Mild Bases in Time-Frequency Analysis: How to Deal with Dirac-like Coherent Systems

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Harmonic Analysis and Beyond
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Outline

Changing the Setting

This talk, which is one in a long series of talks with the goal to promote the idea of "Conceptual Harmonic Analysis" (CHA) the emphasis is on the possibility to use the concept of "unitary Banach Gelfand Triple Isomorphisms" (UBGTIs) in order to come up with a solid description of what is usually called a "continuous ONB", such as the Dirac basis or the Fourier basis.

In short: Gabor Analysis has taught us how important tight frames are (e.g. for the definition of Gabor multipliers) and also the limited usefulness of $\boldsymbol{L}^p$-norms in this context. In contrast, using UBGTIs (as a replacement for invertible matrices) the analogy to finite dimensional vector spaces can be pushed quite far.
Thus we are aiming at a demonstration that the BGT setting allows to extend the intuition taken from Linear Algebra much better to the continuous setting than a pure Hilbert space view.

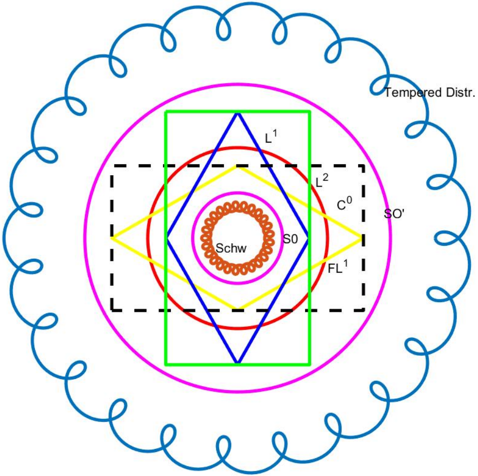
STROBL ABSTRACT:
Mild Bases in Time-Frequency Analysis -
How to Deal with Dirac-like Coherent Systems

Abstract Harmonic Analysis deals with Fourier expansions of functions in terms of pure frequencies, which are the natural eigen-vectors of the corresponding translation operators. At an operator level a similar situation appears in the context of the Wigner-Weyl calculus relevant for pseudo-differential operators or the treatment of time-variant linear systems. Here the building blocks are the time-frequency shifts.

In this talk we will indicate how the popular notion of a "continuous orthonormal basis" which is often used in a physics or engineering context for the family of Dirac measures, can be put on solid mathematical concepts, making use of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

In the background of these considerations stands the characterization of these spaces using Gabor expansions, with coefficients in $\ell^1, \ell^2, \ell^\infty$, and the fact that S_0*, the space of mild distributions, can be viewed as a good model for "signals" and "operators", in analogy two finite vectors and matrix representations of linear mappings in linear algebra. Using such a view-point one can consider the (fractional) Fourier transform as a change of bases in this context, among others.

A Zoo of Banach Spaces for Fourier Analysis



THE Banach Gelfand Triple

Feichtinger Algebra and Mild Distributions

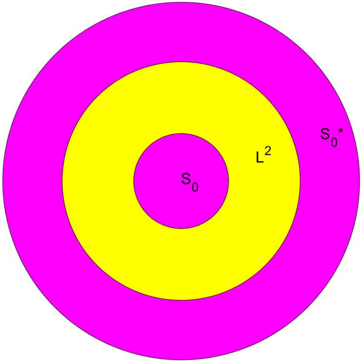
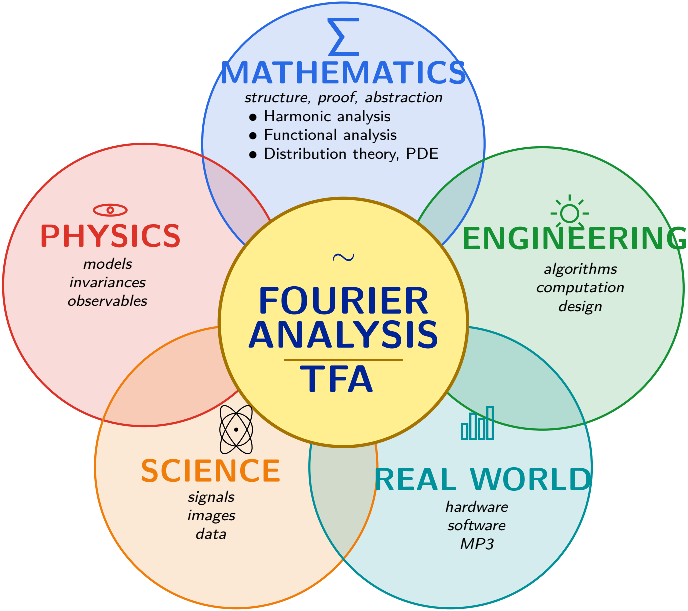


Figure: Feichtinger Algebra and Mild Distributions


MATHEMATICS
structure, proof, abstraction
• Harmonic analysis
• Functional analysis
• Distribution theory, PDE
PHYSICS
models
invariances
observables
ENGINEERING
algorithms
computation
design
FOURIER ANALYSIS
TFA
SCIENCE
signals
images
data
REAL WORLD
hardware
software
MP3

Orientation Questions

If we ask ourselves what the status of FOURIER ANALYSIS is in its third century we have to consider the following basic questions:

1. What are the key results in Fourier Analysis and what are their consequences?
2. What are the "technical tools" that have been developed?
3. Why are we using the tools which have been developed (or which are we using most, etc.)?
4. Are they working well and "serving the purpose"?

Compare: In order to push mobility industry has developed Diesel cars and built up a fossile infra-structure. We will need a lot of mobility also in the future, but is the current system the best possible and the most efficient?

The Role of Time-Frequency Analysis in Different Areas I

Let us just give a short list of examples connecting the world of time-frequency analysis to real-world applications:

- The Physics Nobel-Prize 2017 was given for the detection of *gravitational waves* using time-frequency methods;
- Early contact to colleagues from communication theory (TU-Vienna) was exposing us to slowly time-variant channels, Kohn-Nirenberg calculus and Gabor expansions;
- MP3 is performing a localized DFT in order to discard irrelevant information
- Wavelet theory is used for digital image compression;
- Quantum Harmonic Analysis is of current interest;
- Reconstruction of band-limited signals from irregular samples.

The Situation few Decades Ago (until now?) I

Various observations concerning the use of Fourier Analysis:
First the situation described like 40 years ago:

1. Mathematicians know Carleson's results (almost everywhere convergence of Fourier Series in $\boldsymbol{L}^2$), about Abstract Harmonic Analysis over LCA groups, or the Gelfand Theory of commutative Banach algebras, probably also the existance of the FFT for fast computations of the DFT (Cooley-Tukey). Tempered distributions are good for PDEs.
2. Engineers have been using FFT-based and other discrete algorithms in order to develop methods for signal processing (audio, image processing, compression, like MP3, JPG). Obviously the computer only allows to deal with finite vectors!
3. Physicists describe signals, observables and operators using continuous integrals, using them often just in an intuitive way.

Standard Assumptions I

The established way of looking at the Fourier transform or at operators,; works with the following ideas:

1. The Fourier transform is an *integral transform* and therefore the natural domain is $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$. It fact, also convolution is well defined in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, turning it into a commutative Banach algebra with bounded approximate units. The Fourier transform is the Gelfand transform. Plancherel's Theorem describes (its extension) as unitary automorphism of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Consequently one studies the FT on $\boldsymbol{L}^p(\mathbb{R}^d)$ (H-Y Thm.)
2. When it comes to the study of linear operators it appears to be natural to view the space of all bounded operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ as the universe. Since this space is the dual space for the Banach space of trace class operator $\mathcal{T}^1$, Schatten p-classes, with $\mathcal{HS}$ as central Hilbert space are the analogue.

Standard Assumptions II

3. Gabor Analysis has shown us that - unlike wavelet theory - one cannot have orthogonal, in fact not even Riesz bases of Gaborian type with well localized Gabor atoms. I fact, one of the early versions of the Balian-Low Theorem by Chris Heil confirms that g and $\widehat{g}$ in the Wiener algebra $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ implies that the Gabor system at critical density (with $a = 1 = b$) cannot form a Riesz basis for $\boldsymbol{L}^2(\mathbb{R}^d)$. In fact, as Janssen observed, the Zak transform has to have a zero whenever it is continuous (which is the case here).

   Of course this seems to be in sharp contrast to earlier results (Perelomov etc.) showing that coherent states taken along the von Neumann lattice form a total subset in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$. This difference and the consequences (stability) have given frame theory for Hilbert spaces so much attention.

Unitary matrices: rows, columns, and projections I

Let $U \in \mathbb{C}^{n \times n}$, and write its columns as $u_1, \ldots, u_n \in \mathbb{C}^n$ and its rows as $r_1, \ldots, r_n \in \mathbb{C}^n$. The following conditions are equivalent:

$$U^* U = I_n \quad \Longleftrightarrow \quad UU^* = I_n \quad \Longleftrightarrow \quad U^{-1} = U^*.$$

Equivalently, the columns of U form an orthonormal basis of $\mathbb{C}^n$, i.e. $\langle u_j, u_k \rangle = \delta_{j,k}$, or, what is the same for a square matrix, the rows of U form an orthonormal basis of $\mathbb{C}^n$, i.e. $\langle r_j, r_k \rangle = \delta_{j,k}$. Viewed as a linear operator on $\mathbb{C}^n$, such a matrix preserves the Hilbert-space norm and inner product: $|Ux|_2 = |x|_2$ and $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}^n$. Consequently it preserves lengths and angles. This is the reason for calling U a *unitary matrix*. In the real case this reduces to an *orthogonal matrix*, and the inverse is the transpose, $U^{-1} = U^T$.

Unitary matrices: rows, columns, and projections II

Equivalently, if $u_1, \ldots, u_n$ denotes the orthonormal system given by the columns of U, then the identity operator has the resolution

$$I_n x = \sum_{k=1}^{n} \langle x, u_k \rangle u_k, \qquad x \in \mathbb{C}^n, \quad \text{or briefly} \quad I_n = \sum_{k=1}^{n} u_k \otimes u_k.$$

Thus the identity is the sum of the rank-one orthogonal projections onto the one-dimensional components generated by the vectors u_k. Consequently one should not describe frames as generalizations of ONBs but rather tight frames, whereas frames (i.e. generating systems) are generalizations of (oblique) bases.

Wrong generalizations to Hilbert spaces I

When passing from finite-dimensional linear algebra to the setting of a separable Hilbert space $\mathcal{H}$, some basic notions have to be generalized with care. The naive extensions of "generating system" and "linear independence" are often misleading.

- A family $(g_i)_{i \in I}$ is called *total* if its closed linear span is all of $\mathcal{H}$, i.e. $\overline{\mathrm{span}}\{g_i : i \in I\} = \mathcal{H}$. This is the analogue of spanning, but by itself it gives no stable expansion procedure.
- Ordinary linear independence only tests finite subcollections: $\sum_{k=1}^{N} c_k g_{i_k} = 0$ implies $c_k = 0$. In infinite-dimensional Hilbert spaces this is too weak to guarantee stable reconstruction or useful coefficient control.

Wrong generalizations to Hilbert spaces II

- The correct replacement for a stable generating system is a *frame*: there exist constants $0 < A \leq B < \infty$ such that $A\|f\|_{\mathcal{H}}^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B\|f\|_{\mathcal{H}}^2$ for all $f \in \mathcal{H}$. Frames allow stable, redundant expansions and are therefore the right infinite-dimensional substitute for well-conditioned spanning systems.
- The correct replacement for a stable independent system is a *Riesz basic sequence*: the synthesis map from square-summable coefficient sequences onto the closed span of $(g_i)_{i \in I}$ is an isomorphism. Equivalently, there are constants $0 < A \leq B < \infty$ such that $A \sum |c_i|^2 \leq \| \sum c_i g_i \|_{\mathcal{H}}^2 \leq B \sum |c_i|^2$ for all finitely supported coefficient families.

Wrong generalizations to Hilbert spaces III

- Thus, in Hilbert spaces, the useful dichotomy is not simply “spanning versus independent”, but rather *frames* for stable analysis and reconstruction, and *Riesz basic sequences* for stable independent systems.

Wrong generalizations to Hilbert spaces I

When passing from finite-dimensional linear algebra to a separable Hilbert space $\mathcal{H}$, one should not simply replace "spanning" by *totality* and keep ordinary finite linear independence as the counterpart of independence.

- Totality, i.e. $\overline{\operatorname{span}}\{g_i : i \in I\} = \mathcal{H}$, is only a qualitative density condition. It does not provide stable coefficients or stable reconstruction.
- Finite linear independence is likewise too weak in infinite dimensions. It does not control the synthesis map and is not stable under limiting processes.
- The useful Hilbert-space replacements are therefore: *frames* for stable generating systems, and *Riesz basic sequences* for stable independent systems.

Wrong generalizations to Hilbert spaces II

- Thus the finite-dimensional dichotomy "spanning versus independent" should be replaced by the stable operator-theoretic dichotomy "frame versus Riesz basic sequence".

This is the conceptual point behind many frame-theoretic results: one should not merely preserve algebraic properties, but rather preserve stable analysis and stable synthesis.

The critical Gaussian Gabor family with lattice $\mathbb{Z} \times \mathbb{Z}$ gives and example of a total and linear independent, countable family (of coherent states) in the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$ which is neither a frame nor a Riesz basic sequence.

Frames in Hilbert spaces

Since this section of the conference is devoted to Pete Casazza we have to mention the concept of frames, which - according to my knowledge - caught his interest after a long-term visit by Ole Christensen. According to Pete (email of 2025) he was asking me around 2003 whether it would be OK to call the conjecture (which arose from the study of Gaborian Gabor frames with "good atoms") that any such (equal norm) frame is a finite union of Riesz basic sequences the Feichtinger Conjecture. Of course I am immensely grateful for him, not only for setting this terminology but above all for the investigations surrounding this question and after all establishing the (unexpected but exciting) connection to the older and already well-known Kadison-Singer conjecture.

Still let me mention right away that the concentration of frame theory on the setting of Hilbert spaces has some weakness!

Kadison and Feichtinger in Oberwolfach

Figure: Kadison in Feichtinger in Oberwolfach

The Key-players for Time-Frequency Analysis (TFA)

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

–

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t),$$

with $x, \omega, t \in \mathbb{R}^d$, compatible with the Fourier transform

$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \ \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\boldsymbol{M}^1(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ *(dense embedding);*
2. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ *is a Banach algebra under both pointwise multiplication and convolution;*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Benefits of the BGT point of view I

The Banach Gelfand Triple $(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$ provides a natural functional-analytic setting in which many stability properties of Gabor analysis can be formulated in a precise and robust way.

- If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and the Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$, then the corresponding Gabor frame operator $S_{g,\Lambda}$ is a BGT-isomorphism of $(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$. Consequently the canonical dual Gabor atom $\gamma = S_{g,\Lambda}^{-1} g$ again belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$. The same applies to the canonical tight Gabor atom $S_{g,\Lambda}^{-1/2} g$, which also belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$.
- The canonical dual atom depends continuously on the time-frequency lattice Λ. Thus small perturbations of Λ lead to small changes of the dual atom, in the topology of $\boldsymbol{S}_0(\mathbb{R}^d)$, as long as the Gabor frame property is preserved.

Benefits of the BGT point of view II

- The computation of canonical dual Gabor atoms can be realized by discrete methods. This remains true in the multi-window case, where the corresponding multi-window Gabor frame operator is again a BGT-isomorphism, and its inverse can be approximated by suitable finite or discrete matrix computations.
- Gabor multipliers depend continuously on their ingredients: on the analysis window, the synthesis window, the lattice, and the upper symbol. Moreover, the mapping from upper symbols to the corresponding Gabor multiplier is itself a BGT-morphism. In this sense the BGT setting describes not only the boundedness of such operators on $\boldsymbol{L}^2(\mathbb{R}^d)$, but also their regularity-preserving behavior on $\boldsymbol{S}_0(\mathbb{R}^d)$ and their natural extension to $\boldsymbol{S}_0'(\mathbb{R}^d)$.

The universe of mild distributions I

The dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ to the Banach algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ is large enough to contain all the objects which are needed in the context of signal processing. The universe of mild distributions contains continuous, discrete, periodic, and non-periodic models.

- Continuous periodic functions (Fourier Series, multi-dim.);
- Continuous non-periodic functions from $\boldsymbol{L}^p(\mathbb{R}^d), 1 \leq \infty$, as well as almost periodic functions.
- Discrete non-periodic functions (discrete groups);
- Discrete and periodic functions: finite-dimensional setting (DFT/FFT) $>>$ computational methods

Thus $\boldsymbol{S}_0'(\mathbb{R}^d)$ provides a common ambient space for classical Fourier analysis, Fourier series, discrete Fourier analysis, and finite time-frequency analysis. The BGT $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ allows a coherent functional-analytic framework.

The universe of mild distributions II

In particular, it allows transitions between different settings, via sampling and quasi-interpolation, using distributional, i.e. w^*-convergence in this setting. I call it *mild convergence*. While norm convergence in $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is just uniform convergence of the STFT with respect to some/any window $0 \neq g \in \boldsymbol{S}_0(\mathbb{R}^d)$, mild convergence corresponds to the more natural *local uniform convergence*. A sequence of signals is thus very similar if for every finite set of measurements (e.g. Gabor coefficients) one observes a small difference of the outcome, for large indices.

This is much stronger than distributional convergence in $\mathcal{S}'(\mathbb{R}^d)$, because $\mathcal{S}(\mathbb{R}^d)$ is a small, still dense subspace of $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$!

The universe of mild distributions III

Dirac combs over lattices) belong to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$. By Poisson's formula (for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$!) $\sqcup\!\sqcup$ is Fourier invariant!
Pointwise multiplication by $\sqcup\!\sqcup$ corresponds to regular sampling. Via the extended FT this turns into convolution with $\sqcup\!\sqcup$ and thus periodization. This allows an easy prove of practical versions of the Shannon Sampling Theorem.
Even the combination of sampling (fine lattice) with periodization (along a coarse grid), or a product-convolution operator, called PS-operators, allows to map $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ and can be well treated in this context (e.g. period $h = 1/p, p = 4^k, k \in \mathbb{N}$).
It may be no surprise that $\mathrm{PS}_p(f) \to f$ in the sense of mild distributions for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. Quasi-interpolation even allows norm reconstruction from fine sampled, periodized versions of f (for $k \to \infty$) (results with N. Kaiblinger).

True Signals can be modeled as mild distributions

A central message of this talks reads:

Fourier analysis becomes cleaner when formulated in the context of the *mild* Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ (**mBDT**), with

$$\boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{L}^2(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d).$$

The space $\boldsymbol{S}_0(\mathbb{R}^d)$ provides enough regularity (continuity plus absolute Riemann integrability) while $\boldsymbol{S}_0'(\mathbb{R}^d)$ provides enough generality to include all kinds of "signals": continuous and discrete, periodic and non-periodic, as treated usually in the context of Abstract Harmonic Analysis.

We have come to the conclusion that the elements $\sigma \in \boldsymbol{S}_0'$ provide a good model for *signals* while the bilinear evaluation of σ on test functions $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ can be viewed as (legitimate) measurement.

THE Banach Gelfand Triple Situation

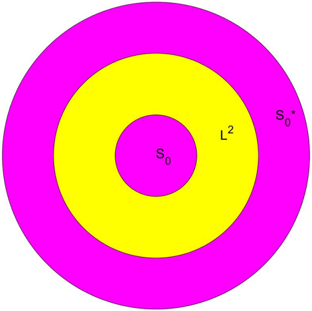


Figure: Three layers of Banach spaces, with the Hilbert space in the middle

BGT Morphism: Four Layers

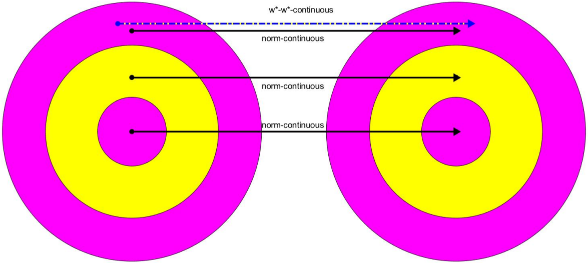


Figure: BGT3morph1.jpg: Note that there are four continuity requirements!

Regularizing operators on BGTs

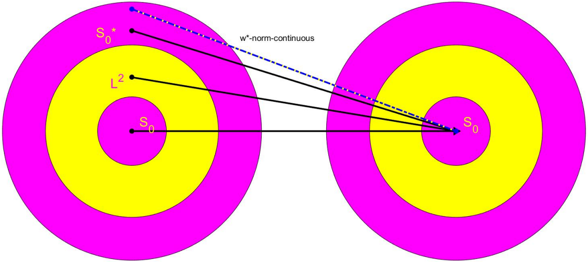


Figure: Regularizing operators: w^*-convergent nets to norm convergent ones.

Good operators compared to Trace-Class Operators

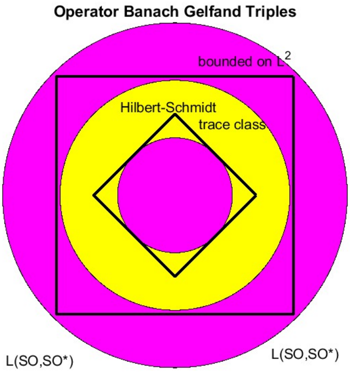


Figure: QRCtrace26a.jpg: Larger predual (trace class) implies smaller dual space: only operators bounded on L2!

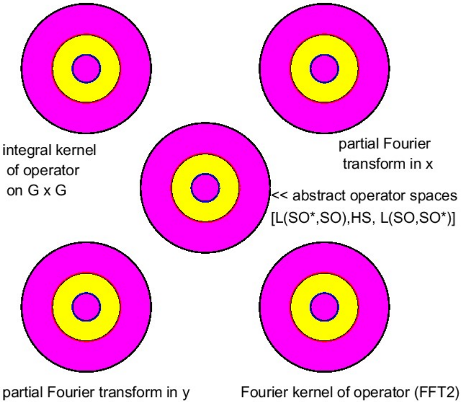


Figure: The spaces of operators $(\boldsymbol{B}, \mathcal{H}S, \boldsymbol{B}')$ can be identified with four different representations of operators.

The Fourier transform as a unitary BGT-automorphism I

The Fourier transform is the basic example of a unitary automorphism of the Banach Gelfand Triple $(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$. With the usual normalization,

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(t)\, \mathrm{e}^{-2\pi i\, \omega\cdot t}\, dt, \qquad f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

it maps $\boldsymbol{S}_0(\mathbb{R}^d)$ onto itself, extends uniquely to a unitary operator on $\boldsymbol{L}^2(\mathbb{R}^d)$, and further extends by duality to a weak-star continuous automorphism of $\boldsymbol{S}_0'(\mathbb{R}^d)$.

The FT is a unitary BGT-automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. Thus all three levels of the triple are respected: $\mathcal{F} : \boldsymbol{S}_0(\mathbb{R}^d) \to \boldsymbol{S}_0(\mathbb{R}^d)$ is a Banach-space isomorphism, $\mathcal{F} : \boldsymbol{L}^2(\mathbb{R}^d) \to \boldsymbol{L}^2(\mathbb{R}^d)$ is unitary, and $\mathcal{F} : \boldsymbol{S}_0'(\mathbb{R}^d) \to \boldsymbol{S}_0'(\mathbb{R}^d)$ is the corresponding distributional extension. This provides the conceptual reason why the Fourier transform is not merely an integral transform on suitable functions, but a

The Fourier transform as a unitary BGT-automorphism II

structural symmetry of the whole BGT. It preserves the Hilbert-space geometry on $\boldsymbol{L}^2(\mathbb{R}^d)$ and at the same time preserves the test-function space $\boldsymbol{S}_0(\mathbb{R}^d)$ and the space of mild distributions $\boldsymbol{S}_0'(\mathbb{R}^d)$.
Equivalently, the inverse transform is again a BGT-automorphism and is given by $\mathcal{F}^{-1} = \mathcal{F}^*$ on $\boldsymbol{L}^2(\mathbb{R}^d)$. In particular, one has $\|\mathcal{F}f\|_2 = \|f\|_2$ for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, while the same operator also gives topological isomorphisms on the test-function and distribution levels.
Being defined via duality, i.e. using $\widehat{\sigma}(f) = \sigma(\widehat{f})$, $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ it is also w^*-w^*--continuous.

Retracts in Interpolation Theory

A linear mapping $\mathcal{C} : \boldsymbol{X} \to \boldsymbol{Y}$ defines a retract from $\boldsymbol{X}$ into $\boldsymbol{Y}$ if there exists a bounded linear mapping $\mathcal{R} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$\mathcal{R} \circ \mathcal{C} = \mathrm{Id}_{\boldsymbol{X}} .$$

Set $\boldsymbol{Y}_0 := \mathrm{range}(\mathcal{C})$. Then $P := \mathcal{C} \circ \mathcal{R}$ is an idempotent operator on $\boldsymbol{Y}$, and therefore $\boldsymbol{Y}_0$ is a closed complemented subspace of $\boldsymbol{Y}$.

Moreover, $\mathcal{C}$ establishes an isomorphism $\mathcal{C} : \boldsymbol{X} \to \boldsymbol{Y}_0$.
Thus $\boldsymbol{X}$ may be identified with $\boldsymbol{Y}_0$, and also with the quotient

$$\boldsymbol{Y} / \ker(\mathcal{R}).$$

Retract diagram

$$
\begin{tikzcd}
 & \boldsymbol{Y} \arrow[dl, "\mathcal{R}"'] \arrow[d, "P"] \\
\boldsymbol{X} \arrow[r, shift right, "\mathcal{C}"'] & \boldsymbol{Y}_0 \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \ell^2(I) \arrow[dl, "\mathcal{R}"'] \arrow[d, "\mathbf{P}"] \\
\mathcal{H} \arrow[r, shift right, "\mathcal{C}"'] & C(\mathcal{H}) \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Banach Frames for the Banach Gelfand Triples

$$
\begin{tikzcd}
 & & (\ell^1, \ell^2, \ell^\infty)(\Lambda) \arrow[dll, "\mathcal{R}"'] \arrow[d, "\mathbf{P}"] \\
(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0') \arrow[rr, shift right, "\mathcal{C}"'] & & C((\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')) \arrow[ll, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Basic Properties

Theorem

For $d \in \mathbb{N}$ the collection $\mathrm{Aut}_{\mathrm{BGT}}(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ of all BGT-automorphisms of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ is a group under composition. Its elements are precisely the BGT-isomorphisms mapping $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ onto itself. In analogy with the notation for the general linear group of matrices, we write $\mathrm{GL}(d, \boldsymbol{S}_0)$.
The subset of all unitary BGT-automorphisms, i.e. those which are unitary operators at the Hilbert-space level forms a closed subgroup of $\mathrm{GL}(d, \boldsymbol{S}_0)$. denoted by $\mathrm{U}(d, \boldsymbol{S}_0)$. Its elements will be called unitary BGT-automorphisms, or briefly UBGT-automorphisms. Thus $T \in \mathrm{U}(d, \boldsymbol{S}_0)$ means that T is a BGT-automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ whose Hilbert-space extension to $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ is unitary (like FT).

Mild Bases, Definition

Definition

We call a family $(\tau_x)_{x\in\mathbb{R}^d}$ a *mild basis*[a] for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ if there exists some BGT-automorphism α such that $\tau_x = \alpha(\delta_x), x \in \mathbb{R}^d$. We call it a *unitary mild basis* (a **UMB**) for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ if there exists some unitary BGT-automorphism.

[a] This terminology relates well to the use of the Dirac basis in physics and other application areas. Strictly speaking it would be more correct to call it a *tight, mild frame*, because a basis is supposed to be minimal, while frames allow some redundancy. All the *relevant* properties of the $(\delta_x)_{x\in\mathbb{R}^d}$ would be valid also for the family (for illustration only) $(\delta_q)_{q\in\mathbb{Q}}$ as well, since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. One can consider the *tight Dirac frame* $(\delta_x)_{x\in\mathbb{R}^d}$ as the limit of Gabor frames of the form $(\mathrm{St}_\rho g_0, \rho a, b/\rho)$, using compressed Gaussians as atoms, for $\rho \to 0$, starting from any pair (a, b) with $ab < 1$. Thus the Dirac basis consists of in infinitely fine grid in time and an infinitely sparse grid in frequency, and the concept of redundancy does not make much sense anymore for this limiting case.

Signals, operators, and BGT coordinate systems I

The Banach Gelfand Triple point of view allows one to say, in a precise functional-analytic sense, that signals are mild distributions: a signal is represented at the outer level by an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$, with the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ and the test-function space $\boldsymbol{S}_0(\mathbb{R}^d)$ as distinguished inner levels.
The same principle applies to operators. By the kernel theorem, suitable linear mappings from $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$ can be identified, in the spirit of matrix representations, with signals over the product domain, that is, with mild distributions of two variables, belonging naturally to a space of the form $\boldsymbol{S}_0'(\mathbb{R}^{2d})$.
Thus an operator on signals may be studied through its kernel, symbol, spreading function, or other phase-space representation.
In each case there are various continuous coordinate systems.
Unitary Banach Gelfand Triple automorphisms provide the correct language for passing from one such coordinate system to another.

Signals, operators, and BGT coordinate systems II

Transformations and partial Fourier transforms act first at the inner level and then extend canonically to the Hilbert-space and distribution levels, exactly as in the case of the Fourier transform. Important examples are provided by members of the metaplectic group, by the spreading representation, and by the Kohn–Nirenberg representation of operators. These representations give continuous coordinate systems for describing operators, while the BGT formalism guarantees that the transitions between them are compatible with $\boldsymbol{S}_0(\mathbb{R}^d)$, $\boldsymbol{L}^2(\mathbb{R}^d)$, and $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Thus the BGT framework unifies three viewpoints:

- signals as mild distributions in $\boldsymbol{S}_0'(\mathbb{R}^d)$;
- operators as mild distributions on product domains;
- coordinate changes as unitary BGT-automorphisms, implemented by transformations, partial Fourier transforms, and metaplectic operators.

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TEST 777
Change of bases
Composition laws
dual frames (self-dual!)
distributional convergence: different levels
citefeko98, groups of unitary operators
Both K and K' have to leave $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ invariant plus
UNITARY property, which is essentially

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With the group structure in the background we have a situation which is completely analog to the case of matrices and bases:

Theorem

- *Given two mild bases there exists exactly one UBGT mapping the first one into the second one.*
- *Given two different mild bases and the $\boldsymbol{S}_0'(\mathbb{R}^{2d})$-kernel of a linear mapping $T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ with respect to one of these bases the corresponding kernel can be obtained by a standard "change of basis" (conjugation) procedure.*
- *The (multiple composition) of operators can be realized by taking the mild limits of compositions of the standard product of integral operators in analogy to matrix multiplication.*

Fractional Fourier Transform I

The α-order fractional Fourier transform of a function $f(t)$ is defined as:

$$\mathcal{F}_\alpha\{f(t)\} = F_\alpha(u) = \int_{-\infty}^{\infty} f(t) K_\alpha(t,u)\, dt,$$

where the kernel $K_\alpha(t,u)$ is given by:

$$K_\alpha(t,u) = \begin{cases} \sqrt{1 - i\cot\alpha}\, e^{i\pi(t^2+u^2)\cot\alpha - 2i\pi tu\csc\alpha}, & \alpha \notin n\pi, \\ \delta(t-u), & \alpha = 0, \\ \delta(t+u), & \alpha = \pi. \end{cases}$$

Final message: mild bases and CHA I

The aim of this talk is not to add another technical variant of Fourier analysis, but to propose a change of viewpoint. The Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ provides a setting in which the intuition from finite-dimensional linear algebra can be transferred to the continuous world more faithfully than in the pure Hilbert-space setting.

- The role of vectors is played by mild distributions: signals are elements of $\mathbf{S}_0'(\mathbb{R}^d)$, while test functions in $\mathbf{S}_0(\mathbb{R}^d)$ provide legitimate measurements.
- The role of matrices is played by kernels, symbols, spreading functions, or other representations of operators, all living naturally as mild distributions on product domains.
- The role of unitary matrices is played by unitary BGT-automorphisms: they preserve $\mathbf{S}_0(\mathbb{R}^d)$, act unitarily on $\mathbf{L}^2(\mathbb{R}^d)$, and extend weak-star continuously to $\mathbf{S}_0'(\mathbb{R}^d)$.

Final message: mild bases and CHA II

- The Fourier transform, fractional Fourier transforms, metaplectic operators, and the spreading representation are therefore not merely formulas, but coordinate changes inside the same BGT universe.
- What is often called a "continuous orthonormal basis", such as the Dirac or Fourier basis, should be understood as a unitary mild basis: a Dirac-like coherent system obtained from the standard Dirac family by a unitary BGT-automorphism.

It allows allows to express the way how continuous operations can be a approximated by discrete ones, say the FT by making use of the FFT, applied to samples of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. A quantitative version has been published recently by Ehler/Groechenig/Klotz in SIAM.

Final message: mild bases and CHA III

Thus the BGT keeps the useful Hilbert-space geometry, but adds the test-function and distribution levels which are needed for time-frequency analysis, operators, and applications. The crux is the bilinear pairing between test functions which extends to mild distribution (in the first OR the second argument).
The setting is less general than that of tempered distributions, hence more precise when it comes to convergence and less restrictive in the choice of test functions. BUT it still allows to give (mostly integral) expressions in physics or engineering books a correct interpretation, in the spirit of Dirac's Bra-Ket notation.

From linear algebra to Gabor frames without inequalities I

The finite-dimensional model. Let $A \in \mathbb{R}^{n\times n}$. In linear algebra the following conditions are equivalent:

$$A \text{ is invertible} \iff A \text{ is one-to-one} \iff A \text{ is onto.}$$

Equivalently, if $v_1, \dots, v_n$ are the columns of A, then

$$(v_1, \dots, v_n) \text{ is a basis of } \mathbb{R}^n$$

if and only if every vector has a unique expansion

$$f = \sum_{k=1}^{n} c_k v_k.$$

Thus, in finite dimensions, basis properties can be tested without estimating norms or proving inequalities.

From linear algebra to Gabor frames without inequalities II

The infinite-dimensional difficulty. For operators on infinite-dimensional spaces this equivalence usually fails: one-to-one does not imply invertible, and dense range does not imply onto. Hence the usual Gabor frame condition

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2, \qquad f \in \boldsymbol{L}^2(\mathbb{R}^d),$$

contains the genuinely difficult part: the lower frame bound.

From linear algebra to Gabor frames without inequalities III

Gröchenig's observation. In the paper *Gabor frames without inequalities*, Gröchenig shows that, for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, part of the finite-dimensional intuition returns. The Gabor frame property of

$$\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$$

is equivalent to several qualitative conditions which do not involve an explicit frame inequality.
With the analysis operator

$$C_{g,\Lambda} f = \left(\langle f, \pi(\lambda)g \rangle\right)_{\lambda \in \Lambda}$$

one has, in particular,

$$\mathcal{G}(g,\Lambda) \text{ is a frame for } \boldsymbol{L}^2(\mathbb{R}^d)$$

$$\iff \quad C_{g,\Lambda} : \boldsymbol{S}_0'(\mathbb{R}^d) \to \ell^\infty(\Lambda) \text{ is one-to-one.}$$

From linear algebra to Gabor frames without inequalities IV

Thus the frame property can be detected by injectivity on the large distribution space $\boldsymbol{S}_0'(\mathbb{R}^d)$.

There is also a corresponding synthesis statement. The operator

$$D_{g,\Lambda}c = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g$$

has dense range in $\boldsymbol{S}_0(\mathbb{R}^d)$, and in fact is surjective

$$D_{g,\Lambda} : \ell^1(\Lambda) \to \boldsymbol{S}_0(\mathbb{R}^d),$$

precisely in the frame case.

From linear algebra to Gabor frames without inequalities V

The adjoint lattice. Using the adjoint lattice Λ°, Gröchenig obtains the dual formulation

$$D_{g,\Lambda^\circ} : \ell^\infty(\Lambda^\circ) \to \boldsymbol{S}_0'(\mathbb{R}^d) \text{ is one-to-one.}$$

Equivalently, the analysis operator on the adjoint lattice satisfies the surjectivity condition

$$C_{g,\Lambda^\circ} : \boldsymbol{S}_0(\mathbb{R}^d) \to \ell^1(\Lambda^\circ) \text{ is onto.}$$

Thus the Hilbert-space frame property is encoded by endpoint mapping properties between

$$\boldsymbol{S}_0(\mathbb{R}^d), \quad \boldsymbol{L}^2(\mathbb{R}^d), \quad \boldsymbol{S}_0'(\mathbb{R}^d)$$

From linear algebra to Gabor frames without inequalities VI

and the corresponding sequence spaces

$$\ell^1, \quad \ell^2, \quad \ell^\infty.$$

Moral for this talk. In finite dimensions, invertibility of a matrix can be tested by injectivity or surjectivity. For Gabor frames, this principle becomes true again in a surprising way: not merely inside $\boldsymbol{L}^2(\mathbb{R}^d)$, but in the natural Banach Gelfand triple

$$\boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{L}^2(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d).$$

This is one reason why mild distributions are not only a technical extension, but part of the natural language of Gabor analysis.

Links and Refs I

Thanks for your attention!

Let us provide a few links:
At the page www.nuhag.eu/feitalks:
you find my talks and also recorded talks from some courses, e.g. FROM LINEAR ALGEBRA TO GELFAND TRIPLES.

Various Lecture Notes can be downloaded from
https://www.univie.ac.at/nuhag-php/home/skripten.php
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The almost complete list of my publications can be downloaded from www.nuhag.eu/feipub !!