some redundancy. All the *relevant* properties of the $(\delta_x)_{x\in\mathbb{R}^d}$ would be valid also for the family (for illustration only) $(\delta_q)_{q\in\mathbb{Q}}$ as well, since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. In a way, one can consider the *tight Dirac frame* $(\delta_x)_{x\in\mathbb{R}^d}$ as the limit of Gabor frames of the form $(\mathrm{St}_\rho g_0, \rho a, b/\rho)$, using compressed Gaussians as atoms, for $\rho \to 0$, starting from any pair (a, b) with $ab < 1$. Thus the Dirac basis consists of an infinitely fine grid in the time and an infinitely sparse grid in the frequency direction. Thus the concept of redundancy does not make sense anymore for this limiting case.

Following the habit of the applied literature one might describe a UMB as *mild orthonormal basis* (write **MOB**) for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.

Benefits of This Approach I

- The setting is universal and technically simple yet mathematically well founded (basic functional analysis);
- The extended Fourier transform for $S_0'(\mathbb{R}^d)$ contains all the relevant cases for engineering applications (proof of Shannon's Theorem, characterization of translation invariant systems);
- Flexible enough to deal with real-world signals which are neither periodic not decaying (music, heartbeat, etc.);
- analogy to the finite dimensional case (FFT, etc.), piecewise linear interpolation allows approximation in $S_0(\mathbb{R})$;
- existence of a kernel theorem for $\mathcal{L}(S_0, S_0')$ by kernels in $S_0'(\mathbb{R}^{2d})$, like matrix representation over $\mathbb{R}^n$.
- Justify heuristic principles, such as transition from Fourier Series to the Fourier Transform (period to ∞!?)

Links and Refs

Let us provide a few links:

ETH Course Notes are found at: www.nuhag.eu/ETH20
At the page www.nuhag.eu/feitalks:
you find my talks and also recorded talks from some courses, e.g. FROM LINEAR ALGEBRA TO GELFAND TRIPLES.
Various Lecture Notes can be downloaded from
https://www.univie.ac.at/nuhag-php/home/skripten.php
The public version of my publications can be provided upon request, via hans.feichtinger@univie.ac.at !!

This talk provides some thought related to the CONCEPTUAL HARMONIC ANALYSIS idea.